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Research Article

Value Distribution of Certain Type of Difference Polynomials

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We investigate the value distribution of difference product $f(z)^n \sum_{i=1}^{k} a_i f(z+c_i)$, for $n \geq 2$ and $n = 1$, respectively, where $f(z)$ is a transcendental entire function of finite order and a_i, c_i are constants satisfying $\sum_{i=1}^{k} a_i f(z+c_i) \not\equiv 0$.

1. Introduction

In this paper, we assume that the reader is familiar with the basic notions of Nevanlinna's value distribution theory (see [1–3]). The notation $S(r, f)$ is defined to be any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \to \infty$, possibly outside a set of finite linear measures. In addition, we use the notation $\sigma(f)$ to denote the order of growth of the meromorphic function $f(z)$ and $\lambda(f)$ to denote the exponent of convergence of zeros of $f(z)$.

Hayman proved the following theorem in [4].

Theorem 1. *Let $f(z)$ be a transcendental integral function and let $n \geq 2$ be an integer; then $f^n f'(z)$ assumes all values except possibly zero infinitely often.*

Clunie proved that if $n = 1$, then Theorem 1 remains valid.

Recently, many papers (see [5–17]) focus on complex difference. They obtain many new results on difference using the value distribution theory of meromorphic functions.

In [12], Laine and Yang found a difference analogue of Hayman's result as follows.

Theorem 2. *Let $f(z)$ be a transcendental entire function of finite order and c a nonzero complex constant. Then for $n \geq 2$, $f(z)^n f(z+c)$ assumes every nonzero value $a \in \mathbb{C}$ infinitely often.*

Liu and Yang [14] proved the following theorem.

Theorem 3. *Let $f(z)$ be a transcendental entire function of finite order and let c be a nonzero complex constant, $\Delta f(z) = f(z+c) - f(z) \not\equiv 0$. Then for $n \geq 2$, $f(z)^n \Delta f(z) - p(z)$ has infinitely many zeros, where $p(z) \not\equiv 0$ is a polynomial in z.*

Chen [6] proved the following theorem.

Theorem 4. *Let $f(z)$ be a transcendental entire function of finite order and let $c \in \mathbb{C}\backslash\{0\}$ be a constant satisfying $f(z+c) \not\equiv f(z)$. Set $H_n(z) = f(z)^n \Delta f(z)$ where $\Delta f(z) = f(z+c) - f(z)$, and $n \geq 2$ is an integer. Then the following statements hold.*

(i) *If $f(z)$ satisfies $\sigma(f) \neq 1$ or has infinitely many zeros, then $H_n(z)$ has infinitely many zeros.*

(ii) *If $f(z)$ has only finitely many zeros and $\sigma(f) = 1$, then $H_n(z)$ has only finitely many zeros.*

It is natural to ask what condition will guarantee that

$$f(z)^n L(f) \tag{1}$$

assumes every nonzero and zero value infinitely often, where $L(f)$ is a linear kth order difference operator with varying shifts, operating on a transcendental entire function of finite order.

In this paper, we consider the above question for $n \geq 2$ and $n = 1$, respectively, and obtain the following results.

Theorem 5. *Let f be a transcendental entire function of finite order and let a_i, c_i $(i = 1, \ldots, k)$ be constant satisfying $\sum_{i=1}^{k} a_i f(z+c_i) \not\equiv 0$ and $c_i \neq c_j$ when $i \neq j$. Set $H_n(z) = f(z)^n \sum_{i=1}^{k} a_i f(z+c_i)$, where $n, k \geq 2$ are integers. Then the following statements hold.*

(i) *If $f(z)$ satisfies $\sigma(f)\neq 1$ or has infinitely many zeros, then $H_n(z)$ has infinitely many zeros.*

(ii) *If $f(z)$ has only finitely many zeros and $\sigma(f)=1$, then $H_n(z)$ has only finitely many zeros.*

(iii) *$H_n(z)-\alpha(z)$ has infinitely many zeros, and $\lambda(H_n(z)-\alpha(z))=\sigma(f)$, where $\alpha(z)\not\equiv 0$ is a small function of f.*

Remark 6. The result of Theorem 5 may be false if $k=1$. For example, if $f(z)=e^{z^2}$, we have that $f(z)^2 f(z+c)=e^{3z^2+2cz+c^2}$ (where $c\in\mathbf{C}\setminus\{0\}$ is a constant satisfying $f(z+c)\not\equiv f(z)$) has no zero, but $f(z)^2(f(z+c)-f(z))=e^{3z^2}(e^{2cz+c^2}-1)$ has infinitely many zeros. This also shows that the restriction $\sigma(f)=1$ in Theorem 5(ii) is sharp. The following example shows that the assumption $\sigma(f)\neq 1$ in Theorem 5(i) cannot be deleted. In fact, let $f(z)=e^z$; we have $H_2=f^2(f(z+c)-f(z))=e^{2z}(e^{z+1}-e^z)=e^{3z}(e-1)\neq 0$.

By (i) and (iii) of Theorem 5, we can easily obtain the following corollary.

Corollary 7. *Let f be a transcendental entire function of finite order and let a_i, c_i $(i=1,\dots,k)$ be constants satisfying $\sum_{i=1}^k a_i f(z+c_i)\not\equiv 0$ and $c_i\neq c_j$ when $i\neq j$. Set $H_n(z)=f(z)^n\sum_{i=1}^k a_i f(z+c_i)$, where $n,k\geq 2$ are integers. If $\sigma(f)\neq 1$ or has infinitely many zeros, then $H_n(z)$ takes every value $a\in\mathbf{C}$ infinitely often.*

Theorem 8. *Let f be a finite-order transcendental entire function with a finite Borel exceptional value d, and let a_i, c_i be constants satisfying $\sum_{i=1}^k a_i f(z+c_i)\not\equiv 0$ where $\sum_{i=1}^k a_i=0$. Set $H(z)=f(z)\sum_{i=1}^k a_i f(z+c_i)$. Then the following statements hold.*

(i) *$H(z)$ takes every nonzero value $a\in\mathbb{C}$ infinitely often and satisfies $\lambda(H-a)=\sigma(f)$.*

(ii) *If $d\neq 0$, then $H(z)$ has no finite Borel exceptional value.*

(iii) *If $d=0$, then 0 is also the Borel exceptional value of $H(z)$. So that $H(z)$ has no nonzero finite Borel exceptional value.*

Theorem 9. *Let f be a transcendental entire function of finite order and let a_i, c_i be constants satisfying $\sum_{i=1}^k a_i f(z+c_i)\not\equiv 0$. Set $H(z)=f(z)\sum_{i=1}^k a_i f(z+c_i)$.*

If there exists an infinite sequence $\{z_n\}$ satisfying $f(z_n)=\sum_{i=1}^k a_i f(z_n+c_i)=0$, then $H(z)$ takes every value $a\in\mathbb{C}$ (including $a=0$) infinitely often.

Theorem 10. *Let f be a transcendental entire function of finite order and let c_i be distinct constants satisfying $\sum_{i=1}^k a_i f(z+c_i)\not\equiv 0$. Set $H(z)=f(z)\sum_{i=1}^k a_i f(z+c_i)$, where $k\geq 2$ is an integer.*

(i) *If $f(z)$ has only finitely many zeros and $\sigma(f)\neq 1$ or has infinitely many zeros, then $H(z)$ has infinitely many zeros.*

(ii) *If $f(z)$ has only finitely many zeros and $\sigma(f)=1$, then $H(z)$ has only finitely many zeros.*

Example 11. An entire function $f(z)=e^{z^2}$ satisfies Theorem 8 (iii), it has Borel exceptional value 0, and let $a_1=a_2=1$, $a_3=-2$, $a_4=\cdots=a_k=0$, $c_1=1$, $c_2=-1$, and $c_3=0$. Then

$$\begin{aligned} H(z)&=f(z)\left(f(z+1)+f(z-1)-2f(z)\right)\\ &=e^{2z^2}\left(\left(e+\frac{1}{e}\right)e^{2z}-2\right) \end{aligned}\tag{2}$$

has also the Borel exceptional value 0 since $\lambda(H)=1<\sigma(H)=2$.

Simultaneously, $f(z)=e^{z^2}$ also satisfies Theorem 10(i), although $f(z)$ has no zero, we can also get $H(z)$ has infinitely many zeros since $\sigma(f)\neq 1$.

Example 12. An entire function $f(z)=e^z+1$ satisfies Theorem 8(ii), it has Borel exceptional value 1, and let $a_1=a_2=1$, $a_3=-2$, $a_4=\cdots=a_k=0$, $c_1=1$, $c_2=-1$, and $c_3=0$. Then

$$\begin{aligned} H(z)&=f(z)\left(f(z+1)+f(z-1)-2f(z)\right)\\ &=e^z\left(e^z+1\right)\left(e+\frac{1}{e}-2\right) \end{aligned}\tag{3}$$

has no finite Borel exceptional value.

2. Some Lemmas

Lemma 13 (see [9]). *Let $f(z)$ be a meromorphic function of finite order, $c\in\mathbb{C}\setminus\{0\}$, $\delta<1$. Then*

$$m\left(r,\frac{f(z+c)}{f(z)}\right)=o\left(\frac{T(r+|c|,f)}{r^\delta}\right)=S(r,f),\tag{4}$$

for all r outside an exceptional set of finite logarithmic measures.

Lemma 14 (see [7]). *Let $f(z)$ be a nonconstant, finite-order meromorphic solution of*

$$f^n P_1(z,f)=Q_1(z,f),\tag{5}$$

where $P_1(z,f)$, $Q_1(z,f)$ are difference polynomials in $f(z)$ with meromorphic coefficients $a_j(z)$ $(j=1,\dots,s)$, and let $\delta<1$. If the degree of $Q_1(r,f)$ as a polynomial in $f(z)$ and its shifts is at most n, then

$$\begin{aligned} m(r,P_1(z,f))&=o\left(\frac{T(r+|c|,f)}{r^\delta}\right)+o(T(r,f))\\ &\quad+O\left(\sum_{j=1}^s m(r,a_j)\right)\\ &=S(r,f)+O\left(\sum_{j=1}^s m(r,a_j)\right), \end{aligned}\tag{6}$$

for all r outside an exceptional set of finite logarithmic measures.

Lemma 15 (see [3]). *Let $f_j(z)$ $(j = 1, \ldots, n)$ $(n \geq 2)$ be meromorphic functions, and let $g_j(z)$ $(j = 1, \ldots, n)$ be entire functions that satisfy the following:*

(i) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0$;

(ii) *when* $1 \leq j < k \leq n$, $g_j(z) - g_k(z)$ *is not a constant;*

(iii) *when* $1 \leq j \leq n$, $1 \leq h < k \leq n$,

$$T\left(r, f_j\right) = o\left\{T\left(r, e^{g_h - g_k}\right)\right\} \quad (r \longrightarrow \infty,\ r \notin E), \tag{7}$$

where $E \subset (1, \infty)$ is of finite linear measure or finite logarithmic measure. Then $f_j(z) \equiv 0$ $(j = 1, \ldots, n)$.

Lemma 16. *Let f be a transcendental entire function of finite order and let a_i, c_i be constants satisfying $\sum_{i=1}^{k} a_i f(z + c_i) \not\equiv 0$. Then $H_n(z) = f(z)^n \sum_{i=1}^{k} a_i f(z+c_i)$ $(n \geq 1)$ is transcendental.*

Proof. If $H_n(z) \equiv 0$, then $\sum_{i=1}^{k} a_i f(z + c_i) \equiv 0$ which contradicts our condition $\sum_{i=1}^{k} a_i f(z + c_i) \not\equiv 0$. Now we suppose that

$$H_n(z) = f(z)^n \sum_{i=1}^{k} a_i f\left(z + c_i\right) = P(z), \tag{8}$$

where $P(z) \not\equiv 0$ is a polynomial. Applying Lemma 14 to (8), we obtain that

$$T\left(r, \sum_{i=1}^{k} a_i f\left(z + c_i\right)\right) = m\left(r, \sum_{i=1}^{k} a_i f\left(z + c_i\right)\right) = S(r, f). \tag{9}$$

Thus by (8), (9), and the first fundamental theorem of Nevanlinna theory, we obtain that

$$T\left(r, f(z)^n\right) = T\left(r, \frac{P(z)}{\sum_{i=1}^{k} a_i f\left(z + c_i\right)}\right) = S(r, f). \tag{10}$$

Since $n \geq 1$, this is a contradiction. Hence $H_n(z)$ is a transcendental entire function. $\square$

Lemma 17 (see [17]). *Let $f(z)$ be a nonconstant finite-order meromorphic function and let $c \neq 0$ be an arbitrary complex number. Then*

$$T\left(r, f(z+c)\right) = T\left(r, f(z)\right) + S(r, f). \tag{11}$$

3. Proof of Theorems 5 and 10

Proof of Theorem 5. (i) If $f(z)$ has infinitely many zeros, then $H_n(z)$ has infinitely many zeros since $\sum_{i=1}^{k} a_i f(z + c_i)$ is an entire function and $\sum_{i=1}^{k} a_i f(z + c_i) \not\equiv 0$.

Now we suppose that $f(z)$ has only finitely many zeros and $\sigma(f) \neq 1$. Thus since f is transcendental, $f(z)$ can be written as follows:

$$f(z) = g(z)e^{h(z)}, \tag{12}$$

where $g(z)(\not\equiv 0)$, $h(z)$ are polynomials, $\deg h(z) \geq 2$. Thus

$$f\left(z + c_i\right) = g\left(z + c_i\right)e^{h(z+c_i)}. \tag{13}$$

Now we suppose that $H_n(z)$ has only finitely many zeros. By Lemma 16, we see that $H_n(z)$ is transcendental. So $H_n(z)$ can be written as

$$\begin{aligned} H_n(z) &= g(z)^n \sum_{i=1}^{k} a_i g\left(z + c_i\right)e^{nh(z)+h(z+c_i)} \\ &= g_1(z)e^{h_1(z)}, \end{aligned} \tag{14}$$

where $g_1(z)(\not\equiv 0)$, $h_1(z)$ are polynomials, $\deg h_1(z) \geq 1$. Set

$$h(z) = b_m z^m + b_{m-1}z^{m-1} + \cdots + b_0, \quad b_m \neq 0, \tag{15}$$

where $b_m, \ldots, b_0$ are constants and $m \geq 2$. Thus

$$\begin{aligned} h\left(z + c_i\right) &= b_m z^m + \left(b_m m c_i + b_{m-1}\right)z^{m-1} \\ &\quad + b'_{m-2}z^{m-2} + \cdots + b'_0, \end{aligned} \tag{16}$$

where $b'_{m-2}, \ldots, b'_0$ are constants. Since $m \geq 2$ and

$$h\left(z + c_i\right) - h\left(z + c_j\right) = b_m m\left(c_i - c_j\right)z^{m-1} + \cdots \quad (i \neq j), \tag{17}$$

we see that $nh(z) + h(z + c_i) - (nh(z) + h(z + c_j))$ $(i \neq j)$ are not constants.

Case 1. If for any i, $nh(z) + h(z + c_i) - h_1(z)$ are not constants, then by Lemma 15 and (14), we see that

$$a_i g(z)^n g\left(z + c_i\right) \equiv 0, \qquad g_1(z) \equiv 0, \tag{18}$$

which is a contradiction.

Case 2. If there exists a j satisfying $nh(z)+h(z+c_j)-h_1(z) = \delta$ where δ is a constant, then by (14), we have

$$\begin{aligned} &\left(g(z)^n a_j g\left(z + c_j\right) - e^{-\delta}g_1(z)\right)e^{nh(z)+h(z+c_j)} \\ &\quad + g(z)^n \sum_{i \neq j} a_i g\left(z + c_i\right)e^{nh(z)+h(z+c_i)} = 0. \end{aligned} \tag{19}$$

By (19), Lemma 15, and $k \geq 2$, we obtain that

$$\begin{aligned} &a_i g(z)^n g\left(z + c_i\right) \equiv 0 \quad (i \neq j), \\ &g(z)^n a_j g\left(z + c_j\right) - e^{-\delta}g_1(z) \equiv 0, \end{aligned} \tag{20}$$

which is also a contradiction. Hence, $H_n(z)$ has infinitely many zeros.

(ii) Suppose that $f(z)$ has only finitely many zeros and $\sigma(f) = 1$. Then $f(z)$ can be written as

$$f(z) = g_2(z)e^{bz+d}, \tag{21}$$

where $g_2(z)(\not\equiv 0)$ is a polynomial and $b(\neq 0), d$ are constants. Thus

$$f(z+c_i) = g_2(z+c_i)e^{bc_i}e^{bz+d},$$
$$H_n(z) = \sum_{i=1}^{k} a_i g_2(z)^n g_2(z+c_i)e^{bc_i}e^{(n+1)(bz+d)}. \tag{22}$$

By the condition $\sum_{i=1}^{k} a_i f(z+c_i) \not\equiv 0$, we see that $\sum_{i=1}^{k} a_i g_2(z+c_i)e^{bc_i} \not\equiv 0$.

Hence $H_n(z)$ has only finitely many zeros.

(iii) Case 1. $\sigma(f) = 0$. From $0 \le \lambda(H_n(z) - \alpha(z)) \le \sigma(H_n(z) - \alpha(z)) \le \sigma(f) = 0$, we get $\lambda(H_n(z) - \alpha(z)) = \sigma(H_n(z) - \alpha(z)) = \sigma(f) = 0$. If $H_n(z) - \alpha(z)$ has only finitely zeros, then $H_n(z) - \alpha(z)$ can be written as

$$H_n(z) - \alpha(z) = p(z), \quad \text{i.e.,} \; H_n(z) = p(z) + \alpha(z), \tag{23}$$

where $p(z)$ is a polynomial. By using a similar method as in the proof of Lemma 16, we get a contradiction. Thus $H_n(z) - \alpha(z)$ has infinitely many zeros.

Case 2. $\sigma(f) > 0$. Suppose on contrary to the assertion that $\lambda(H_n(z)-\alpha(z)) < \sigma(f)$. If $f(z)^n \sum_{i=1}^{k} a_i f(z+c_i) - \alpha(z) \equiv 0$, that is, $f(z)^n \sum_{i=1}^{k} a_i f(z+c_i) \equiv \alpha(z)$. By using a similar method as in the proof of Lemma 16, we get a contradiction. So we have $f(z)^n \sum_{i=1}^{k} a_i f(z+c_i) - \alpha(z) \not\equiv 0$. Thus, by Hadamard's theorem, $H_n(z) - \alpha(z)$ can be written as

$$\begin{aligned} H_n(z) - \alpha(z) &= f(z)^n \sum_{i=1}^{k} a_i f(z+c_i) - \alpha(z) \\ &= \frac{P(z)}{Q(z)} e^{h(z)}, \end{aligned} \tag{24}$$

where $h(z)$ is a polynomial and $P(z)(\not\equiv 0), Q(z)(\not\equiv 0)$ are the canonical products formed by zeros and poles of $H_n(z)-\alpha(z)$, respectively, such that

$$\lambda(P(z)) = \sigma(P(z)) = \lambda(H_n(z) - \alpha(z)) < \sigma(f) = \sigma. \tag{25}$$

Since $T(r, \alpha(z)) = S(r, f)$, we get that

$$\lambda(Q(z)) = \sigma(Q(z)) = \lambda\left(\frac{1}{\alpha(z)}\right) < \sigma(f) = \sigma. \tag{26}$$

We set $g(z) = P(z)/Q(z)$; then from (25) and (26), we get

$$\sigma(g) = \max\{\sigma(P(z)), \sigma(Q(z))\} < \sigma(f) = \sigma. \tag{27}$$

Differentiating (24) and eliminating $e^{h(z)}$, we get

$$f(z)^{n-1} F(z, f) = \alpha'(z) g(z) - \alpha(z)\left(g(z) h'(z) + g'(z)\right), \tag{28}$$

where

$$\begin{aligned} F(z, f) = {} & n f'(z) g(z) \sum_{i=1}^{k} a_i f(z+c_i) \\ & + f(z) g(z) \sum_{i=1}^{k} a_i f'(z+c_i) \\ & - \left(g(z) h'(z) + g'(z)\right) f(z) \sum_{i=1}^{k} a_i f(z+c_i). \end{aligned} \tag{29}$$

Case 2.1. $F(z, f) \equiv 0$. Then from (28), we have

$$\alpha'(z) g(z) - \alpha(z)\left(g(z) h'(z) + g'(z)\right) \equiv 0. \tag{30}$$

By integrating, we have

$$\alpha(z) = c g(z) e^{h(z)}, \tag{31}$$

where c is a nonzero constant. From (24) and (31), we have

$$f(z)^n \sum_{i=1}^{k} a_i f(z+c_i) = \left(1 + \frac{1}{c}\right)\alpha(z). \tag{32}$$

By using a similar method as in the proof of Lemma 16, we get a contradiction.

Case 2.2. $F(z, f) \not\equiv 0$. Let

$$\begin{aligned} F^*(z, f) = \frac{F(z)}{f(z)^2} = {} & n \frac{f'(z)}{f(z)} g(z) \sum_{i=1}^{k} a_i \frac{f(z+c_i)}{f(z)} \\ & + g(z) \sum_{i=1}^{k} a_i \frac{f'(z+c_i)}{f(z+c_i)} \cdot \frac{f(z+c_i)}{f(z)} \\ & - \left(g(z) h'(z) + g'(z)\right) \sum_{i=1}^{k} a_i \frac{f(z+c_i)}{f(z)}. \end{aligned} \tag{33}$$

Then from (28), we have

$$f(z)^{n+1} F^*(z, f) = \alpha'(z) g(z) - \alpha(z)\left(g(z) h'(z) + g'(z)\right). \tag{34}$$

From Lemma 13 and Lemma 14, we have

$$\begin{aligned} m\left(r, f(z)^k F^*(z, f)\right) \le {} & S(r, f) + O\left(m(r, g)\right) \\ & + O\left(\sum_{i=1}^{k} m\left(r, \frac{f'(z+c_i)}{f(z+c_i)}\right)\right), \quad k = 1, 2. \end{aligned} \tag{35}$$

Now for any given ε $(0 < \varepsilon < 1)$, we obtain from Lemma 17 and (27) that

$$\begin{aligned} m\left(r, \frac{f'(z+c_i)}{f(z+c_i)}\right) &= S\left(r, f(z+c_i)\right) \\ &= S\left(r, f(z)\right), T(r, g) = O\left(r^{\sigma-\varepsilon}\right). \end{aligned} \tag{36}$$

It follows from (35) and (36) that

$$m(r, f(z)F^*(z, f)) = O(r^{\sigma-\varepsilon}) + S(r, f), \quad (37)$$

$$m\left(r, f(z)^2F^*(z, f)\right) = O(r^{\sigma-\varepsilon}) + S(r, f). \quad (38)$$

We obtain from the definition of $F(z, f)$ that

$$N(r, F(z, f)) = O(N(r, g(z))) = O(r^{\sigma-\varepsilon}). \quad (39)$$

Thus from (38) and (39), we have

$$\begin{aligned} T\left(r, f(z)^2F^*(z, f)\right) &= T(r, F(z, f)) \\ &= O(r^{\sigma-\varepsilon}) + S(r, f). \end{aligned} \quad (40)$$

Note that a zero of $f(z)$ which is not a pole of $g(z)$ is a pole of $f(z)F^*(z, f)$ with the multiplicity at most 1, so from (34) and (27) we get that, for ε (> 0) sufficiently small,

$$\begin{aligned} &(n-1)N\left(r, \frac{1}{f(z)}\right) \\ &\quad \le N\left(r, \frac{1}{\alpha'(z)g(z) - \alpha(z)(g(z)h'(z) + g'(z))}\right) \\ &\quad + O(N(r, g(z))) = O(r^{\sigma-\varepsilon}) + S(r, f). \end{aligned} \quad (41)$$

Hence from (33) and the above formula, we have

$$\begin{aligned} N(r, f(z)F^*(z, f)) &= O\left(N\left(r, \frac{1}{f(z)}\right) + N(r, g(z))\right) \\ &= O(r^{\sigma-\varepsilon}) + S(r, f). \end{aligned} \quad (42)$$

It follows from (37) and (42) that

$$T(r, f(z)F^*(z, f)) = O(r^{\sigma-\varepsilon}) + S(r, f). \quad (43)$$

Therefore, from (40) and (43), we have

$$T(r, f(z)) = O(r^{\sigma-\varepsilon}) + S(r, f), \quad (44)$$

which contradicts the assumption that $f(z)$ is a transcendental entire function of finite order σ. This completes the proof of Theorem 5.

By using the same methods as in the proof of Theorem 5 (i) and (ii), we complete the proof of Theorem 10. □

4. Proof of Theorem 8

Proof. Firstly, we prove (ii) and (iii). (ii) Suppose that $d(\neq 0)$ is the Borel exceptional value of $f(z)$. Then $f(z)$ can be written as follows:

$$f(z) = d + p(z)e^{\alpha z^k}, \quad (45)$$

where k is a positive integer, α ($\neq 0$) is a constant, and $p(z)(\not\equiv 0)$ is an entire function satisfying

$$\sigma(p) < \sigma(f) = k. \quad (46)$$

Thus

$$f(z + c_i) = d + p(z + c_i)p_i(z)e^{\alpha z^k}, \quad (47)$$

where $p_i(\not\equiv 0)$ is an entire function satisfying $\sigma(p_i) = k - 1$. So by using $\sum_{i=1}^{k} a_i = 0$, we have

$$\begin{aligned} H(z) &= \sum_{i=1}^{k} a_i\left(d + p(z)e^{\alpha z^k}\right)\left(d + p(z + c_i)p_i(z)e^{\alpha z^k}\right) \\ &= \sum_{i=1}^{k} da_ip(z + c_i)p_i(z)e^{\alpha z^k} \\ &\quad + \sum_{i=1}^{k} a_ip(z)p(z + c_i)p_i(z)e^{2\alpha z^k}. \end{aligned} \quad (48)$$

Since $\sum_{i=1}^{k} a_if(z + c_i) \not\equiv 0$, we see that

$$\sum_{i=1}^{k} a_ip(z + c_i)p_i(z) \not\equiv 0. \quad (49)$$

By (48) and (49), we see that

$$\sigma(H) = \sigma(f) = k. \quad (50)$$

If $H(z)$ has the Borel exceptional value d^*, then

$$H(z) = d^* + p^*(z)e^{\beta z^k}, \quad (51)$$

where $\beta(\neq 0)$ is a constant and $p^*(z)(\not\equiv 0)$ is an entire function satisfying

$$\sigma(p^*(z)) < \sigma(H) = k. \quad (52)$$

By (48) and (51), we have

$$\begin{aligned} &\sum_{i=1}^{k} da_ip(z + c_i)p_i(z)e^{\alpha z^k} + \sum_{i=1}^{k} a_ip(z)p(z + c_i)p_i(z)e^{2\alpha z^k} \\ &\quad - p^*(z)e^{\beta z^k} - d^* = 0. \end{aligned} \quad (53)$$

Case 1. If $\beta \neq 2\alpha$ and $\beta \neq \alpha$, then by Lemma 15 and (53), we can obtain that

$$\sum_{i=1}^{k} da_ip(z + c_i)p_i(z) \equiv 0. \quad (54)$$

This contradicts with (49).
Case 2. If $\beta = 2\alpha$ or $\beta = \alpha$, then using the same method as above, we can also obtain a contradiction. Hence $H(z)$ has no Borel exceptional value.

(iii) Suppose that $d = 0$ is the Borel exceptional value of $f(z)$. Using the same method as above, we obtain

$$H(z) = \sum_{i=1}^{k} a_ip(z)p(z + c_i)p_i(z)e^{2\alpha z^k}. \quad (55)$$

From (49) and

$$\sigma\left(\sum_{i=1}^{k} a_i p(z)\, p(z+c_i)\, p_i(z)\right) < k, \tag{56}$$

we see that 0 is the finite Borel exceptional value of $H(z)$. Thus, $H(z)$ has no nonzero finite Borel exceptional value.

Finally, we prove (i). By the assertion of (ii) and (iii), we see that if $f(z)$ has the finite Borel exceptional value, then any nonzero finite value a must not be the Borel exceptional value of $H(z)$. Hence $H(z)$ takes the value a infinitely often. By (50), we obtain $\lambda(H-a) = \sigma(H) = \sigma(f)$. □

5. Proof of Theorem 9

Proof. Clearly, if $a = 0$, then $H(z)$ has infinitely many zeros since $\sum_{i=1}^{k} a_i f(z+c_i)\ (\not\equiv 0)$ is an entire function and $f(z)$ has infinitely many zeros.

Now we suppose that $a \neq 0$. Suppose that $H(z)-a$ has only finitely many zeros. Then $H(z)-a$ can be written as follows:

$$H(z) - a = \sum_{i=1}^{k} a_i f(z)\, f(z+c_i) - a = p(z)\, e^{q(z)}, \tag{57}$$

where $p(z)$, $q(z)$ are polynomials. By Lemma 16, we see that $p(z) \not\equiv 0$, $\deg q(z) \geq 1$. Differentiating (57) and eliminating $e^{q(z)}$, we obtain

$$\frac{\left(f(z)\sum_{i=1}^{k} a_i f(z+c_i)\right)'}{f(z)\sum_{i=1}^{k} a_i f(z+c_i)} = \frac{p'(z) + p(z)\, q'(z)}{p(z)} \times\left(1 - \frac{a}{f(z)\sum_{i=1}^{k} a_i f(z+c_i)}\right). \tag{58}$$

Since there exists an infinite sequence $\{z_n\}$ satisfying $f(z_n) = \sum_{i=1}^{k} a_i f(z_n+c_i) = 0$, we see that there is a sufficiently large point z_0 such that $f(z_0) = \sum_{i=1}^{k} a_i f(z_0+c_i) = 0$ and $p'(z_0) + p(z_0)q'(z_0) \neq 0$, $p(z_0) \neq 0$ at the same time.

From observation, we have the following: $(f(z)\sum_{i=1}^{k} a_i f(z+c_i))' / f(z)\sum_{i=1}^{k} a_i f(z+c_i)$ has a simple pole at z_0 and $a/f(z)\sum_{i=1}^{k} a_i f(z+c_i)$ has pole at z_0 of multiplicity at least 2. This shows that (58) is a contradiction. Hence $H(z)$ takes every value a infinitely often. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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