Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 264049, 4 pages
http://dx.doi.org/10.1155/2014/264049

Letter to the Editor

Comment on "Conservation Laws of Two (2 + 1)-Dimensional Nonlinear Evolution Equations with Higher-Order Mixed Derivatives"

Long Wei and Yang Wang

Department of Mathematics, Hangzhou Dianzi University, Zhejiang 310018, China

Correspondence should be addressed to Yang Wang; yangwang79@126.com

Received 30 October 2013; Accepted 28 November 2013; Published 28 January 2014

Academic Editor: Chaudry M. Khalique

In a recent paper (Zhang (2013)), the author claims that he has proposed two rules to modify Ibragimov's theorem on conservation laws to "ensure the theorem can be applied to nonlinear evolution equations with any mixed derivatives." In this letter, we analysis the paper. Indeed, the so-called "modification rules" are needless and the theorem of Ibragimov can be applied to construct conservation laws directly for nonlinear equations with any mixed derivatives as long as the formal Lagrangian is rewritten in symmetric form. Moreover, the conservation laws obtained by the so-called "modification rules" in the paper under discussion are equivalent to the one obtained by Ibragimov's theorem.

1. Introduction

Noether's famous theorem about symmetries and conservation laws is now almost a century old and has been discussed in literally hundreds of papers and is covered in many textbooks, for example, in [1, 2]. Noether's theorem allows one to construct conservation laws for differential equations following a straightforward algorithm. Although Noether's approach provides an elegant algorithm for finding conservation laws, it possesses a strong limitation: it can only be applied to equations having variational structure. However, a large number of differential equations without variational structure admit conservation laws. Recently, Ibragimov proved a result in [3] which allows one to construct conservation laws for equations without variational structure. Essentially, Ibragimov's theorem is an extension of Noether's theorem by introducing formal Lagrangian to get rid of the variational limitation. There have been abundant papers on constructing conservation laws using Ibragimov's theorem; see [4–9] and references therein.

In [10], to construct conservation laws for some nonlinear equations with higher-order mixed derivatives, the author proposed two so-called "modification rules" to modify Ibragimov's theorem and gave "new" formulae of conservation laws for ANNV equation and KP-BBM equation. The two rules in [10] are uncalled for and illusive; indeed, Ibragimov's theorem can be applied to construct conservation laws directly for nonlinear equations with higher-order mixed derivatives. To do so, we only need to rewrite the corresponding form Lagrangian in symmetric form. Moreover, by direct calculations, we find that the conservation laws obtained by so-called "modification rules" in [10] are equivalent to the one obtained by Ibragimov's theorem.

For simplicity, we only illuminate our statement for the ANNV equation [10, (1)] in the following section.

2. Formulae of Conservation Laws for ANNV Equation and the Equivalence

In this section, we first apply Ibragimov's theorem to construct conservation laws for ANNV equation [10, (1)]

$$u_{yt} + u_{xxxy} - 3u_{xx}u_y - 3u_x u_{xy} = 0. \qquad (1)$$

The form Lagrangian of (1) and the corresponding adjoint equation is

$$L = v\left(u_{yt} + u_{xxxy} - 3u_{xx}u_y - 3u_x u_{xy}\right), \qquad (2)$$

where v is the solution of the adjoint equation. We rewrite L in the symmetric form as follows:

$$\mathscr{L} = v\left[\frac{1}{2}\left(u_{yt}+u_{ty}\right) + \frac{1}{4}\left(u_{xxxy}+u_{xxyx}+u_{xyxx}+u_{yxxx}\right) - 3u_{xx}u_y - \frac{3}{2}u_x\left(u_{xy}+u_{yx}\right)\right]. \tag{3}$$

For any Lie point, Lie-Bäcklund, and nonlocal symmetry of (1),

$$V = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \tau\frac{\partial}{\partial t} + \phi\frac{\partial}{\partial u}; \tag{4}$$

from Ibragimov's theorem, we obtain the general formulae of conservation laws for the system consisting of (1) and the adjoint equation as follows:

$$\begin{aligned} X = {} & \xi\mathscr{L} + W\left[\mathscr{L}_{u_x} - D_x\left(\mathscr{L}_{u_{xx}}\right) - D_y\left(\mathscr{L}_{u_{xy}}\right) - D_{xxy}\left(\mathscr{L}_{u_{xxxy}}\right) - D_{xyx}\left(\mathscr{L}_{u_{xxyx}}\right) - D_{yxx}\left(\mathscr{L}_{u_{xyxx}}\right)\right] \\ & + D_x(W)\left[\mathscr{L}_{u_{xx}} + D_{xy}\left(\mathscr{L}_{u_{xxxy}}\right) + D_{yx}\left(\mathscr{L}_{u_{xxyx}}\right)\right] \\ & + D_y(W)\left[\mathscr{L}_{u_{xy}} + D_{xx}\left(\mathscr{L}_{u_{xyxx}}\right)\right] \\ & + D_{xx}(W)\left[-D_y\left(\mathscr{L}_{u_{xxxy}}\right)\right] + D_{xy}(W) \\ & \times\left[-D_x\left(\mathscr{L}_{u_{xxyx}}\right)\right] + D_{yx}(W)\left[-D_x\left(\mathscr{L}_{u_{xyxx}}\right)\right] \\ & + D_{xxy}(W)\mathscr{L}_{u_{xxxy}} + D_{xyx}(W)\mathscr{L}_{u_{xxyx}} \\ & + D_{yxx}(W)\mathscr{L}_{u_{xyxx}}, \end{aligned} \tag{5}$$

$$\begin{aligned} Y = {} & \eta\mathscr{L} + W\left[\mathscr{L}_{u_y} - D_x\left(\mathscr{L}_{u_{yx}}\right) - D_t\left(\mathscr{L}_{u_{yt}}\right) - D_{xxx}\left(\mathscr{L}_{u_{yxxx}}\right)\right] \\ & + D_x(W)\left[\mathscr{L}_{u_{yx}} + D_{xx}\left(\mathscr{L}_{u_{yxxx}}\right)\right] + D_t(W)\mathscr{L}_{u_{yt}} \\ & + D_{xx}(W)\left[-D_x\left(\mathscr{L}_{u_{yxxx}}\right)\right] + D_{xxx}(W)\mathscr{L}_{u_{yxxx}}, \end{aligned} \tag{6}$$

$$T = \tau\mathscr{L} + W\left(-D_y\mathscr{L}_{u_{ty}}\right) + D_y(W)\mathscr{L}_{u_{ty}}, \tag{7}$$

where $W = \phi - \xi u_x - \eta u_y - \tau u_t$ is the Lie characteristic function, $\mathscr{L}$ is the formal Lagrangian in the symmetric form given by (3), and $\mathscr{L}_w = \partial\mathscr{L}/\partial w$. Thus, the conservation laws for (1) can be derived from above formulae (5), (6), and (7) if Lie symmetries of (1) are known. For example, let us construct the conserved vector corresponding to the generator

$$V_1 = g(y)\frac{\partial}{\partial y} \tag{8}$$

as follows:

$$\begin{aligned} X_1 = {} & \frac{3}{2}g(y)u_yu_{xy}v - 3g(y)u_y^2v_x - \frac{3}{2}g(y)u_xu_yv_y \\ & + \frac{3}{4}g(y)u_yv_{xxy} - \frac{1}{2}g(y)u_{xy}v_{xy} - \frac{1}{4}g'(y)u_yv_{xx} \\ & + \frac{3}{2}g'(y)u_xu_yv - \frac{1}{4}g(y)u_{yy}v_{xx} + \frac{3}{2}g(y)u_xu_{yy}v \\ & + \frac{1}{4}g(y)u_{xxy}v_y + \frac{1}{2}v_xg'(y)u_{xy} + \frac{1}{2}g(y)v_xu_{xyy} \\ & - \frac{3}{4}g'(y)u_{xxy}v - \frac{3}{4}g(y)u_{xxyy}v, \end{aligned} \tag{9}$$

$$\begin{aligned} Y_1 = {} & \frac{1}{4}g(y)\left(2u_{ty}v + 3vu_{xxxy} - 6vu_{xx}u_y - 6vu_xu_{xy} - 6u_xu_yv_x + 2u_yv_t + u_yv_{xxx} - u_{xy}v_{xx} + u_{xxy}v_x\right), \end{aligned} \tag{10}$$

$$T_1 = \frac{1}{2}g(y)u_yv_y - \frac{1}{2}g'(y)u_yv - \frac{1}{2}g(y)u_{yy}v. \tag{11}$$

We should point out here that the conservation laws obtained in [10] $(X_{11j}, Y_{11j}, T_{11j})$, $(X_{2j1}, Y_{2j1}, T_{2j1})$, $(X_{3j1}, Y_{3j1}, T_{3j1})$, $(X_{4j1}, Y_{4j1}, T_{4j1})$ $(j = 1,2,3,4)$ are equivalent to our result (X_1, Y_1, T_1) given above if we transfer the terms $D_y(\cdots)$, $D_x(\cdots)$ from X_1, Y_1 to each other and transfer the terms $D_t(\cdots)$, $D_y(\cdots)$ from Y_1, T_1 to each other, respectively.

To explain the fact, in general, we show the equivalence of the formulae of conservation laws (5), (6), (7), and the ones (X_{ij}, Y_{ij}, T_{ij}) in Theorem 3.1 in [10].

Now let us consider the formulae of conserved vector (X, Y, T) of (1). Observe the facts that in (5)

$$\begin{aligned} & -WD_y\left(\mathscr{L}_{u_{xy}}\right) + D_y(W)\mathscr{L}_{u_{xy}} = D_y\left(W\mathscr{L}_{u_{xy}}\right) - 2WD_y\left(\mathscr{L}_{u_{xy}}\right), \\ & WD_{xxy}\left(\mathscr{L}_{u_{xxxy}}\right) = D_y\left[WD_{xx}\left(\mathscr{L}_{u_{xxxy}}\right)\right] - D_y(W)D_{xx}\left(\mathscr{L}_{u_{xxxy}}\right), \\ & D_x(W)D_{xy}\mathscr{L}_{u_{xxxy}} = D_y\left[D_x(W)D_x\left(\mathscr{L}_{u_{xxxy}}\right)\right] - D_{xy}(W)D_x\left(\mathscr{L}_{u_{xxxy}}\right), \\ & D_{xx}(W)D_y\left(\mathscr{L}_{xxxy}\right) = D_y\left[D_{xx}(W)\mathscr{L}_{u_{xxxy}}\right] - D_{xxy}(W)\mathscr{L}_{u_{xxxy}}, \\ & WD_{xxy}\left(\mathscr{L}_{u_{xxxy}}\right) = WD_{xyx}\left(\mathscr{L}_{u_{xxyx}}\right) = WD_{yxx}\left(\mathscr{L}_{u_{xyxx}}\right), \\ & D_x(W)D_{xy}\mathscr{L}_{u_{xxxy}} = D_x(W)D_{yx}\mathscr{L}_{u_{xxyx}}, \end{aligned} \tag{12}$$

in (6)

$$\begin{aligned}
&-WD_t\left(\mathscr{L}_{u_{yt}}\right)+D_t(W)\mathscr{L}_{u_{yt}}\\
&\quad=D_t\left[W\mathscr{L}_{u_{yt}}\right]-2WD_t\left(\mathscr{L}_{u_{yt}}\right),\\
&WD_x\left(\mathscr{L}_{u_{yx}}\right)=D_x\left(W\mathscr{L}_{u_{yx}}\right)-D_x(W)\mathscr{L}_{u_{yx}},\\
&D_x(W)D_{xx}\left(\mathscr{L}_{u_{yxxx}}\right)=D_x\left[WD_{xx}\left(\mathscr{L}_{u_{yxxx}}\right)\right]\\
&\quad-WD_{xxx}\left(\mathscr{L}_{u_{yxxx}}\right),\\
&D_{xx}(W)D_x\left(\mathscr{L}_{u_{yxxx}}\right)=D_x\left[D_x(W)D_x\left(\mathscr{L}_{u_{yxxx}}\right)\right]\\
&\quad-D_x(W)D_{xx}\left(\mathscr{L}_{u_{yxxx}}\right),\\
&D_{xxx}W\mathscr{L}_{u_{yxxx}}=D_x\left[D_{xx}(W)\mathscr{L}_{u_{yxxx}}\right]\\
&\quad-D_{xx}(W)D_x\left(\mathscr{L}_{u_{yxxx}}\right),
\end{aligned}\tag{13}$$

and in (7)

$$\begin{aligned}
&-WD_y\left(\mathscr{L}_{u_{ty}}\right)+D_y(W)\mathscr{L}_{u_{ty}}\\
&\quad=-D_y\left[W\mathscr{L}_{u_{ty}}\right]+2D_y(W)\mathscr{L}_{u_{ty}}\\
&\quad=D_y\left[W\mathscr{L}_{u_{ty}}\right]-2WD_y\left(\mathscr{L}_{u_{ty}}\right).
\end{aligned}\tag{14}$$

Thus, we have that

$$\begin{aligned}
X&=\xi\mathscr{L}+W\mathscr{L}_{u_x}-WD_x\left(\mathscr{L}_{u_{xx}}\right)-2WD_y\left(\mathscr{L}_{u_{xy}}\right)\\
&\quad+D_x(W)\mathscr{L}_{u_{xx}}-4D_{xy}(W)D_x\left(\mathscr{L}_{u_{xxxy}}\right)\\
&\quad+4D_{xxy}(W)\mathscr{L}_{u_{xxxy}}+4D_y(W)D_{xx}\mathscr{L}_{u_{xxxy}}\\
&\quad+D_y\left[W\mathscr{L}_{u_{xy}}\right]-3D_y\left[WD_{xx}\left(\mathscr{L}_{u_{xxxy}}\right)\right]\\
&\quad+2D_y\left[D_x(W)D_x\left(\mathscr{L}_{u_{xxxy}}\right)\right]\\
&\quad-D_y\left[D_{xx}(W)\mathscr{L}_{u_{xxxy}}\right],\\
Y&=\eta\mathscr{L}+W\mathscr{L}_{u_y}-2WD_t\left(\mathscr{L}_{u_{yt}}\right)\\
&\quad+2D_x(W)\mathscr{L}_{u_{xy}}-4WD_{xxx}\left(\mathscr{L}_{u_{xxxy}}\right)-D_x\left[W\mathscr{L}_{u_{yx}}\right]\\
&\quad+3D_x\left[WD_{xx}\left(\mathscr{L}_{u_{xxxy}}\right)\right]-2D_x\left[D_x(W)D_x\left(\mathscr{L}_{u_{xxxy}}\right)\right]\\
&\quad+D_x\left[D_{xx}(W)\mathscr{L}_{u_{xxxy}}\right]+D_t\left[W\mathscr{L}_{u_{yt}}\right],\\
T&=\tau\mathscr{L}+2D_y(W)\mathscr{L}_{u_{ty}}-D_y\left[W\mathscr{L}_{u_{ty}}\right].
\end{aligned}\tag{15}$$

Therefore, we can deduce the conserved quality as follows:

$$\begin{aligned}
X&=\xi\mathscr{L}+W\mathscr{L}_{u_x}-WD_x\left(\mathscr{L}_{u_{xx}}\right)-2WD_y\left(\mathscr{L}_{u_{xy}}\right)\\
&\quad+D_x(W)\mathscr{L}_{u_{xx}}-4D_{xy}(W)D_x\left(\mathscr{L}_{u_{xxxy}}\right)\\
&\quad+4D_{xxy}(W)\mathscr{L}_{u_{xxxy}}+4D_y(W)D_{xx}\mathscr{L}_{u_{xxxy}},\\
Y&=\eta\mathscr{L}+W\mathscr{L}_{u_y}-2WD_t\left(\mathscr{L}_{u_{yt}}\right)+2D_x(W)\mathscr{L}_{u_{xy}}\\
&\quad-4WD_{xxx}\left(\mathscr{L}_{u_{xxxy}}\right),\\
T&=\tau\mathscr{L}+2D_y(W)\mathscr{L}_{u_{ty}}.
\end{aligned}\tag{16}$$

Rewriting above result in L, we have

$$\begin{aligned}
X&=\xi L+WL_{u_x}-WD_x\left(L_{u_{xx}}\right)-WD_y\left(L_{u_{xy}}\right)\\
&\quad+D_x(W)L_{u_{xx}}-D_{xy}(W)D_x\left(L_{u_{xxxy}}\right)\\
&\quad+D_{xxy}(W)L_{u_{xxxy}}+D_y(W)D_{xx}L_{u_{xxxy}},\\
Y&=\eta L+WL_{u_y}-WD_t\left(L_{u_{yt}}\right)+D_x(W)L_{u_{xy}}\\
&\quad-WD_{xxx}\left(L_{u_{xxxy}}\right),\\
T&=\tau\mathscr{L}+D_y(W)L_{u_{yt}};
\end{aligned}\tag{17}$$

this is the result $(X_{11},Y_{11},T_{11})=(X^1,Y^1,T^1)+(B_1^X,B_1^Y,0)$ in [10].

Similarly, if we rewrite (5), (6), and (7) in the following way:

$$\begin{aligned}
X&=\xi\mathscr{L}+W\mathscr{L}_{u_x}-WD_x\left(\mathscr{L}_{u_{xx}}\right)+D_x(W)\mathscr{L}_{u_{xx}}\\
&\quad-4WD_{xxy}\left(\mathscr{L}_{u_{xxxy}}\right)+2D_y(W)\mathscr{L}_{u_{xy}}\\
&\quad+4D_x(W)D_{xy}\left(\mathscr{L}_{u_{xxxy}}\right)+4D_{xxy}(W)\mathscr{L}_{u_{xxxy}}\\
&\quad-D_y\left[W\mathscr{L}_{u_{xy}}\right]+D_y\left[WD_{xx}\left(\mathscr{L}_{u_{xyxx}}\right)\right]\\
&\quad-D_y\left[D_{xx}(W)\mathscr{L}_{u_{xyxx}}\right]-2D_y\left[D_x(W)D_x\left(\mathscr{L}_{u_{xyxx}}\right)\right],\\
Y&=\eta\mathscr{L}+W\mathscr{L}_{u_y}-2WD_x\left(\mathscr{L}_{u_{yx}}\right)+2D_t(W)\mathscr{L}_{u_{yt}}\\
&\quad-4D_{xx}(W)D_x\left(\mathscr{L}_{u_{yxxx}}\right)+D_x\left[W\mathscr{L}_{u_{yx}}\right]\\
&\quad-D_x\left[WD_{xx}\left(\mathscr{L}_{u_{yxxx}}\right)\right]+D_x\left[D_{xx}(W)\mathscr{L}_{u_{yxxx}}\right]\\
&\quad+2D_x\left[D_x(W)D_x\left(\mathscr{L}_{u_{yxxx}}\right)\right]-D_t\left[W\mathscr{L}_{u_{yt}}\right],\\
T&=\tau\mathscr{L}-2WD_y\left(\mathscr{L}_{u_{ty}}\right)+D_y\left[W\mathscr{L}_{u_{ty}}\right],
\end{aligned}\tag{18}$$

then the conservation law (X, Y, T) can be deduced in L as follows:

$$\begin{aligned} X &= \xi L + WL_{u_x} - WD_x\left(L_{u_{xx}}\right) + D_x(W)L_{u_{xx}} \\ &\quad - WD_{xxy}\left(L_{u_{xxxy}}\right) + D_y(W)L_{u_{xy}} \\ &\quad + D_x(W)D_{xy}\left(L_{u_{xxxy}}\right) + D_{xxy}(W)L_{u_{xxxy}}, \\ Y &= \eta L + WL_{u_y} - WD_x\left(L_{u_{xy}}\right) + D_t(W)L_{u_{yt}} \\ &\quad - D_{xx}(W)D_x\left(L_{u_{xxxy}}\right), \\ T &= \tau L - WD_y\left(L_{u_{yt}}\right); \end{aligned} \tag{19}$$

this is the conservation law $(X_{43}, Y_{43}, T_{43}) = (X^4, Y^4, T^4) + (B_3^X, B_3^Y, 0)$ in [10].

Adopting the same procedure as above, we see that all the conservation laws (X_{ij}, Y_{ij}, T_{ij}) obtained in Theorem 3.1 in [10] can be deduced from our result (X, Y, T) given by (5), (6), and (7); we have checked them and omit the details here.

3. Conclusions

In this letter, we have analyzed the paper [10]. We see that the two so-called "modification rules" in [10] are needless and Ibragimov's theorem can be applied to construct conservation laws directly for nonlinear equations with higher-order mixed derivatives. By direct calculations, we have found that the conservation laws obtained by the so-called "modification rules" in [10] are equivalent to the one obtained by Ibragimov's theorem. Therefore, Ibragimov's theorem on conservation laws need not be modified.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The first author is partly supported by Zhejiang Provincial Natural Science Foundation of China under Grant no. LY12A01003 and subjects research and development foundation of Hangzhou Dianzi University under Grant no. ZX100204004-6. The second author is partly supported by NSFC under Grant no. 11101111.

References

[1] G. W. Bluman and S. Kumei, *Symmetries and differential equations*, Springer, New York, NY, USA, 1989.

[2] P. J. Olver, *Applications of Lie groups to differential equations*, Springer, Berline, Germany, 1986.

[3] N. H. Ibragimov, "A new conservation theorem," *Journal of Mathematical Analysis and Applications*, vol. 333, no. 1, pp. 311–328, 2007.

[4] N. H. Ibragimov, M. Torrisi, and R. Tracinà, "Self-adjointness and conservation laws of a generalized Burgers equation," *Journal of Physics A*, vol. 44, no. 14, article 145201, 2011.

[5] M. L. Gandarias, "Weak self-adjoint differential equations," *Journal of Physics A*, vol. 44, article 262001, 2011.

[6] M. L. Gandarias, M. Redondo, and M. S. Bruzón, "Some weak self-adjoint Hamilton-Jacobi-Bellman equations arising in financial mathematics," *Nonlinear Analysis: Real World Applications*, vol. 13, no. 1, pp. 340–347, 2012.

[7] N. H. Ibragimov, "Nonlinear self-adjointness and conservation laws," *Journal of Physics A: Mathematical and Theoretical*, vol. 44, no. 43, Article ID 432002, 2011.

[8] I. L. Freire, "New classes of nonlinearly self-adjoint evolution equations of third- and fifth-order," *Communications in Nonlinear Science and Numerical Simulation*, vol. 18, no. 3, pp. 493–499, 2013.

[9] Y. Wang and L. Wei, "Self-adjointness, symmetries, and conservation laws for a class of wave equations incorporating dissipation," *Abstract and Applied Analysis*, vol. 2013, Article ID 407908, 6 pages, 2013.

[10] L.-H. Zhang, "Conservation laws of two (2+1)-dimensional nonlinear evolution equations with higher-order mixed derivatives," *Abstract and Applied Analysis*, vol. 2013, Article ID 594975, 10 pages, 2013.