Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 267510, 6 pages
http://dx.doi.org/10.1155/2014/267510

Research Article

Extended Nonsingular Terminal Sliding Surface for Second-Order Nonlinear Systems

Ji Wung Jeong, Ho Suk Yeon, and Kang-Bak Park

Department of Control and Instrumentation Engineering, Korea University, 2511 Sejong-ro, Sejong City 339-700, Republic of Korea

Correspondence should be addressed to Kang-Bak Park; kbpark@korea.ac.kr

Received 29 January 2014; Accepted 18 April 2014; Published 11 May 2014

Academic Editor: Hui Zhang

An extended nonsingular terminal sliding surface is proposed for second-order nonlinear systems. It is shown that the proposed surface is a superset of a conventional nonsingular terminal sliding surface which guarantees that the system state gets to zero in finite time. The conventional nonsingular sliding surfaces have been designed using a power function whose exponent is a rational number with positive odd numerator and denominator. The proposed nonsingular terminal sliding surface overcomes the restriction on the exponent of a power function; that is, the exponent can be a positive real number. Simulation results are provided to show the validity of the main result.

1. Introduction

According to the progress of control schemes, a variety of nonlinear control systems have been proposed: H-infinity optimal control, fuzzy control, neural network control, controller using genetic algorithms, and so on [1–4]. Sliding mode control (SMC) method, which is also called a variable structure system (VSS), is one of nonlinear robust control schemes. It has been widely used because of its invariance properties to parametric uncertainties and external disturbances [5–8]. In conventional sliding mode control systems, sliding surfaces have been designed such that the overall system in the sliding mode is asymptotically stable. However, the asymptotic stability does not guarantee a finite time convergence.

Although most of control systems proposed so far have been designed such that the closed-loop system is asymptotic, finite time stabilization is very important to many actual applications, such as motor, power, robot, and aerospace systems because, in the actual applications, the main objective of a control system is to make system's state to a desired one in a finite time interval which is determined a priori. Thus, many recent studies have focused on the finite time stabilization [9–13].

Finite time control systems based on sliding mode control schemes have been called terminal sliding mode control systems since their sliding surfaces have been designed as terminal attractors [14]. Recently, terminal sliding mode control systems have been studied to achieve a finite time convergence in many applications [15–17]. Conventional terminal sliding mode controllers used terminal sliding surfaces which were designed using a power function of a system state. They guaranteed that the system state reached zero in finite time. On the contrary, however, they suffered from singularity problems and had restrictions on the range of the exponent of a power function. The exponent should be a rational number with positive odd numerator and denominator [18]. In order to avoid the singularity problem in conventional terminal sliding mode control systems, nonsingular terminal sliding surfaces have been proposed very recently for robot manipulators [19, 20]. However, the same restriction on the exponent of a power function in the nonsingular sliding surface still remained: the exponent should be a rational number with a positive odd numerator and denominator.

Thus, in this paper, a novel nonsingular terminal sliding surface is proposed for second-order nonlinear systems. It is shown that the proposed scheme guarantees that the system state gets to zero in finite time and it does not suffer from singularity problems. Furthermore, the proposed nonsingular terminal sliding surface overcomes the restriction on the exponent of a power function by being a superset of a conventional nonsingular terminal sliding surface. That is, we extend the range of an exponent of a power function in the nonsingular terminal sliding surface from a rational number with an odd numerator and an odd denominator to a real number.

Simulation results and experimental results are given to show the validity of the main result.

2. Main Results

Consider a second-order nonlinear system of the following form:

$$\ddot{x} = f(x, \dot{x}, t) + g(x, \dot{x}, t)(u + d(x, \dot{x}, t)), \quad (1)$$

where x and $\dot{x}$ are state variables, $f(x, \dot{x}, t)$ is a nonlinear term, u is a scalar input, $g(x, \dot{x}, t) \neq 0$ $\forall x, \dot{x} \in R$, $\forall t \in R_+$, and $d(x, \dot{x}, t)$ represents the uncertainties and external disturbances. It is assumed that the following assumption holds.

Assumption 1. The uncertainty $d(x, \dot{x}, t)$ is bounded as follows:

$$|d(x, \dot{x}, t)| \leq D(x, \dot{x}, t), \quad \forall x, \dot{x} \in R, \ \forall t \in R_+, \quad (2)$$

where $D(x, \dot{x}, t)$ is a known positive function.

In previous works on terminal sliding mode control systems, the conventional terminal sliding surfaces have been designed as

$$s_1 = \dot{x} + c_1 x^{q_1/r_1}, \quad (3)$$

where $c_1 > 0$, $0 < q_1/r_1 < 1$, and q_1 and r_1 are positive odd integers [8–10]. However, though the conventional terminal sliding surface in (3) ensures finite time convergence, it suffers from the singularity problem and has a restriction on the exponent of the power function [14]. In the phase space with x and $\dot{x}$ axes, the set of singular points is a vertical axis except for at the origin; that is,

$$S = \{(x, \dot{x}) \mid x = 0, \ \dot{x} \neq 0\}. \quad (4)$$

Recently, a nonsingular terminal sliding surface was proposed to overcome the singularity problem [12]

$$s_2 = \dot{x}^{q_2/r_2} + c_2 x, \quad (5)$$

where $c_2 > 0$, $1 < q_2/r_2 < 2$, and q_2, r_2 are positive odd integers

However, this surface still has the restrictions on the exponent of the power function; that is, q_2, r_2 should be positive odd integers. Thus, we propose an extended nonsingular terminal sliding surface whose exponent can be any real number.

$$s = |\dot{x}|^{1/p} \operatorname{sgn}(\dot{x}) + c \cdot x, \quad (6)$$

where $\operatorname{sgn}(\cdot)$ is the signum function, c is a positive constant, and $1/2 \leq p < 1$ is a real number.

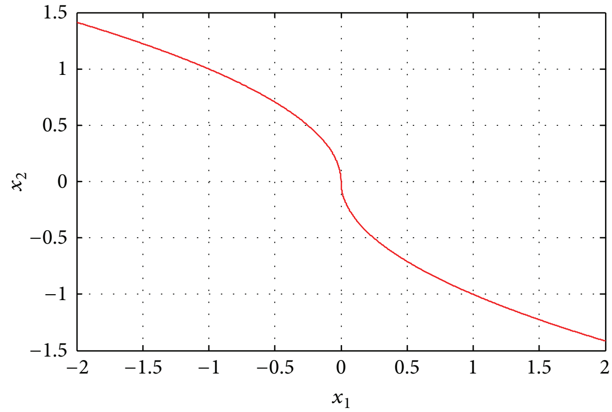

Figure 1: Example of a nonsingular terminal sliding surface.

Figure 1 shows a typical nonsingular terminal sliding surface. Clearly, for any $c > 0$ and $1/2 \leq p < 1$, the phase portrait is in the second and fourth quadrants in the phase space with the axes of $x_1 = x$ and $x_2 = \dot{x}$.

Remark 2. It is clear that the proposed nonsingular terminal sliding surface is stable because the sliding surface is in the second and fourth quadrants in the phase space with the axes of x and $\dot{x}$, where this property implies that $x \cdot \dot{x} < 0$ for all $x \neq 0$.

For the proposed nonsingular terminal sliding surface, we derived the following theorem for the finite time convergence.

Theorem 3. *The proposed nonsingular terminal sliding surface* (6) *guarantees that the system state gets to zero in finite time in the sliding mode, $s = 0$, and the relaxation time [18] is*

$$t_r = \frac{|x(0)|^{1-p}}{c^p (1-p)}. \quad (7)$$

Proof. If the system is in the sliding mode and $\dot{x} \geq 0$, the proposed sliding surface (6) can be rewritten as follows:

$$\dot{x}^{(1/p)} + cx = 0. \quad (8)$$

From the above equation, if $\dot{x}(0) \geq 0$, that is, $x(0) \leq 0$, the following equations can be easily derived:

$$\begin{aligned} &\dot{x}^{(1/p)} + cx = 0 \\ &\Longleftrightarrow \frac{dx}{dt} = c^p (-x)^p \\ &\Longleftrightarrow \frac{dx}{(-x)^p} = c^p dt \\ &\Longleftrightarrow -\frac{d(-x)}{(-x)^p} = c^p dt \\ &\Longleftrightarrow -\int_{-x=-x(0)}^{-x=0} \frac{d(-x)}{(-x)^p} = \int_0^{t_r} c^p dt \end{aligned}$$

$$\begin{aligned}
&\Longleftrightarrow \frac{(-x)^{(1-p)}}{1-p}\bigg|_{-x=-x(0)}^{-x=0} = c^p t_r \\
&\Longleftrightarrow t_r = \frac{(-x(0))^{(1-p)}}{c^p(1-p)} = \frac{|x(0)|^{(1-p)}}{c^p(1-p)},
\end{aligned} \tag{9}$$

where t_r represents the relaxation time [18].

If $\dot{x}(0) < 0$, that is, $x(0) > 0$, t_r can be derived in a similar way:

$$\begin{aligned}
&-(-\dot{x})^{(1/p)} + cx = 0 \\
&\Longleftrightarrow -\dot{x} = c^p x^p \\
&\Longleftrightarrow -\frac{dx}{dt} = c^p x^p \\
&\Longleftrightarrow -\frac{dx}{x^p} = c^p dt \\
&\Longleftrightarrow -\int_{x(0)}^{0} x^{-p} dx = \int_0^{t_r} c^p dt \\
&\Longleftrightarrow -\frac{x^{(1-p)}}{1-p}\bigg|_{x=x(0)}^{x=0} = c^p t_r \\
&\Longleftrightarrow t_r = \frac{x(0)^{(1-p)}}{c^p(1-p)} = \frac{|x(0)|^{(1-p)}}{c^p(1-p)}.
\end{aligned} \tag{10}$$

This completes the proof. □

In the above theorem, we proved that the system state gets to zero in finite time if it is in the sliding mode. Thus, in the following theorem, we propose a controller that guarantees that the sliding mode existence condition holds such that the overall system will be in the sliding mode.

Theorem 4. *For system* (1) *with the proposed nonsingular terminal sliding surface* (6)*, the following controller guarantees that the sliding mode existence condition holds:*

$$\begin{aligned}
u = &\frac{1}{g(x,\dot{x})} \\
&\times \left(-f(x,\dot{x}) - cp|\dot{x}|^{(2-(1/p))}\operatorname{sgn}(\dot{x}) - k_1 s - k_2 \operatorname{sgn}(s)\right),
\end{aligned} \tag{11}$$

where $k_1 > 0$, $k_2 > |g|(D+\alpha)$, α *is a positive constant and* $1/2 \le p < 1$.

Proof. Let the Lyapunov function candidate be

$$V(s) = \frac{1}{2}s^2. \tag{12}$$

Applying (1) and (6) to $dV/dt = \dot{V}$, the following equations can be obtained if $\dot{x} > 0$; that is, $x < 0$:

$$\begin{aligned}
\dot{V}(s) = s\dot{s} &= s\left(\frac{1}{p}\dot{x}^{((1/p)-1)}\ddot{x} + c\dot{x}\right) \\
&= s\left(\frac{1}{p}\dot{x}^{((1/p)-1)}(f + g(u+d)) + c\dot{x}\right) \\
&\le s\left(\frac{1}{p}\dot{x}^{((1/p)-1)}\right)(-k_1 s - \alpha \operatorname{sgn}(s)) \\
&\le \frac{1}{p}\dot{x}^{((1/p)-1)}\left(-k_1 s^2 - \alpha|s|\right).
\end{aligned} \tag{13}$$

Similarly, if $\dot{x} < 0$, that is, $x > 0$, the following equations can be derived:

$$\begin{aligned}
\dot{V}(s) = s\dot{s} &= s\left(\frac{1}{p}(-\dot{x})^{((1/p)-1)}(-\ddot{x})(-1) + c\dot{x}\right) \\
&= s\left(\frac{1}{p}(-\dot{x})^{((1/p)-1)}(f + g(u+d)) + c\dot{x}\right) \\
&\le s\left(\frac{1}{p}\dot{x}^{((1/p)-1)}\right)(-k_1 s - \alpha \operatorname{sgn}(s)) \\
&\le \frac{1}{p}(-\dot{x})^{((1/p)-1)}\left(-k_1 s^2 - \alpha|s|\right).
\end{aligned} \tag{14}$$

Equations (13) and (14) can be represented as

$$\dot{V}(s) = -\frac{1}{p}|\dot{x}|^{((1/p)-1)}\left(k_1 s^2 + \alpha|s|\right). \tag{15}$$

Here, $\dot{V}(s) < 0$ if $|\dot{x}| \ne 0$ and $s \ne 0$.

If $|\dot{x}| = 0$, from (1) and (11), the following equation can be obtained:

$$\ddot{x} = \frac{d\dot{x}}{dt} = -k_1 s - k_2 \operatorname{sgn}(s) + g\cdot d \tag{16}$$

which implies that

$$s\frac{d\dot{x}}{dt}\bigg|_{\dot{x}=0} \le -k_1 s^2 - \alpha|s|. \tag{17}$$

Thus, it is clear that if $|\dot{x}| = 0$ and $s \ne 0$, $\dot{x}$ deviates from zero; that is, the set of points, $|\dot{x}| = 0$ and $s \ne 0$, is not an attractor. In phase space, these points are all points on the horizontal axis that save the origin.

Therefore, we can conclude that s goes to zero. □

Remark 5. Since the proposed controller (11) has a term $|\dot{x}|^{(2-(1/p))}$, it is clear that in order to avoid the singularity problem one should choose the control parameter, p, such that $1/2 \le p < 1$, even though the finite time convergence can be obtained for $0 < p < 1$ in (6).

3. Examples

To show the validity of the proposed method, the simulation results for the following system are given:

$$\ddot{x} = 20\,\dot{x}|\dot{x}| + 10\sin(5t)\sqrt{|x|} + 10(u+d), \tag{18}$$

where $d(x,\dot{x},t) = 2\sqrt{x} + \sin(10t)$. The control parameters were chosen as follows: $c = 5$, $k_1 = 2$, $k_2 = 10(2\sqrt{|x|} + 1 + \alpha)$, and $\alpha = 1$. When the initial condition is $x(0) = 1$, from

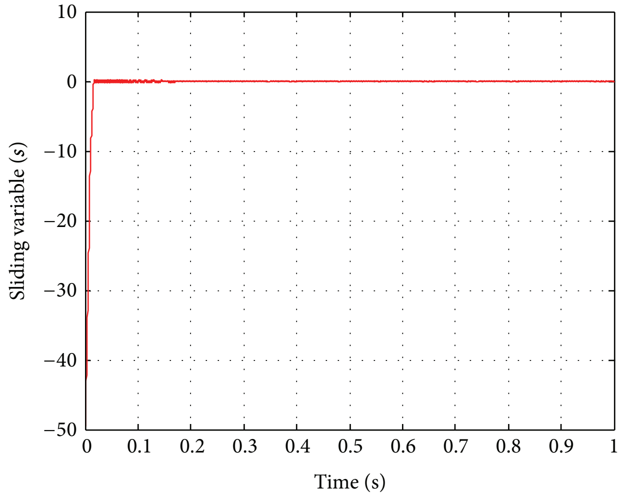

FIGURE 2: Sliding variable (s).

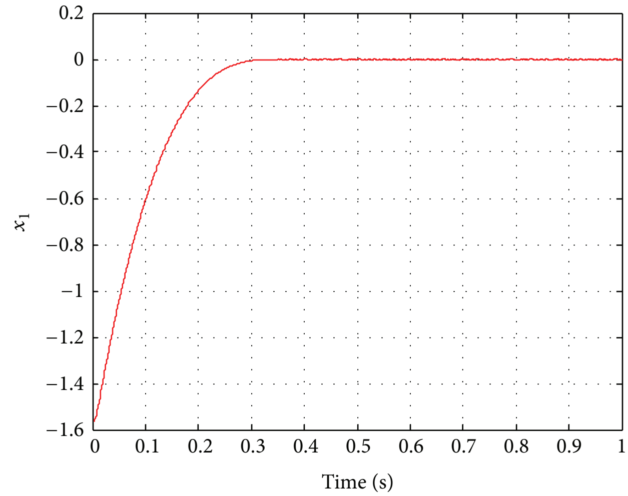

FIGURE 4: Output (x).

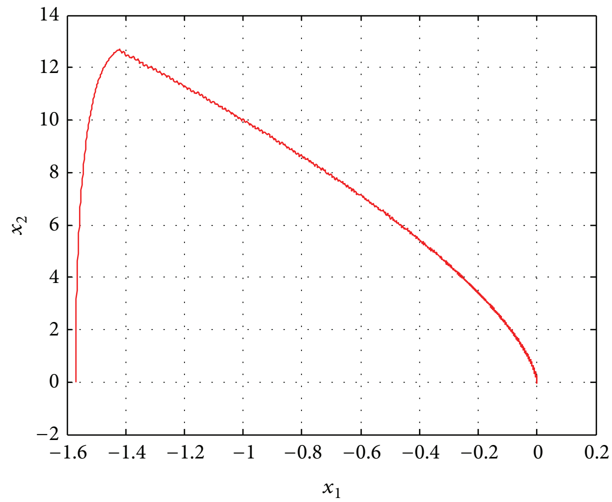

FIGURE 3: Phase portrait (x versus $\dot{x}$).

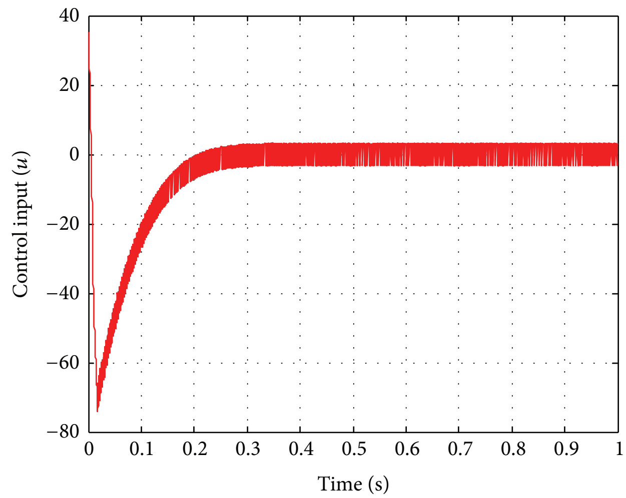

FIGURE 5: Control input (u) for signum function.

(18), $p = 1/2$ is obtained for the minimum relaxation time. Clearly, $p = 1/2$ is out of a conventional nonsingular terminal sliding surface. In the following figures, x_1, x_2 represent $x, \dot{x}$, respectively.

Figures 2–5 show the simulation results. Figure 2 represents the profile of the sliding variable, s. It is shown that the system state stayed on the nonsingular terminal sliding surface all the time after it hit the sliding surface for the first time.

It is clear that the sliding mode existence condition, $s \cdot \dot{s} < 0$, was satisfied all the time and it was also known from the phase portrait given in Figure 3.

Figure 4 shows that the output zeroed in finite time.

The control input signal can be seen in Figure 5.

To show the nonsingular property, we simulated the system with the initial condition near the vertical axis in the phase space because, from (4), a set of singular points for the conventional terminal sliding surface is a vertical axis in the phase space. Figures 6 and 7 show the results. Figure 6 represents that the system state zeroed even though it crossed over the vertical axis, a set of singular points for a conventional terminal sliding surface. In the proof of Theorem 4, we showed that the horizontal axis is not an attractor so that the system state on this axis moves out of the axis. This is also verified from Figure 6, since the system state also crossed the horizontal axis. The control input is shown in Figure 7. It can be seen that it did not suffer from the singularity problem.

In addition, we also applied the proposed method to the actual DC motor system. For the control unit in the experimental system, a TMS320F2812 DSP processor was used. The sampling time was set to 1 msec. The following model was used for the DC motor system:

$$\ddot{x} = -40.65\dot{x} + 46.67u. \quad (19)$$

Figure 8 shows that the motor position converges zero in finite time.

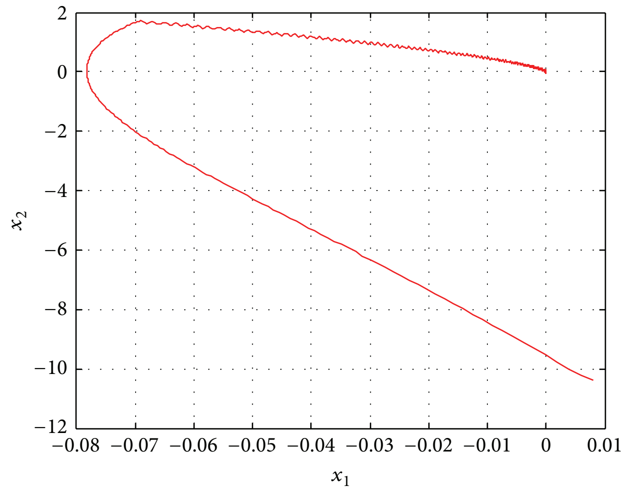

Figure 6: Phase portrait (x versus $\dot{x}$).

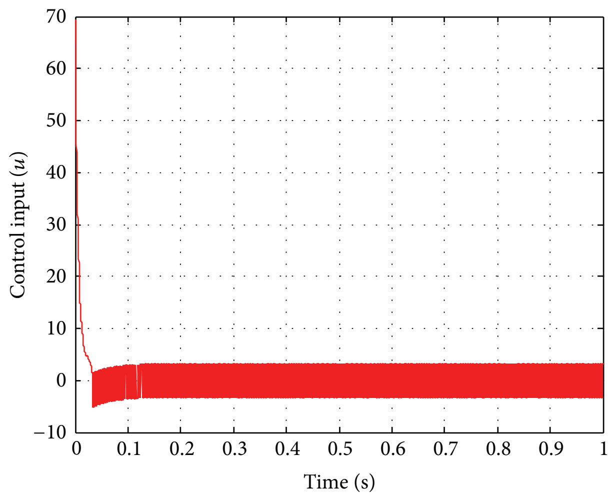

Figure 7: Control input (u).

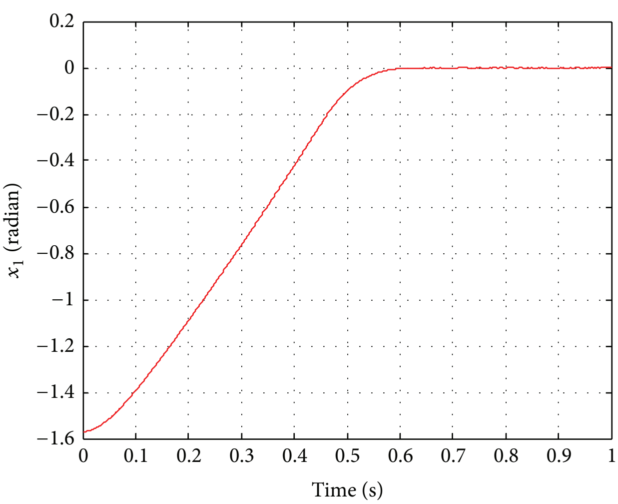

Figure 8: Output angle (x).

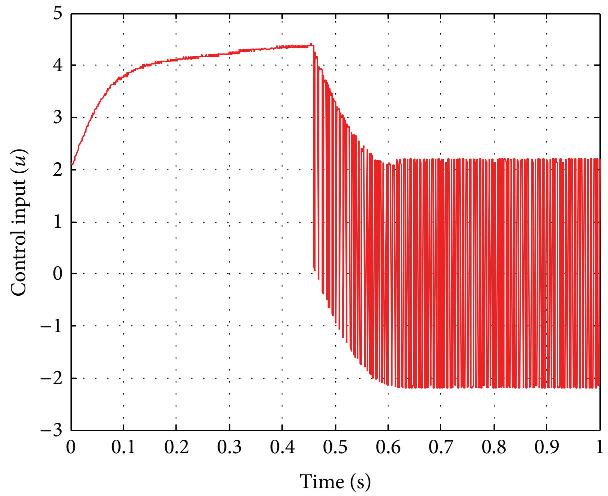

Figure 9: Control input (u).

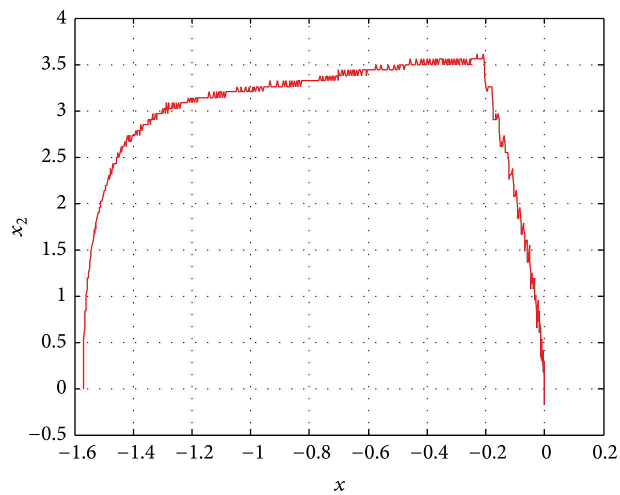

Figure 10: Phase portrait (x versus $\dot{x}$).

The control input signal is given in Figure 9. It represents that the control signal is bounded all the time.

Figure 10 shows a phase portrait. It is clear that the sliding mode existence condition, $s \cdot \dot{s} < 0$, was satisfied all the time.

4. Conclusions

In this paper, the extended nonsingular terminal sliding surface for second-order nonlinear systems has been proposed. It has been shown that the proposed nonsingular terminal sliding surface guarantees finite time convergence and is singularity-free. Furthermore, the exponent of the power function in the proposed sliding surface can be a real number in contrast to conventional nonsingular terminal sliding surfaces where the exponent should be a rational number with an odd numerator and odd denominator. Simulation and experimental results have shown the validity of the main result.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

This work was supported by a Korea University Grant.

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