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Research Article

Modified Reduced Differential Transform Method for Partial Differential-Algebraic Equations

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This work presents the application of the reduced differential transform method (RDTM) to find solutions of partial differential-algebraic equations (PDAEs). Two systems of index-two and index-three are solved to show that RDTM can provide analytical solutions for PDAEs in convergent series form. In addition, we present the posttreatment of the power series solutions with the Laplace-Padé resummation method as a useful technique to find exact solutions. The main advantage of the proposed technique is that it is based on a few straightforward steps and does not generate secular terms or depend on a perturbation parameter.

1. Introduction

As widely known, the importance of research on partial differential-algebraic equations (PDAEs) is that many phenomena, practical or theoretical, can be easily modelled by such equations. Those kinds of equations arise in fields like nanoelectronics [1], electrical networks [2–4], and mechanical systems [5], among others.

In recent years, PDAEs have received much attention. Nevertheless, the theory in this field is still young. For linear PDAEs, the convergence of Runge-Kutta method is investigated in [6]. The numerical solution to linear PDAEs with constant coefficients and the study of indices are given in [7–10]. Linear and nonlinear PDAEs are characterized by means of indices which play an important role in the treatment of these equations. The differentiation index is defined as the minimum number of times that all or part of the PDAE must be differentiated with respect to time, in order to obtain the time derivative of the solution, as a continuous function of the solution and its space derivatives [11].

Higher-index PDAEs (differentiation index greater than one) are known to be difficult to treat even numerically. Often such problems are first transformed to index-one systems before applying numerical integration methods. This procedure called index-reduction can be very expensive and may change the properties of the solution. Since application problems in science and engineering often lead to higher-index PDAEs, new techniques are required to solve these problems efficiently.

Modern methods like differential transform method (DTM) [12, 13], reduced differential transform method (RDTM) [14–16], homotopy perturbation method (HPM) [17, 18], homotopy analysis method (HAM) [19], variational iteration method (VIM) [20], and generalized homotopy method [21], among others, are powerful tools to approximate linear and nonlinear problems. Recently, the modifications of the HPM have been used to solve DAEs [22–25]. Besides, the multivariate Padé series [26] was applied to solve PDAEs. Analytical solutions aid researchers in studying the effect of different variables or parameters of functions under consideration easily [27].

Among the abovementioned methods, the DTM is highlighted by its simplicity and versatility to solve nonlinear differential equations. This method does not rely on a perturbation parameter or a trial function like other popular approximation methods. In [12], the DTM was introduced to the engineering field as a tool to find approximate solutions of electrical circuits. DTM produces approximations based on an iterative procedure derived from the Taylor series expansion. This method is very effective and powerful for solving various kinds of differential equations as nonlinear biochemical reaction model [13], two-point boundary-value problems [28], differential-algebraic equations [29], the KdV and mKdV equations [30], the Schrodinger equations [31], fractional differential equations [32], and the Riccati differential equation [33], among others. Later, the RDTM [14–16] was proposed in order to provide a simplified (but not less powerful) version of DTM. The computation resources required by RDTM are much less than those for DTM. Moreover, using RDTM, the solution to initial valued problems can be expressed as an infinite power series. Later, taking advantage of the resummation methods capabilities [34–38], the domain of convergence of such power series can be extended leading in some cases to the exact solution. As well, the RDTM has been applied successfully to problems such as fractional differential equations [39], generalized KdV equations [16], generalized Hirota-Satsuma coupled KdV equation [40], Fornberg-Whitham equations [41], Newell-Whitehead-Segel equation [42], time-fractional telegraphic equation [43], radial diffusivity equation [44], and nonlinear evolutions equations [45].

Therefore, in this paper we present the application of a hybrid technique combining RDTM, Laplace transform, and Padé approximant [46] to find analytical solutions for PDAEs [34–38]. Solutions to PDAEs are first obtained in convergent series form using the RDTM. To improve the solution obtained from RDTM's truncated series, we apply Laplace transform to it and then convert the transformed series into a meromorphic function by forming its Padé approximant. Finally, we take the inverse Laplace transform of the Padé approximant to obtain the analytical solution. This hybrid method (LPRDTM) which combines RDTM with Laplace-Padé posttreatment greatly improves RDTM's truncated series solutions in convergence rate. In fact, the Laplace-Padé resummation method enlarges the domain of convergence of the truncated power series and often leads to the exact solution.

It is important to remark that LPRDTM can obtain exact solutions without requiring an index-reduction for the PDAEs. The proposed method does not produce noise terms also known as secular terms as the homotopy perturbation based techniques [22]. This property of RDTM greatly reduces the volume of computation and improves the efficiency of the method in comparison to the perturbation based methods. What is more, LPRDTM does not require a perturbation parameter like the perturbation based techniques (including HPM). Finally, LPRDTM is straightforward and can be programmed using computer algebra packages like Maple or Mathematica.

The rest of this paper is organized as follows. In the next section, we describe how the RDTM can be applied to solve PDAEs. The main idea behind the Padé approximant is given in Section 3. In Section 4, we give the basic concept of the Laplace-Padé resummation method. In Section 5, we apply LPRDTM to solve two PDAEs problems of index-two and index-three. In Section 6, we give a brief discussion. Finally, a conclusion is drawn in the last section.

2. Reduced Differential Transform Method (RDTM)

In this section we will describe the reduced differential transform method to solve PDAEs.

Definition 1. If a function $u(t,x)$ is analytical and continuously differentiable with respect to time t and space x in the domain of interest Ω, then

$$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(t,x)\right]_{t=0}, \quad x \in \Omega, \tag{1}$$

is the transformed function of $u(t,x)$.

Definition 2. The differential inverse transform of $\{U_k(x)\}_{k=0}^{n}$ is defined by

$$u(t,x) = \sum_{k=0}^{\infty} U_k(x)\, t^k. \tag{2}$$

Substituting (1) into (2), we deduce that

$$u(t,x) = \sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(t,x)\right]_{t=0} t^k. \tag{3}$$

From the above definitions, it is easy to see that the concept of the RDTM is obtained from the power series expansion. To illustrate the application of the proposed RDTM to solve PDAEs, we consider the following nonlinear PDAE system:

$$A\frac{\partial}{\partial t}u(t,x) + B\frac{\partial^2}{\partial x^2}u(t,x) + N(u(t,x)) = g(t,x), \quad t \geq 0,\ x \in \Omega, \tag{4}$$

where A and B are $n \times n$ square matrices with A singular, $N\ (u(t,x))$ is a nonlinear differential operator, and g is a known analytical function.

PDAE (4) is supplied with some consistent initial conditions:

$$u(0,x) = f(x), \quad x \in \Omega. \tag{5}$$

In contrast to parabolic or hyperbolic initial value problems, initial conditions for PDAEs cannot be prescribed for all components of the solution vector arbitrarily as initial conditions have to fulfill certain consistency conditions.

RDTM establishes that the solution to a differential equation can be written as

$$u(t,x) = \sum_{k=0}^{\infty} U_k(x)\, t^k, \tag{6}$$

where $U_0(x), U_1(x), \ldots$ are unknown functions to be determined by RDTM.

Applying the RDTM to initial condition (5) and PDAE (4), respectively, we obtain the transformed initial condition

$$U_0(x) = f(x), \quad x \in \Omega, \tag{7}$$

and the recursion system

$$(1+k) A U_{k+1}(x) + B \frac{d^2}{dx^2} U_k(x) + N(U_0(x), \ldots, U_k(x)) = G(x), \quad x \in \Omega, \ k = 0, 1, 2, \ldots, \tag{8}$$

where $(d^2/dx^2)U_k(x)$, $N(U_0(x), \ldots, U_k(x))$, and $G(x)$ are the reduced differential transforms of $(\partial^2/\partial x^2)u(t,x)$, $N(u(t,x))$, and $g(t,x)$, respectively.

Substituting (7) into (8) and solving the resulting system, we determine the unknown functions $U_k(x)$, $k = 0, 1, 2, \ldots$. Then, the differential inverse transformation of the set of functions $\{U_k(x)\}_{k=0}^{n}$ gives the approximate solution

$$\overline{u}(t,x) = \sum_{k=0}^{n} U_k(x) t^k, \tag{9}$$

where n is the approximation order of the solution. The exact solution to problem (4)-(5) is then given by

$$u(t,x) = \sum_{k=0}^{\infty} U_k(x) t^k. \tag{10}$$

If $U_k(x)$ and $V_k(x)$ are the reduced differential transforms of $u(t,x)$ and $v(t,x)$, respectively, then the main operations of RDTM are shown in Table 1.

The process of RDTM can be described as follows.

(1) Apply the reduced differential transform to the initial conditions.

(2) Apply the reduced differential transform to the PDAE to obtain a recursion system for the unknown functions $U_0(x), U_1(x), \ldots$.

(3) Use the transformed initial conditions and solve the recursion system for the unknown functions $U_0(x)$, $U_1(x), \ldots$.

(4) Use differential inverse transform formula (9) to obtain an approximate or exact solution for the PDAE.

The solutions series obtained from RDTM may have limited regions of convergence, even if we take a large number of terms. Therefore, we propose to apply the Laplace-Padé resummation method to RDTM truncated series to enlarge the convergence region as depicted in the next section.

3. Padé Approximant

Let $u(t)$ be an analytical function with Maclaurin's expansion:

$$u(t) = \sum_{n=0}^{\infty} u_n t^n, \quad 0 \le t \le T. \tag{11}$$

Table 1: Main operations of RDTM.

Function	Reduced differential transform
$\alpha u(t,x) \pm \beta v(t,x)$	$\alpha U_k(x) \pm \beta V_k(x)$
$u(t,x)v(t,x)$	$\sum_{r=0}^{k} U_r(x) V_{k-r}(x)$
$\frac{\partial}{\partial t}[u(t,x)]$	$(k+1) U_{k+1}(x)$
$\frac{\partial}{\partial x}[u(t,x)]$	$\frac{d}{dx} U_k(x)$
$x^m t^n$	$\begin{cases} x^m, & k = n \\ 0, & k \neq n \end{cases}$
$x^m t^n u(t,x)$	$x^m U_{k-n}(x)$
$e^{\lambda t}$	$\frac{\lambda^k}{k!}$
$\sin(\omega t + \alpha)$	$\frac{\omega^k}{k!} \sin\left(\frac{\pi k}{2} + \alpha\right)$
$\cos(\omega t + \alpha)$	$\frac{\omega^k}{k!} \cos\left(\frac{\pi k}{2} + \alpha\right)$

Then the Padé approximant to $u(t)$ of order $[L, M]$ which we denote by $[L/M]_u(t)$ is defined by [46]

$$\left[\frac{L}{M}\right]_u (t) = \frac{p_0 + p_1 t + \cdots + p_L t^L}{1 + q_1 t + \cdots + q_M t^M}, \tag{12}$$

where we considered $q_0 = 1$, and the numerator and denominator have no common factors.

The numerator and the denominator in (12) are constructed so that $u(t)$ and $[L/M]_u(t)$ and their derivatives agree at $t = 0$ up to $L + M$. That is,

$$u(t) - \left[\frac{L}{M}\right]_u (t) = O\left(t^{L+M+1}\right). \tag{13}$$

From (13), we have

$$u(t) \sum_{n=0}^{M} q_n t^n - \sum_{n=0}^{L} p_n t^n = O\left(t^{L+M+1}\right). \tag{14}$$

From (14), we get the following algebraic linear systems:

$$\begin{aligned} u_L q_1 + \cdots + u_{L-M+1} q_M &= -u_{L+1}, \\ u_{L+1} q_1 + \cdots + u_{L-M+2} q_M &= -u_{L+2}, \\ &\vdots \\ u_{L+M-1} q_1 + \cdots + u_L q_M &= -u_{L+M}, \end{aligned} \tag{15}$$

$$\begin{aligned} p_0 &= u_0 \\ p_1 &= u_1 + u_0 q_1 \\ &\vdots \\ p_L &= u_L + u_{L-1} q_1 + \cdots + u_0 q_L. \end{aligned} \tag{16}$$

From (15), we calculate first all the coefficients q_n, $1 \le n \le M$. Then, we determine the coefficients p_n, $0 \le n \le L$, from (16).

Note that, for a fixed value of $L + M + 1$, error (13) is the smallest when the numerator and denominator of (12) have the same degree or when the numerator has one degree higher than the denominator.

4. Laplace-Padé Resummation Method

Several approximate methods provide power series solutions (polynomial). Nevertheless, sometimes, this type of solutions lacks large domains of convergence. Therefore, Laplace-Padé [34–38] resummation method is used in literature to enlarge the domain of convergence of solutions or inclusive to find exact solutions.

The Laplace-Padé method can be explained as follows.

(1) First, Laplace transformation is applied to power series (9).

(2) Next, s is substituted by $1/t$ in the resulting equation.

(3) After that, we convert the transformed series into a meromorphic function by forming its Padé approximant of order $[N/M]$. N and M are arbitrarily chosen, but they should be of smaller values than the order of the power series. In this step, the Padé approximant extends the domain of the truncated series solution to obtain better accuracy and convergence.

(4) Then, t is substituted by $1/s$.

(5) Finally, by using the inverse Laplace s transformation, we obtain the exact or approximate solution.

5. Test Problems

In this section, we will demonstrate the effectiveness and accuracy of the LPRDTM described in the previous sections through two PDAE systems of index-two and index-three.

5.1. Nonlinear First Order Index-Two PDAE. Consider the following nonlinear index-two PDAE:

$$\frac{\partial u_1}{\partial t} - \frac{\partial^2 u_1}{\partial x^2} - 2u_1 \frac{\partial u_1}{\partial x} + \left(1 + x^2\right) u_3 = 0, \tag{17}$$

$$\frac{\partial u_2}{\partial t} - \frac{\partial^2 u_2}{\partial x^2} - 2u_2 \frac{\partial u_2}{\partial x} + \left(1 + x^2\right) u_3 = 0, \tag{18}$$

$$5u_1 - 3u_2 - 2e^{-t} \cos x = 0, \tag{19}$$

a coupled system of two parabolic equations and one algebraic equation, with $-\infty < x < +\infty$ and $t \ge 0$.

This PDAE is subject to the following initial conditions:

$$u_1(0, x) = \cos x, \quad u_2(0, x) = \cos x, \quad -\infty < x < +\infty. \tag{20}$$

Note here that no initial condition is prescribed for the variable u_3 as this is determined by the PDAEs (17)–(19) and (20). Moreover, since differentiating (19) twice with respect to time and using (17)–(19) determine $\partial u_3/\partial t$ in terms of u_1, u_2 and their space derivatives, then the index of PDAEs (17)–(19) is two. Therefore, this PDAE is difficult to solve numerically.

Applying the reduced differential transform to initial conditions (20) and PDAEs (17)–(19), respectively, we get

$$U_{1,0}(x) = \cos x, \quad U_{2,0}(x) = \cos x, \quad -\infty < x < +\infty, \tag{21}$$

and the recursion system

$$\begin{gathered}
(k+1) U_{1,k+1}(x) - \frac{d^2}{dx^2} U_{1,k}(x) \\
- 2\sum_{r=0}^{k} U_{1,r}(x) \frac{d}{dx} U_{1,k-r}(x) + \left(1 + x^2\right) U_{3,k}(x) = 0, \\
(k+1) U_{2,k+1}(x) - \frac{d^2}{dx^2} U_{2,k}(x) \\
- 2\sum_{r=0}^{k} U_{2,r}(x) \frac{d}{dx} U_{2,k-r}(x) + \left(1 + x^2\right) U_{3,k}(x) = 0, \\
5U_{1,k}(x) - 3U_{2,k}(x) - \frac{2(-1)^k}{k!} \cos x = 0,
\end{gathered} \tag{22}$$

for $k = 0, 1, 2 \ldots$.

System (22) can be written as

$$\begin{gathered}
kU_{1,k}(x) - \frac{d^2}{dx^2} U_{1,k-1}(x) - 2\sum_{r=0}^{k-1} \left(U_{1,r}(x) \times \frac{d}{dx} U_{1,k-1-r}(x)\right) \\
+ \left(1 + x^2\right) U_{3,k-1}(x) = 0, \\
kU_{2,k}(x) - \frac{d^2}{dx^2} U_{2,k-1}(x) - 2\sum_{r=0}^{k-1} \left(U_{2,r}(x) \times \frac{d}{dx} U_{2,k-1-r}(x)\right) \\
+ \left(1 + x^2\right) U_{3,k-1}(x) = 0, \\
5U_{1,k}(x) - 3U_{2,k}(x) - \frac{2(-1)^k}{k!} \cos x = 0,
\end{gathered} \tag{23}$$

for $k = 1, 2, 3 \ldots$.

Using (21) and solving the 3×3 algebraic linear system (23) for $U_{1,k}(x)$, $U_{2,k}(x)$, and $U_{3,k-1}(x)$ for $k = 1, 2, 3 \ldots$, we have

$$\begin{aligned}
&U_{1,1}(x) = U_{2,1}(x) = -\cos x, && U_{3,0}(x) = -\frac{\sin 2x}{1+x^2},\\
&U_{1,2}(x) = U_{2,2}(x) = \frac{1}{2}\cos x, && U_{3,1}(x) = \frac{2\sin 2x}{1+x^2},\\
&U_{1,3}(x) = U_{2,3}(x) = -\frac{1}{3!}\cos x, && U_{3,2}(x) = -\frac{2\sin 2x}{1+x^2},\\
&U_{1,4}(x) = U_{2,4}(x) = \frac{1}{4!}\cos x, && U_{3,3}(x) = \frac{3\sin 2x}{4(1+x^2)},\\
&U_{1,5}(x) = U_{2,5}(x) = -\frac{1}{5!}\cos x, && U_{3,4}(x) = -\frac{2\sin 2x}{3(1+x^2)},\\
&\vdots
\end{aligned} \tag{24}$$

Then, using (9) and (24), we get the fourth order approximation solution:

$$\begin{aligned}
u_1(t,x) = u_2(t,x) &\cong \sum_{k=0}^{4} U_{1,k}(x)\,t^k\\
&= \left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!}\right)\cos x,\\
u_3(t,x) &\cong \sum_{k=0}^{4} U_{3,k}(x)\,t^k\\
&= -\left(1 - 2t + \frac{2^2t^2}{2!} - \frac{2^3t^3}{3!} + \frac{2^4t^4}{4!}\right)\frac{\sin 2x}{1+x^2}.
\end{aligned} \tag{25}$$

The solutions series obtained from the RDTM may have limited regions of convergence, even if we take a large number of terms. Therefore, we propose to apply the t-Padé approximation technique to these series to increase the convergence region. First t-Laplace transform is applied to (25). Then, s is substituted by $1/t$ and the t-Padé approximant is applied to the transformed series. Finally, t is substituted by $1/s$ and the inverse Laplace s-transform is applied to the resulting expression to obtain the approximate or exact solution.

Applying Laplace transforms to $u_1(t,x)$, $u_2(t,x)$, and $u_3(t,x)$ yields

$$\begin{aligned}
\mathscr{L}[u_1(t,x)] = \mathscr{L}[u_2(t,x)] &= \left(\frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^3} - \frac{1}{s^4}\right)\cos x,\\
\mathscr{L}[u_3(t,x)] &= -\left(\frac{1}{s} - \frac{2}{s^2} + \frac{4}{s^3} - \frac{8}{s^4}\right)\frac{\sin 2x}{1+x^2}.
\end{aligned} \tag{26}$$

For simplicity let $s = 1/t$; then

$$\begin{aligned}
\mathscr{L}[u_1(t,x)] = \mathscr{L}[u_2(t,x)] &= \left(t - t^2 + t^3 - t^4\right)\cos x,\\
\mathscr{L}[u_3(t,x)] &= -\left(t - 2t^2 + 4t^3 - 8t^4\right)\frac{\sin 2x}{1+x^2}.
\end{aligned} \tag{27}$$

All of the $[L/M]$ t-Padé approximants of (27) with $L \geq 1$ and $M \geq 1$ and $L + M \leq 4$ yield

$$\begin{aligned}
\left[\frac{L}{M}\right]_{u_1}(t,x) = \left[\frac{L}{M}\right]_{u_2}(t,x) &= \left(\frac{t}{1+t}\right)\cos x,\\
\left[\frac{L}{M}\right]_{u_3}(t,x) &= -\left(\frac{t}{1+2t}\right)\frac{\sin 2x}{1+x^2}.
\end{aligned} \tag{28}$$

Now since $t = 1/s$, we obtain $[L/M]_{u_1}$, $[L/M]_{u_2}$, and $[L/M]_{u_3}$ in terms of s as follows:

$$\begin{aligned}
\left[\frac{L}{M}\right]_{u_1}(t,x) = \left[\frac{L}{M}\right]_{u_2}(t,x) &= \left(\frac{1}{1+s}\right)\cos x,\\
\left[\frac{L}{M}\right]_{u_3}(t,x) &= -\left(\frac{1}{2+s}\right)\frac{\sin 2x}{1+x^2}.
\end{aligned} \tag{29}$$

Finally, applying the inverse Laplace transform to Padé approximants (29) yields an approximate solution which in this case is the exact solution:

$$\begin{aligned}
u_1(t,x) &= e^{-t}\cos x,\\
u_2(t,x) &= e^{-t}\cos x,\\
u_3(t,x) &= -\frac{e^{-2t}\sin 2x}{1+x^2},\\
-\infty &< x < +\infty, \quad t \geq 0.
\end{aligned} \tag{30}$$

5.2. Linear Second Order Index-Three PDAE. Consider the following index-three PDAE [47]:

$$\frac{\partial^2 u_1}{\partial t^2} - \frac{\partial^2 u_1}{\partial x^2} - u_3 \sin \pi x = 0, \tag{31}$$

$$\frac{\partial^2 u_2}{\partial t^2} - \frac{\partial^2 u_2}{\partial x^2} - u_3 \cos \pi x = 0, \tag{32}$$

$$u_1 \sin \pi x + u_2 \cos \pi x - e^{-t} = 0, \tag{33}$$

a coupled system of two hyperbolic equations and one algebraic equation, with $-\infty < x < +\infty$ and $t \geq 0$.

System (31)–(33) is subject to the following initial conditions:

$$\begin{aligned}
u_1(0,x) &= \sin \pi x, && \frac{\partial u_1}{\partial t}(0,x) = -\sin \pi x,\\
u_2(0,x) &= \cos \pi x, && \frac{\partial u_2}{\partial t}(0,x) = -\cos \pi x,\\
& && -\infty < x < +\infty.
\end{aligned} \tag{34}$$

Note here that no initial condition is prescribed for the variable u_3 as this is determined by the PDAEs (31)–(33) and

(34). Moreover, since differentiating (33) three times with respect to time and using (31)–(33) determine $\partial u_3/\partial t$ in terms of u_1, u_2 and their space derivatives, then the index of PDAEs (31)–(33) is three. Therefore, this PDAE is difficult to solve numerically.

Applying the reduced differential transform to initial conditions (34) and PDAEs (31)–(33), respectively, we obtain

$$\begin{gathered} U_{1,0}(x) = \sin \pi x, \qquad U_{1,1}(x) = -\sin \pi x, \\ U_{2,0}(x) = \cos \pi x, \qquad U_{2,1}(x) = -\cos \pi x, \\ -\infty < x < +\infty, \end{gathered} \tag{35}$$

and the recursion system

$$\begin{gathered} (k+1)(k+2)U_{1,k+2}(x) - \frac{d^2}{dx^2}U_{1,k}(x) - U_{3,k}(x)\sin \pi x = 0, \\ (k+1)(k+2)U_{2,k+2}(x) - \frac{d^2}{dx^2}U_{2,k}(x) - U_{3,k}(x)\cos \pi x = 0, \\ U_{1,k}(x)\sin \pi x + U_{2,k}(x)\cos \pi x - \frac{(-1)^k}{k!} = 0, \\ \text{for } k = 0, 1, 2, \ldots. \end{gathered} \tag{36}$$

System (36) can be written as

$$\begin{gathered} (k-1)kU_{1,k}(x) - \frac{d^2}{dx^2}U_{1,k-2}(x) - U_{3,k-2}(x)\sin \pi x = 0, \\ (k-1)kU_{2,k}(x) - \frac{d^2}{dx^2}U_{2,k-2}(x) - U_{3,k-2}(x)\cos \pi x = 0, \\ U_{1,k}(x)\sin \pi x + U_{2,k}(x)\cos \pi x - \frac{(-1)^k}{k!} = 0, \\ \text{for } k = 2, 3, \ldots. \end{gathered} \tag{37}$$

Using (35) and solving the 3×3 algebraic linear system (37) for $U_{1,k}(x)$, $U_{2,k}(x)$, and $U_{3,k-2}(x)$ for $k = 2, 3, 4 \ldots$, we have

$$\begin{gathered} U_{1,2}(x) = \frac{1}{2}\sin \pi x, \qquad U_{2,2}(x) = \frac{1}{2}\cos \pi x, \\ U_{3,0}(x) = 1 + \pi^2, \\ U_{1,3}(x) = -\frac{1}{6}\sin \pi x, \qquad U_{2,3}(x) = -\frac{1}{6}\cos \pi x, \\ U_{3,1}(x) = -\left(1 + \pi^2\right), \\ U_{1,4}(x) = \frac{1}{24}\sin \pi x, \qquad U_{2,4}(x) = \frac{1}{24}\cos \pi x, \\ U_{3,2}(x) = \frac{1}{2}\left(1 + \pi^2\right), \\ \vdots \end{gathered} \tag{38}$$

Using (9) and (38), we get the approximate solution

$$\begin{aligned} u_1(t,x) &\cong \sum_{k=0}^{4} U_{1,k}(x)t^k \\ &= \left(1 - t + \frac{1}{2}t^2 - \frac{1}{3!}t^3 + \frac{1}{4!}t^4\right)\sin \pi x, \\ u_2(t,x) &\cong \sum_{k=0}^{4} U_{2,k}(x)t^k \\ &= \left(1 - t + \frac{1}{2}t^2 - \frac{1}{3!}t^3 + \frac{1}{4!}t^4\right)\cos \pi x, \\ u_3(t,x) &\cong \sum_{k=0}^{2} U_{3,k}(x)t^k = \left(1 + \pi^2\right)\left(1 - t + \frac{1}{2}t^2\right). \end{aligned} \tag{39}$$

Similarly, the coefficients $U_{1,k}(x)$, $U_{2,k}(x)$, and $U_{3,k-2}(x)$ for $k \geq 5$ can be found from (37). The solutions series obtained from the RDTM may have limited regions of convergence, even if we take a large number of terms. Therefore, we propose to apply the t-Padé approximation technique to these series to increase the convergence region. First t-Laplace transform is applied to (39). Then, s is substituted by $1/t$ and the t-Padé approximant is applied to the transformed series. Finally, t is substituted by $1/s$ and the inverse Laplace s-transform is applied to the resulting expressions to get the approximate or exact solutions.

Applying Laplace transforms to $u_1(t,x)$, $u_2(t,x)$, and $u_3(t,x)$ yields

$$\begin{aligned} \mathscr{L}[u_1(t,x)] &= \left(\frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^3}\right)\sin \pi x, \\ \mathscr{L}[u_2(t,x)] &= \left(\frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^3}\right)\cos \pi x, \\ \mathscr{L}[u_3(t,x)] &= \left(1 + \pi^2\right)\left(\frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^3}\right). \end{aligned} \tag{40}$$

For simplicity let $s = 1/t$; then

$$\begin{aligned} \mathscr{L}[u_1(t,x)] &= \left(t - t^2 + t^3\right)\sin \pi x, \\ \mathscr{L}[u_2(t,x)] &= \left(t - t^2 + t^3\right)\cos \pi x, \\ \mathscr{L}[u_3(t,x)] &= \left(1 + \pi^2\right)\left(t - t^2 + t^3\right). \end{aligned} \tag{41}$$

All of the $[L/M]$ t-Padé approximants of (41) with $L \geq 1$ and $M \geq 1$ and $L + M \leq 3$ yield

$$\begin{aligned} \left[\frac{L}{M}\right]_{u_1}(t,x) &= \left(\frac{t}{1+t}\right)\sin \pi x, \\ \left[\frac{L}{M}\right]_{u_2}(t,x) &= \left(\frac{t}{1+t}\right)\cos \pi x, \\ \left[\frac{L}{M}\right]_{u_3}(t,x) &= \left(1 + \pi^2\right)\left(\frac{t}{1+t}\right). \end{aligned} \tag{42}$$

Now since $t = 1/s$, we obtain $[L/M]_{u_1}$, $[L/M]_{u_2}$, and $[L/M]_{u_3}$ in terms of s as follows:

$$\begin{aligned} \left[\frac{L}{M}\right]_{u_1}(t,x) &= (1+s)^{-1}\sin\pi x, \\ \left[\frac{L}{M}\right]_{u_2}(t,x) &= (1+s)^{-1}\cos\pi x, \\ \left[\frac{L}{M}\right]_{u_3}(t,x) &= \left(1+\pi^2\right)(1+s)^{-1}. \end{aligned} \tag{43}$$

Finally, applying the inverse Laplace transform to Padé approximants (43) we obtain an approximate solution which in this case is the exact solution:

$$\begin{aligned} u_1(t,x) &= e^{-t}\sin\pi x, \\ u_2(t,x) &= e^{-t}\cos\pi x, \\ u_3(t,x) &= \left(1+\pi^2\right)e^{-t}, \\ -\infty < x &< +\infty, \quad t \geq 0. \end{aligned} \tag{44}$$

6. Discussion

In this paper, we presented the reduced differential transform method (RDTM) as a useful analytical tool to solve partial differential-algebraic equations (PDAEs). The coupling of RDTM and Laplace-Padé enabled us to obtain the exact solution to two PDAE problems of index-two and index-three without the need for a preprocessing step of index-reduction. This is a relevant result given the fact that a higher-index PDAE is often difficult to treat numerically without reducing its index to one and that the index-reduction can be very expensive and may not preserve the properties of the solution to the original PDAE. For each of the two problems solved here, the RDTM transformed the PDAE into an easily solvable linear algebraic recursion system for the coefficient functions of the power series solution. As aforementioned, in order to enlarge the domain of convergence of the RDTM power series solution, a Laplace-Padé resummation was applied to the RDTM's truncated series leading to the exact solution.

The RDTM solution procedure does not involve unnecessary computation like that related to noise terms [22], which is a common problem for approximation methods like the HPM or others. This property of RDTM greatly reduces the volume of computation and improves the efficiency of the method. It should be noted that the high complexity of these problems was effectively handled by LPRDTM method due to the malleability of RDTM and resummation capability of Laplace-Padé. What is more, there is not any standard analytical or numerical method to solve higher-index PDAEs, converting the LPRDTM method into an attractive tool to solve such problems.

On the one hand, semianalytic methods like HPM, HAM, and VIM, among others, require an initial approximation for the solutions sought and the computation of one or several adjustment parameters. If the initial approximation is properly chosen, then the results can be highly accurate. Nonetheless, there is no general method to choose such initial approximation. This issue motivates the use of adjustment parameters obtained by minimizing the least-squares error with respect to the numerical solution. On the other hand, RDTM or LPRDTM methods do not require any trial equation or a procedure for least-squares error minimization. As well, RDTM obtains its coefficients using an easily computable straightforward procedure that can be implemented into programmes like Maple or Mathematica.

It is important to remark that even if the Laplace-Padé resummation strategy fails to obtain the exact solution to the PDAE under study, it can still produce a good approximation with an enlarged domain of convergence. The treatment of higher-index PDAEs is still an open issue in science and requires further research.

7. Conclusion

This work presented LPRDTM method as a combination of the RDTM and a resummation method based on Laplace transforms and the Padé approximant. Firstly, the solutions of PDAEs are obtained in convergent series forms using RDTM. Next, in order to enlarge the domain of convergence of the truncated power series, a posttreatment combining Laplace transforms and the Padé approximant is applied. This technique that we call LPRDTM greatly improves RDTM's truncated series solutions in convergence rate and often leads to the exact solution. Additionally, RDTM is an attractive tool, because it does not require a perturbation parameter to work and it does not generate secular terms (noise terms) as other semianalytical methods like HPM, HAM, or VIM.

By solving two problems, we presented the LPRDTM as a handy tool with a great potential to solve linear/nonlinear higher-index PDAEs. Additionally, the LPRDTM does not require an index-reduction to solve higher-index PDAEs. Furthermore, we obtained successfully the exact solutions of such two problems highlighting the efficiency of the LPRDTM. The proposed method is based on a straightforward procedure, producing highly accurate approximations. Therefore, it is suitable for engineers and in particular for those in fields of mechanics, electronics, and electrical engineering where application problems give rise to higher-index PDAEs. Finally, further research should be performed to solve other higher-index nonlinear PDAE systems.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] A. Bartel and R. Pulch, "A concept for classification of partial differential-algebraic equations in nanoelectronics," Preprint BUW-AMNA 06/07, 2006, http://www.math.uni-wuppertal.de/org/Num/.

[2] M. Günther, "A joint DAE/PDE model for interconnected electrical networks," *Mathematical and Computer Modelling of Dynamical Systems*, vol. 6, no. 2, pp. 114–128, 2000.

[3] G. Alì, A. Bartel, and M. Günther, "Parabolic differential-algebraic models in electrical network design," *Multiscale Modeling & Simulation*, vol. 4, no. 3, pp. 813–838, 2005.

[4] G. Alì, A. Bartel, M. Günther, and C. Tischendorf, "Elliptic partial differential-algebraic multiphysics models in electrical network design," *Mathematical Models & Methods in Applied Sciences*, vol. 13, no. 9, pp. 1261–1278, 2003.

[5] B. Simeon, "Modelling a flexible slider crank mechanism by a mixed system of DAEs and PDEs," *Mathematical Modelling of Systems*, vol. 2, no. 1, pp. 1–18, 1996.

[6] K. Debrabant and K. Strehmel, "Convergence of Runge-Kutta methods applied to linear partial differential-algebraic equations," *Applied Numerical Mathematics*, vol. 53, no. 2–4, pp. 213–229, 2005.

[7] W. Lucht, K. Strehmel, and C. Eichler-Liebenow, "Linear partial differential-algebraic equations part I: indexes, consistent boundary/initial conditions," Report 17, Department of Mathematics and Computer Science, Martin-Luther-University, Halle-Wittenberg, Germany, 1997.

[8] W. Lucht, K. Strehmel, and C. Eichler-Liebenow, "Linear partial differential algebraic equations part II," Report 18, Department of Mathematics and Computer Science, Martin-Luther-University, Halle-Wittenberg, Germany, 1997.

[9] W. Lucht and K. Strehmel, "Discretization based indices for semilinear partial differential algebraic equations," *Applied Numerical Mathematics*, vol. 28, no. 2–4, pp. 371–386, 1998.

[10] W. Lucht, K. Strehmel, and C. Eichler-Liebenow, "Indexes and special discretization methods for linear partial differential algebraic equations," *BIT*, vol. 39, no. 3, pp. 484–512, 1999.

[11] W. S. Martinson and P. I. Barton, "A differentiation index for partial differential-algebraic equations," *SIAM Journal on Scientific Computing*, vol. 21, no. 6, pp. 2295–2315, 2000.

[12] J. K. Zhou, *Differential Transformation and its Applications for Electrical Circuits*, Huazhong University Press, Wuhan, China, 1986, (Chinese).

[13] A. M. Batiha and B. Batiha, "Differential transformation method for a reliable treatment of the nonlinear biochemical reaction model," *Advanced Studies in Biology*, vol. 3, pp. 355–360, 2011.

[14] Y. Keskin and G. Oturanç, "Reduced differential transform method for partial differential equations," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 10, no. 6, pp. 741–749, 2009.

[15] Y. Keskin and G. Oturanc, "The reduced differential transform method: a new approach to fractional partial differential equations," *Nonlinear Science Letters A*, vol. 1, no. 2, pp. 207–217, 2010.

[16] Y. Keskin and G. Oturanç, "Reduced differential transform method for generalized KdV equations," *Mathematical & Computational Applications*, vol. 15, no. 3, pp. 382–393, 2010.

[17] J.-H. He, "Homotopy perturbation technique," *Computer Methods in Applied Mechanics and Engineering*, vol. 178, no. 3-4, pp. 257–262, 1999.

[18] H. Vazquez-Leal, Y. Khan, G. Fernández-Anaya et al., "A general solution for Troesch's problem," *Mathematical Problems in Engineering*, vol. 2012, Article ID 208375, 14 pages, 2012.

[19] F. Guerrero, F. J. Santonja, and R. J. Villanueva, "Solving a model for the evolution of smoking habit in Spain with homotopy analysis method," *Nonlinear Analysis: Real World Applications*, vol. 14, no. 1, pp. 549–558, 2013.

[20] Y. Khan, H. Vazquez-Leal, L. Hernandez-Martinez, and N. Faraz, "Variational iteration algorithm-II for solving linear and non-linear ODEs," *International Journal of Physical Sciences*, vol. 7, no. 25, pp. 3099–4002, 2012.

[21] H. Vazquez-Leal, "Generalized homotopy method for solving nonlinear differential equations," *Computational & Applied Mathematics*, vol. 33, no. 1, pp. 275–288, 2014.

[22] F. Soltanian, M. Dehghan, and S. M. Karbassi, "Solution of the differential algebraic equations via homotopy perturbation method and their engineering applications," *International Journal of Computer Mathematics*, vol. 87, no. 9, pp. 1950–1974, 2010.

[23] M. A. Asadi, F. Salehi, and M. M. Hosseini, "Modification of the homotopy perturbation method for nonlinear system of second-order BVPs," *Journal of Computer Science & Computational Mathematics*, vol. 2, no. 5, pp. 23–28, 2012.

[24] H. Aminikhah and M. Hemmatnezhad, "An effective modification of the homotopy perturbation method for stiff systems of ordinary differential equations," *Applied Mathematics Letters*, vol. 24, no. 9, pp. 1502–1508, 2011.

[25] F. Salehi, M. A. Asadi, and M. M. Hosseini, "Solving system of DAEs by modified homotopy perturbation method," *Journal of Computer Science & Computational Mathematics*, vol. 2, no. 6, pp. 1–5, 2012.

[26] M. Yigider and E. Çelik, "The numerical solution of partial differential-algebraic equations," *Advances in Difference Equations*, vol. 2013, article 8, 2013.

[27] Y. Keskin and G. Oturanc, "Reduced differential transform method for solving linear and nonlinear wave equations," *Iranian Journal of Science and Technology, Transaction A: Science*, vol. 34, no. 2, pp. 113–122, 2010.

[28] C. L. Chen and Y. C. Liu, "Solution of two-point boundary-value problems using the differential transformation method," *Journal of Optimization Theory and Applications*, vol. 99, no. 1, pp. 23–35, 1998.

[29] F. Ayaz, "Applications of differential transform method to differential-algebraic equations," *Applied Mathematics and Computation*, vol. 152, no. 3, pp. 649–657, 2004.

[30] F. Kangalgil and F. Ayaz, "Solitary wave solutions for the KDV and mKDV equations by differential transform method," *Chaos, Solitons and Fractals*, vol. 41, no. 1, pp. 464–472, 2009.

[31] A. S. V. Ravi Kanth and K. Aruna, "Two-dimensional differential transform method for solving linear and non-linear Schrödinger equations," *Chaos, Solitons and Fractals*, vol. 41, no. 5, pp. 2277–2281, 2009.

[32] A. Arikoglu and I. Ozkol, "Solution of fractional differential equations by using differential transform method," *Chaos, Solitons and Fractals*, vol. 34, no. 5, pp. 1473–1481, 2007.

[33] J. Biazar and M. Eslami, "Differential transform method for quadratic riccati differential equation," *International Journal of Nonlinear Science*, vol. 9, no. 4, pp. 444–447, 2010.

[34] Y. Yamamoto, C. Dang, Y. Hao, and Y. C. Jiao, "An aftertreatment technique for improving the accuracy of Adomian's decomposition method," *Computers and Mathematics with Applications*, vol. 43, no. 6-7, pp. 783–798, 2002.

[35] N. H. Sweilam and M. M. Khader, "Exact solutions of some coupled nonlinear partial differential equations using the homotopy perturbation method," *Computers and Mathematics with Applications*, vol. 58, no. 11-12, pp. 2134–2141, 2009.

[36] S. Momani, G. H. Erjaee, and M. H. Alnasr, "The modified homotopy perturbation method for solving strongly nonlinear oscillators," *Computers and Mathematics with Applications*, vol. 58, no. 11-12, pp. 2209–2220, 2009.

[37] P.-Y. Tsai and C.-K. Chen, "An approximate analytic solution of the nonlinear Riccati differential equation," *Journal of the Franklin Institute: Engineering and Applied Mathematics*, vol. 347, no. 10, pp. 1850–1862, 2010.

[38] A. E. Ebaid, "A reliable aftertreatment for improving the differential transformation method and its application to nonlinear oscillators with fractional nonlinearities," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 1, pp. 528–536, 2011.

[39] P. K. Gupta, "Approximate analytical solutions of fractional Benney-Lin equation by reduced differential transform method and the homotopy perturbation method," *Computers & Mathematics with Applications*, vol. 61, no. 9, pp. 2829–2842, 2011.

[40] R. Abazari and M. Abazari, "Numerical simulation of generalized Hirota-Satsuma coupled KdV equation by RDTM and comparison with DTM," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 2, pp. 619–629, 2012.

[41] S. Hesam, A. Nazemi, and A. Haghbin, "Reduced differential transform method for solving the Fornberg-Whitham type equation," *International Journal of Nonlinear Science*, vol. 13, no. 2, pp. 158–162, 2012.

[42] A. Saravanan and N. Magesh, "A comparison between the reduced differential transform method and the Adomian decomposition method for the Newell-Whitehead-Segel equation," *Journal of the Egyptian Mathematical Society*, vol. 21, no. 3, pp. 259–265, 2013.

[43] V. K. Srivastava, M. K. Awasthi, and S. Kumar, "Analytical approximations of two and three dimensional time-fractional telegraphic equation by reduced differential transform method," *Egyptian Journal of Basic and Applied Sciences*, vol. 1, no. 1, pp. 60–66, 2014.

[44] B. Iserom Ita, "The solution of radial diffusivity equation using reduced differential transform method," *International Journal of Conceptions on Computing and Information Technology*, vol. 2, no. 1, pp. 108–124, 2014.

[45] I. Birol and B. Mustafa, "Approximate solutions for some nonlinear evolutions equations by using the reduced differential transform method," *International Journal of Applied Mathematical Research*, vol. 1, no. 3, pp. 288–302, 2012.

[46] G. A. Baker, *Essentials of Padé Approximants*, Academic Press, London, UK, 1975.

[47] B. Benhammouda and H. Vazquez-Leal, "Analytical solutions for systems of partial differential-algebraic equations," *SpringerPlus*, vol. 3, article 137, 9 pages, 2014.