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Research Article

Laplacian Spectral Characterization of Some Unicyclic Graphs

Lijun Yu,[1] Hui Wang,[1] and Jiang Zhou[2]

[1] *College of Automation, Harbin Engineering University, Harbin 150001, China*
[2] *College of Science, Harbin Engineering University, Harbin 150001, China*

Correspondence should be addressed to Jiang Zhou; zhoujiang04113112@163.com

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Let $W(n; q, m_1, m_2)$ be the unicyclic graph with n vertices obtained by attaching two paths of lengths m_1 and m_2 at two adjacent vertices of cycle C_q. Let $U(n; q, m_1, m_2, \ldots, m_s)$ be the unicyclic graph with n vertices obtained by attaching s paths of lengths $m_1, m_2, \ldots, m_s$ at the same vertex of cycle C_q. In this paper, we prove that $W(n; q, m_1, m_2)$ and $U(n; q, m_1, m_2, \ldots, m_s)$ are determined by their Laplacian spectra when q is even.

1. Introduction

Let G be a simple, undirected graph with n vertices. Let A be the adjacency matrix of G and let D be the diagonal matrix of vertex degrees of G. The matrices $L = D - A$ and $Q = D + A$ are called the *Laplacian matrix* and *signless Laplacian matrix* of G, respectively. The multiset of eigenvalues of A and L are called the *A-spectrum* and *L-spectrum* of G, respectively. The eigenvalues of A and L are called the *A-eigenvalues* and *L-eigenvalues* of G, respectively. We use $\lambda_1(G) \geqslant \lambda_2(G) \geqslant \cdots \geqslant \lambda_n(G)$ and $\mu_1(G) \geqslant \mu_2(G) \geqslant \cdots \geqslant \mu_n(G) = 0$ to denote the A-eigenvalues and the L-eigenvalues of G, respectively. Two graphs are said to be *L-cospectral* (*A-cospectral*) if they have the same L-spectrum (A-spectrum). A graph G is said to be *determined by its L-spectrum* (*A-spectrum*) if there is no other nonisomorphic graph L-cospectral (A-cospectral) with G. Let $\phi_A(G, x)$, $\phi_L(G, x)$, and $\phi_Q(G, x)$ denote the characteristic polynomials of the adjacency matrix, the Laplacian matrix, and the signless Laplacian matrix of G, respectively. As usual, P_n, C_n, and K_n stand for the path, the cycle, and the complete graph with n vertices, respectively. Let $\ell(G)$ denote the line graph of G. A tree is called *starlike* if it has exactly one vertex of degree larger than 2. Let $T_{a,b,c}$ denote the starlike tree with a vertex v of degree 3 such that $T_{a,b,c} - v = P_a \cup P_b \cup P_c$.

For a connected graph G with n vertices, G is called a *unicyclic graph* if G has n edges. Which graphs are determined by their spectrum is a difficult problem in the theory of graph spectra. Here, we introduce some results on spectral characterizations of unicyclic graphs. Let $U(n; q, m_1, m_2, \ldots, m_s)$ be the unicyclic graph with n vertices obtained by attaching s paths of lengths $m_1, m_2, \ldots, m_s$ ($m_i \geqslant 1$) at the same vertex of cycle C_q (see Figure 1). Haemers et al. [1] proved that $U(n; q, m_1)$ is determined by its A-spectrum when q is odd, and all $U(n; q, m_1)$ are determined by their L-spectra. It is also known that $U(n; q, m_1)$ is determined by its A-spectrum when q is even [2]. Liu et al. [3] proved that $U(n; q, m_1, m_2)$ is determined by its L-spectrum. It is known that $U(n; q, 1, 1, \ldots, 1)$ is determined by its L-spectrum, and $U(n; q, 1, 1, \ldots, 1)$ is determined by its A-spectrum if q is odd (see [4]). Boulet [5] proved that the sun graph is determined by its L-spectrum. Shen and Hou [6] gave a class of unicyclic graphs with even girth that are determined by their L-spectra.

Let $W(n; q, m_1, m_2)$ be the unicyclic graph with n vertices obtained by attaching two paths of lengths m_1 and m_2 ($m_1, m_2 \geqslant 1$) at two adjacent vertices of cycle C_q (see Figure 1). In this paper, we prove that $W(n; q, m_1, m_2)$ and $U(n; q, m_1, m_2, \ldots, m_s)$ are determined by their L-spectra when q is even.

2. Preliminaries

In this section, we give some lemmas which play important roles throughout this paper.

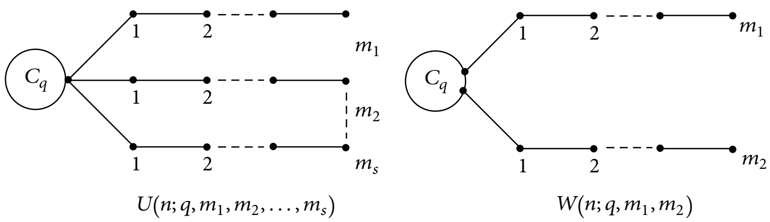

Figure 1: Two classes of unicyclic graphs.

Lemma 1 (see [7]). *Let G be a graph. For the adjacency matrix and the Laplacian matrix, the following can be obtained from the spectrum:*

(i) *the number of vertices,*

(ii) *the number of edges.*

For the adjacency matrix, the following follows from the spectrum:

(iii) *the number of closed walks of any length.*

For the Laplacian matrix, the following follows from the spectrum:

(iv) *the number of components,*

(v) *the number of spanning trees.*

Lemma 2 (see [8]). *For a bipartite graph G, one has* $\phi_L(G, x) = \phi_Q(G, x)$.

Lemma 3 (see [8]). *Let G be a graph with n vertices and m edges. Then*

$$\phi_A(\ell(G), x) = (x+2)^{m-n}\phi_Q(G, x+2). \tag{1}$$

For a graph G with n vertices, let $\phi_L(G, x) = l_0x^n + l_1x^{n-1} + \cdots + l_n$. Oliveira et al. determined the first four coefficients of $\phi_L(G, x)$ as follows.

Lemma 4 (see [9]). *Let G be a graph with n vertices and m edges, and let* $d_1, d_2, \ldots, d_n$ *be the degree sequence of G. Then*

$$\begin{gathered} l_0 = 1, \qquad l_1 = -2m = -\sum_{i=1}^{n} d_i, \\ l_2 = 2m^2 - m - \frac{1}{2}\sum_{i=1}^{n} d_i^2, \\ l_3 = \frac{1}{3}\left[-4m^3 + 6m^2 + 3m^2\sum_{i=1}^{n} d_i^2 \right. \\ \left. -\sum_{i=1}^{n} d_i^3 - 3\sum_{i=1}^{n} d_i^2 + 6N_G(C_3)\right], \end{gathered} \tag{2}$$

where $N_G(C_3)$ *is the number of triangles in G.*

For a graph G, the *subdivision graph* of G, denoted by $S(G)$, is the graph obtained from G by inserting a new vertex in each edge of G.

Lemma 5 (see [8]). *Let G be a graph with n vertices and m edges. Then*

$$\phi_A(S(G), x) = x^{m-n}\phi_Q\left(G, x^2\right). \tag{3}$$

Lemma 6 (see [8]). *Let u be a vertex of G, let N(u) be the set of all vertices adjacent to u, and let C(u) be the set of all cycles containing u. Then*

$$\begin{aligned} \phi_A(G, x) = x\phi_A(G-u, x) &- \sum_{v\in N(u)} \phi_A(G-u-v, x) \\ &- 2\sum_{Z\in C(u)} \phi_A(G - V(Z), x), \end{aligned} \tag{4}$$

where $V(Z)$ *is the vertex set of Z.*

Lemma 7 (see [10]). *Consider* $\phi_A(P_n, 2) = n + 1$.

Lemma 8 (see [1]). *Let G be a graph with n vertices and let v be a vertex of G. Then* $\lambda_1(G) \geqslant \lambda_1(G-v) \geqslant \lambda_2(G) \geqslant \lambda_2(G-v) \geqslant \cdots \geqslant \lambda_{n-1}(G-v) \geqslant \lambda_n(G)$.

Lemma 9 (see [5]). *Let G be a graph with edge set E(G). Then*

$$\mu_1(G) \leqslant \max\{d(u) + d(v) : uv \in E(G)\}, \tag{5}$$

where $d(u)$ *stands for the degree of vertex u.*

Lemma 10 (see [11]). *For a connected graph G with at least two vertices, one has* $\mu_1(G) \geqslant \Delta(G) + 1$, *where* $\Delta(G)$ *denotes the maximum vertex degree of G; equality holds if and only if* $\Delta(G) = n - 1$.

Lemma 11 (see [12]). *Let G be a connected graph with* $n \geqslant 3$ *vertices and let* d_2 *be the second maximum degree of G. Then* $d_2 \leqslant \mu_2(G)$.

Lemma 12 (see [8]). *Let G be a graph with n vertices and let e be an edge of G. Then* $\mu_1(G) \geqslant \mu_1(G-e) \geqslant \mu_2(G) \geqslant \mu_2(G-e) \geqslant \cdots \geqslant \mu_{n-1}(G-e) \geqslant \mu_n(G) = \mu_n(G-e) = 0$.

For a graph G, let $N_G(M)$ denote the number of subgraphs of G which are isomorphic to graph M.

Lemma 13 (see [13]). *Let G be a graph and let* $N_G(k)$ *be the number of closed walks of length k in G. Then*

$$\begin{gathered} N_G(3) = 6N_G(C_3), \\ N_G(5) = 30N_G(C_3) + 10N_G(C_5) + 10N_G(U(4;3,1)). \end{gathered} \tag{6}$$

3. Main Results

Lemma 14. *Let G be a unicyclic graph with n vertices, and G contains an even cycle C_q. Let H be a graph L-cospectral with G. Then the following statements hold.*

(1) *H is a unicyclic graph with n vertices, and the girth of H is q.*

(2) *The line graphs $\ell(G)$ and $\ell(H)$ are A-cospectral.*

(3) *The subdivision graphs $S(G)$ and $S(H)$ are A-cospectral, and $\sqrt{\mu_i(G)} = \lambda_i(S(G))$ $(i = 1, 2, \ldots, n)$.*

Proof. By Lemma 1, H is a unicyclic graph with n vertices, and the girth of H is q. Since q is even, G and H are bipartite. By Lemma 2, one has $\phi_Q(G, x) = \phi_L(G, x) = \phi_L(H, x) = \phi_Q(H, x)$. Lemma 3 implies that line graphs $\ell(G)$ and $\ell(H)$ are A-cospectral. By Lemma 5, subdivision graphs $S(G)$ and $S(H)$ are A-cospectral, and $\sqrt{\mu_i(G)} = \lambda_i(S(G))$ $(i = 1, 2, \ldots, n)$. □

Theorem 15. *The unicyclic graph $G = W(n; q, m_1, m_2)$ is determined by its L-spectrum when q is even.*

Proof. Let H be any graph L-cospectral with G. By Lemma 14, we know that H is a unicyclic graph with n vertices, the girth of H is q, and $\ell(G)$ and $\ell(H)$ are A-cospectral. By Lemmas 1 and 13, we have $N_{\ell(H)}(C_3) = N_{\ell(G)}(C_3) = 2$. So the maximum degree of H does not exceed 3. Suppose that there are a_i vertices of degree i $(i = 1, 2, 3)$ in H. From Lemma 4, we have

$$\sum_{i=1}^{3} a_i = n,$$
$$\sum_{i=1}^{3} i a_i = 2n, \qquad (7)$$
$$\sum_{i=1}^{3} i^2 a_i = 2 \times 3^2 + 4(n-4) + 2 = 4n + 4.$$

Solving the above equations, we get $a_1 = 2$, $a_2 = n - 4$, $a_3 = 2$. So H and G have the same degree sequence. Then, one of the following holds.

(1) H is the unicyclic graph obtained by attaching two paths of lengths l_1 and l_2 at two nonadjacent vertices of cycle C_q.

(2) $H = W(n; q, l_1, l_2)$; that is, H is the unicyclic graph obtained by attaching two paths of lengths l_1 and l_2 at two adjacent vertices of cycle C_q.

(3) H is the graph shown in Figure 2.

Next, we discuss each of these three cases listed above.

Case 1 (H is the unicyclic graph obtained by attaching two paths of lengths l_1 and l_2 at two nonadjacent vertices of cycle C_q). Since $\ell(G)$ and $\ell(H)$ are A-cospectral, by Lemma 1, $\ell(G)$ and $\ell(H)$ have the same number of closed walks of any length. It is not difficult to see that $N_{\ell(G)}(C_5) = N_{\ell(H)}(C_5)$. By Lemma 13, we have $N_{\ell(H)}(U(4; 3, 1)) = N_{\ell(G)}(U(4; 3, 1))$.

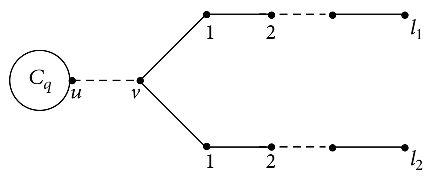

Figure 2: Graph H.

Note that $m_1 + m_2 + q = l_1 + l_2 + q = n$. If $m_1 \geqslant 2$ or $m_2 \geqslant 2$, then $N_{\ell(G)}(U(4; 3, 1)) \geqslant 7$ and $N_{\ell(H)}(U(4; 3, 1)) \leqslant 6$. If $m_1 = m_2 = 1$, then $N_{\ell(G)}(U(4; 3, 1)) = 6$ and $N_{\ell(H)}(U(4; 3, 1)) = 4$. Hence $N_{\ell(H)}(U(4; 3, 1)) \neq N_{\ell(G)}(U(4; 3, 1))$, a contradiction.

Case 2 (H is the unicyclic graph $W(n; q, l_1, l_2)$). From Lemma 14, we know that the subdivision graphs $S(G)$ and $S(H)$ (shown in Figure 3) are A-cospectral. Let $p_f = \phi_A(P_f, x)$; from Lemmas 6 and 7, we have

$$\begin{aligned}
\phi_A(S(G), x) &= x p_{2m_1+2m_2+2q-1} \\
&\quad - \left(p_{2m_1} p_{2q-2+2m_2} + p_{2m_2} p_{2q-2+2m_1}\right) \\
&\quad - 2 p_{2m_1} p_{2m_2}, \\
\phi_A(S(G), 2) &= 2(2m_1 + 2m_2 + 2q) \\
&\quad - (2m_1 + 1)(2q + 2m_2 - 1) \\
&\quad - (2m_2 + 1)(2q + 2m_1 - 1) \\
&\quad - 2(2m_1 + 1)(2m_2 + 1) \\
&= -4(m_1 q + m_2 q + 4m_1 m_2), \\
\phi_A(S(H), x) &= x p_{2l_1+2l_2+2q-1} \\
&\quad - \left(p_{2l_1} p_{2q-2+2l_2} + p_{2l_2} p_{2q-2+2l_1}\right) \\
&\quad - 2 p_{2l_1} p_{2l_2}, \\
\phi_A(S(H), 2) &= 2(2l_1 + 2l_2 + 2q) \\
&\quad - (2l_1 + 1)(2q + 2l_2 - 1) \\
&\quad - (2l_2 + 1)(2q + 2l_1 - 1) \\
&\quad - 2(2l_1 + 1)(2l_2 + 1) \\
&= -4(l_1 q + l_2 q + 4l_1 l_2).
\end{aligned} \qquad (8)$$

By $\phi_A(S(G), 2) = \phi_A(S(H), 2)$, we get $-4(m_1 q + m_2 q + 4m_1 m_2) = -4(l_1 q + l_2 q + 4l_1 l_2)$. By $m_1 + m_2 + q = l_1 + l_2 + q = n$, we get $m_1 m_2 = l_1 l_2$. Hence, $m_1 = l_1, m_2 = l_2$ or $m_1 = l_2, m_2 = l_1$, G and H are isomorphic.

Case 3 (H is the graph shown in Figure 2). It is well known that the largest L-eigenvalue of a path is less than 4, and the largest L-eigenvalue of an even cycle is 4. Lemma 12 implies that $\mu_2(G) < 4$. Let u and v be the two vertices of degree 3 in H (see Figure 2). If u and v are nonadjacent, there exists an edge e of H such that $H - e = C_q \cup T_{l_1, l_2, n-l_1-l_2-q-1}$. By Lemmas

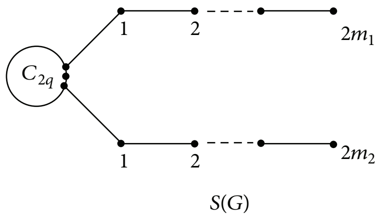

FIGURE 3: Two subdivision graphs.

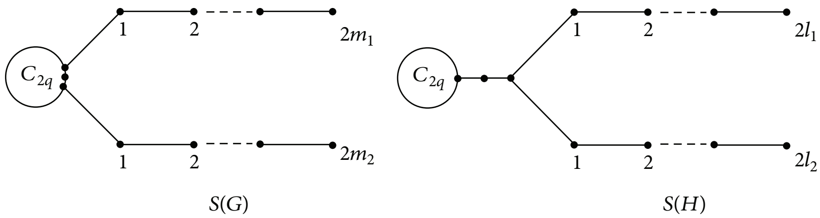

FIGURE 4: Two subdivision graphs.

10 and 12, we get $\mu_2(H) \geqslant 4$, a contradiction to $\mu_2(G) < 4$. So u and v are adjacent.

From Lemma 14, we know that the subdivision graphs $S(G)$ and $S(H)$ (shown in Figure 4) are A-cospectral. Let $p_f = \phi_A(P_f, x)$; from Lemmas 6 and 7, we have

$$\begin{aligned}
&\phi_A(S(G), x) \\
&\quad = x p_{2m_1+2m_2+2q-1} \\
&\qquad - \left(p_{2m_1} p_{2q-2+2m_2} + p_{2m_2} p_{2q-2+2m_1}\right) \\
&\qquad - 2 p_{2m_1} p_{2m_2}, \\
&\phi_A(S(G), 2) \\
&\quad = 2(2m_1 + 2m_2 + 2q) \\
&\qquad - (2m_1 + 1)(2q + 2m_2 - 1) \\
&\qquad - (2m_2 + 1)(2q + 2m_1 - 1) \\
&\qquad - 2(2m_1 + 1)(2m_2 + 1) \\
&\quad = -4(m_1 q + m_2 q + 4 m_1 m_2), \\
&\phi_A(S(H), x) \\
&\quad = x p_{2q-1} \phi_A\left(T_{1,2l_1,2l_2}, x\right) \\
&\qquad - \left(p_{2q-1} p_{2l_1+2l_2+1} + 2 p_{2q-2} \phi_A\left(T_{1,2l_1,2l_2}, x\right)\right) \\
&\qquad - 2\phi_A\left(T_{1,2l_1,2l_2}, x\right), \\
&\phi_A(S(H), 2) \\
&\quad = 2 \times 2q \phi_A\left(T_{1,2l_1,2l_2}, 2\right) \\
&\qquad - \left[2q(2l_1 + 2l_2 + 2) + 2(2q - 1)\phi_A\left(T_{1,2l_1,2l_2}, 2\right)\right] \\
&\qquad - 2\phi_A\left(T_{1,2l_1,2l_2}, 2\right) \\
&\quad = -4q(l_1 + l_2 + 1).
\end{aligned} \tag{9}$$

Since $\phi_A(S(G), 2) = \phi_A(S(H), 2)$, we have $-4q(l_1 + l_2 + 1) = -4(m_1 q + m_2 q + 4 m_1 m_2)$. By $l_1 + l_2 + 1 = m_1 + m_2$, we get $m_1 m_2 = 0$, a contradiction to $m_1, m_2 > 0$. □

Here, we describe a classic method to count the number of closed walks of a given length in a graph (see [2, 13, 14]). For a graph G, $N_G(k)$ stands for the number of closed walks of length k in G and $N_G(M)$ stands for the number of subgraphs of G which are isomorphic to graph M. Let $\omega_k(M)$ be the number of closed walks of length k of graph M which contains all edges of M, and $M_k(G)$ denotes the set of all connected subgraphs M of G such that $\omega_k(M) \neq 0$. Then

$$N_G(k) = \sum_{M \in M_k(G)} N_G(M)\, \omega_k(M). \tag{10}$$

Lemma 16. *Let $G = U(n; q, m_1, m_2, \ldots, m_s)$ and $G' = U(n; q, l_1, l_2, \ldots, l_s)$ be L-cospectral graphs. If q is even, then G and G' are isomorphic.*

Proof. If q is even, by Lemma 14, $\ell(G)$ and $\ell(G')$ are A-cospectral. From Lemma 1, we get $N_{\ell(G)}(k) = N_{\ell(G')}(k)$ for any positive integer k. Suppose $m_1 \leqslant m_2 \leqslant \cdots \leqslant m_s$, $l_1 \leqslant l_2 \leqslant \cdots \leqslant l_s$. Let $r_i = \min\{m_i, l_i\}$ $(i = 1, 2, \ldots, s)$. If $m_1 \neq l_1$, by $m_1 + m_2 + \cdots + m_s = l_1 + l_2 + \cdots + l_s$, we know that $M_{2r_1+3}(\ell(G)) = M_{2r_1+3}(\ell(G'))$. For any $M \in M_{2r_1+3}(\ell(G))$ and $M \neq U(3 + r_1; 3, r_1)$, we have $N_{\ell(G)}(M) = N_{\ell(G')}(M)$. Since $N_{\ell(G)}(U(3 + r_1; 3, r_1)) \neq N_{\ell(G')}(U(3 + r_1; 3, r_1))$, by (10), we get $N_{\ell(G)}(2r_1 + 3) \neq N_{\ell(G')}(2r_1 + 3)$, a contradiction. So we have $m_1 = l_1$. Similar to the above arguments, by counting the number of closed walks of length $2r_i + 3$ $(i = 2, 3, \ldots, s)$, we can get $m_i = l_i$ $(i = 2, 3, \ldots, s)$. Hence G and G' are isomorphic. □

Theorem 17. *The unicyclic graph $G = U(n; q, m_1, m_2, \ldots, m_s)$ is determined by its L-spectrum when q is even.*

Proof. Let G' be any graph L-cospectral with G. By Lemma 14, G' is a unicyclic graph with n vertices, and the girth of G' is q. Let v be the vertex of degree $s+2$ in the subdivision graph $S(G) = U(2n; 2q, 2m_1, 2m_2, \ldots, 2m_s)$; then $S(G) - v = P_{2q-1} \cup P_{2m_1} \cup P_{2m_2} \cup \cdots \cup P_{2m_s}$. Since the largest A-eigenvalue of a path is less than 2, by Lemmas 8 and 14, we get $\sqrt{\mu_2(G)} = \lambda_2(S(G)) < 2$, $\mu_2(G) < 4$. Suppose $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n$ is the degree sequence of G'. By Lemma 11, we have $d_2 \leqslant 3$. From Lemmas 9 and 10, we get $s + 3 < \mu_1(G) \leqslant s + 4$, $d_1 + d_2 \geqslant \mu_1(G) > s + 3$, and $d_1 + 1 < \mu_1(G) \leqslant s + 4$. By $d_2 \leqslant 3$, we have $s < d_1 < s + 3$.

If $d_1 = s + 2$, applying Lemma 4, we have

$$\sum_{i=2}^{n} d_i = \underbrace{2 + 2 + \cdots + 2}_{n-s-1} + \underbrace{1 + 1 + \cdots + 1}_{s},$$
$$\sum_{i=2}^{n} d_i^2 = \underbrace{2^2 + 2^2 + \cdots + 2^2}_{n-s-1} + \underbrace{1^2 + 1^2 + \cdots + 1^2}_{s}. \tag{11}$$

Since $\sum_{i=2}^{n} d_i^2$ is minimal if and only if $|d_i - d_j| \leq 1$ for any $i, j \in \{2, 3, \ldots, n\}$, the degree sequences of G and G' are both $s + 2, \underbrace{2, 2, \ldots, 2}_{n-s-1}, \underbrace{1, 1, \ldots, 1}_{s}$. Lemma 16 implies that G and G' are isomorphic.

If $d_1 = s + 1$, by $d_1 + d_2 > s + 3$ and $d_2 < 4$, we get $d_2 = 3$. Suppose that there are a_3 three, a_2 two, and a_1 one in $d_2, d_3, \ldots, d_n$. By Lemma 4, we have

$$\sum_{i=1}^{3} a_i + 1 = n,$$
$$\sum_{i=1}^{3} i a_i + (s+1) = s + 2(n - s - 1) + (s + 2), \tag{12}$$
$$\sum_{i=1}^{3} i^2 a_i + (s+1)^2 = s + 4(n - s - 1) + (s + 2)^2.$$

Solving the above equations, we get $a_1 = 2s - 1$, $a_2 = n - 3s$, $a_3 = s$. From Lemma 4, we have

$$\sum_{i=1}^{3} i^3 a_i + (s+1)^3 = s + 8(n - s - 1) + (s + 2)^3. \tag{13}$$

$s = 0$ or $s = 1$ is the solution of the above equation. Then $d_1 = 1$ or $d_1 = 2$, a contradiction to $d_2 = 3$. □

The *join* of two graphs G and H, denoted by $G \times H$, is the graph obtained from $G \cup H$ by joining each vertex of G to each vertex of H. Some results on spectral characterizations of graphs obtained by join operation can be found in [15–20]. For a unicyclic graph G, if G is determined by its L-spectrum and $G \neq C_6$, then $G \times K_r$ is determined by its L-spectrum (cf. [18, Theorem 4.4]). Hence, we can obtain the following two results from Theorems 15 and 17.

Corollary 18. *Let $G = W(n; q, m_1, m_2)$. Then $G \times K_r$ is determined by its L-spectrum when q is even.*

Corollary 19. *Let $G = U(n; q, m_1, m_2, \ldots, m_s)$. Then $G \times K_r$ is determined by its L-spectrum when q is even.*

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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