Research Article

Schur m-Power Convexity of a Class of Multiplicatively Convex Functions and Applications

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We investigate the conditions under which the symmetric functions $F_{n,k}(\mathbf{x}, r) = \prod_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} f\left((\sum_{j=1}^{k} x_{i_j}^r)^{1/r} \right)$, $k = 1, 2, \ldots, n$, are Schur m-power convex for $x \in \mathbb{R}_{++}^n$ and $r > 0$. As a consequence, we prove that these functions are Schur geometrically convex and Schur harmonically convex, which generalizes some known results. By applying the theory of majorization, several inequalities involving the pth power mean and the arithmetic, the geometric, or the harmonic means are presented.

1. Introduction

Let $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}, i = 1, 2, \ldots, n\}$ and $\mathbb{R}_{++}^n = \{(x_1, x_2, \ldots, x_n) : x_i > 0, i = 1, 2, \ldots, n\}$. In particular, $\mathbb{R}_{++} = (0, \infty)$. For $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, we denote by

$$
\begin{aligned}
\mathbf{x} &= (x_1, x_2, \ldots, x_n), \\
\ln(\mathbf{x}) &= (\ln x_1, \ln x_2, \ldots, \ln x_n), \\
\frac{1}{\mathbf{x}} &= \left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n} \right).
\end{aligned}
\tag{1}
$$

The Hamy symmetric function [1, 2] is defined as

$$
F_n(\mathbf{x}, r) = F_n(x_1, x_2, \ldots, x_n; r) = \sum_{1 \leq i_1 < i_2 < \cdots < i_r \leq n} \left(\prod_{j=1}^{r} x_{i_j} \right)^{1/r},
$$
$$
r = 1, 2, \ldots, n.
\tag{2}
$$

The properties and applications of Hamy symmetric function can be found in the book of Bullen et al. [1].

Throughout the paper, let $I \subset \mathbb{R}_{++}$ and $I^n \subset \mathbb{R}_{++}^n$.

In 2007, Guan [3] defined a more general symmetric function: let $f : I^n \to \mathbb{R}_{++}$, and

$$
\sum_{n} (f(x)) = \sum_{1 \leq i_1 < i_2 < \cdots < i_r \leq n} f\left(\prod_{j=1}^{r} x_{i_j}^{1/r} \right),
$$
$$
r = 1, 2, \ldots, n,
\tag{3}
$$

where $i_1, i_2, \ldots, i_n$ are nonnegative integers, $r \in N = \{1, 2, 3, \ldots\}$. Guan proved that the above symmetric function is Schur geometrically convex on I^n.

In 2010, Rovenţa [4] defined the following symmetric function. Let $f : I \to \mathbb{R}_{++}$ be a log-convex function

$$
F_k(\mathbf{x}) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \prod_{j=1}^{k} f(x_{i_j}),
$$
$$
k = 1, 2, \ldots, n.
\tag{4}
$$

Rovenţa proved that (4) is a Schur convex function on I^n.

In 2010, Meng et al. [5] proved the dual form of the Hamy symmetric function

$$H_n (\mathbf{x}, r) = \prod_{1 \le i_1 < i_2 < \cdots < i_k \le n} \left(\sum_{j=1}^{r} x_{i_j}^{1/r} \right),$$ (5)

$$k = 1, 2, \ldots, n,$$

was Schur harmonically convex in $\mathbb{R}_{++}^n$.

In 2013, Shi and Zhang [6] investigated the following dual form of $F_k(\mathbf{x})$:

$$F_k^* (\mathbf{x}) = \prod_{1 \le i_1 < i_2 < \cdots < i_k \le n} \sum_{j=1}^{k} f\left(x_{i_j} \right), \quad k = 1, 2, \ldots, n.$$ (6)

They proved that $F_k^*(\mathbf{x})$ is Schur convex, Schur geometrically and harmonically convex on I^n.

Recently, Yang [7–9] generalized the notion of Schur convexity to Schur m-power convexity, which contains the Schur convexity, Schur geometrical convexity, and Schur harmonic convexity. Moreover, he discussed Schur m-power convexity of Stolarsky means [7], Gini means [8], and Daróczy means [9]. Wang and Yang showed that generalized Hamy symmetric function [10] and a class of symmetric functions [11] are Schur m-power convex.

Now we define the more general dual form of symmetric function.

Definition 1. Let $\Omega \subset \mathbb{R}_{++}^n$ be a symmetric convex set with nonempty interior and $f : \Omega \rightarrow \mathbb{R}_{++}$ is continuous on Ω and differentiable in the interior of Ω. For $r > 0$, define the symmetric functions $F_{n,k}(\mathbf{x}, r)$ by

$$F_{n,k} (\mathbf{x}, r) = \prod_{1 \le i_1 < i_2 < \cdots < i_k \le n} f\left(\left(\sum_{j=1}^{k} x_{i_j}^r \right)^{1/r} \right),$$ (7)

$$k = 1, 2, \ldots, n.$$

In this paper, we investigate the Schur m-power convexity of the above more general dual form of symmetric functions. In particular, we obtain that the above more general dual form of symmetric functions is Schur geometrically convex and Schur harmonically convex, which generalizes some known results. As a consequence, we are able to prove a number of new inequalities concerning the pth power mean, the arithmetic mean, and the geometric and the harmonic mean.

2. Definitions and Lemmas

We first recall several definitions as follows.

Definition 2 (see [12, 13]). Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$.

(1) $\mathbf{x} \ge \mathbf{y}$ means $x_i \ge y_i$ for all $i = 1, 2, \ldots, n$.

(2) Let $\Omega \subset \mathbb{R}^n$, $f : \Omega \rightarrow \mathbb{R}$ is said to be increasing if $\mathbf{x} \ge \mathbf{y}$ implies $f(\mathbf{x}) \ge f(\mathbf{y})$. f is said to be decreasing if and only if $-f$ is increasing.

Definition 3 (see [12, 13]). Suppose that $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ are two n-tuples real numbers.

(1) $\mathbf{y}$ majorizes $\mathbf{x}$ (in symbols $\mathbf{x} \prec \mathbf{y}$), if $\sum_{i=1}^{k} x_{[i]} \le \sum_{i=1}^{k} y_{[i]}$, $(k = 1, 2, \ldots, n-1)$ and $\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}$, where $x_{[1]} \ge x_{[2]} \ge \cdots \ge x_{[n]}$, $y_{[1]} \ge y_{[2]} \ge \cdots \ge y_{[n]}$ are rearrangements of $\mathbf{x}$ and $\mathbf{y}$ in a descending order.

(2) A real-valued function $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be Schur convex on Ω if

$$\mathbf{x} \prec \mathbf{y} \quad \text{on } \Omega \Longrightarrow f(\mathbf{x}) \le f(\mathbf{y}).$$ (8)

f is a Schur concave function on Ω if and only if $-f$ is a Schur convex function.

Definition 4 (see [14]). Suppose $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ are two n-tuples real numbers. Let $\Omega \subset \mathbb{R}_{++}^n$. A function $f : \Omega \rightarrow \mathbb{R}_{++}$ is called Schur geometrically convex if

$$\ln \mathbf{x} \prec \ln \mathbf{y} \quad \text{on } \Omega \Longrightarrow \varphi(\mathbf{x}) \le \varphi(\mathbf{y}).$$ (9)

f is Schur geometrically concave if $-f$ is Schur geometrically convex.

The following Theorem is basic and plays an important role in the theory of the Schur geometrically convex function.

Lemma 5 (see [14]). *Let $\varphi(\mathbf{x}) = \varphi(x_1, x_2, \ldots, x_n)$ be symmetric and continuous on $\Omega \subset \mathbb{R}_{++}^n$ and differentiable in Ω^0. Then $\varphi : \Omega \rightarrow \mathbb{R}_{++}$ is Schur geometrically convex (Schur geometrically concave) if and only if*

$$(\ln x_1 - \ln x_2) \left(x_1 \frac{\partial \varphi}{\partial x_1} - x_2 \frac{\partial \varphi}{\partial x_2} \right) \ge 0 \, (\le 0).$$ (10)

Definition 6 (see [15, 16]). Let $\Omega \in \mathbb{R}^n$.

(1) A set Ω is called harmonically convex if $\mathbf{xy}/(\lambda \mathbf{x} + (1 - \lambda \mathbf{y})) \in \Omega$ for every $\mathbf{x}, \mathbf{y} \in \Omega$ and $\lambda \in [0, 1]$, where $\mathbf{xy} = \sum_{i=1}^{n} x_i y_i$ and $1/\mathbf{x} = (1/x_1, \ldots, 1/x_n)$.

(2) A function $\varphi : \Omega \rightarrow \mathbb{R}_{++}$ is called Schur harmonically convex on Ω if $1/\mathbf{x} \prec 1/\mathbf{y}$ implies $\varphi(\mathbf{x}) \le \varphi(\mathbf{y})$. f is Schur harmonically concave if $-f$ is Schur harmonically convex.

Lemma 7 (see [15, 16]). *Let $\Omega \in \mathbb{R}_{++}^n$ be a symmetric and harmonically convex set with inner points and let $\varphi : \Omega \rightarrow \mathbb{R}_{++}$ be a continuously symmetric function which is differentiable in Ω^0. Then φ is Schur harmonically convex (Schur harmonically concave) on Ω if and only if*

$$(x_1 - x_2) \left(x_1^2 \frac{\partial \varphi}{\partial x_1} - x_2^2 \frac{\partial \varphi}{\partial x_2} \right) \ge 0 \, (\le 0).$$ (11)

Schur convex, Schur geometrically convex, and Schur harmonically convex were introduced by Marshall et al. [13], Zhang [14], and Chu and Sun [15], respectively, and played a key role in analytic inequalities [1–36]. Moreover, the theory of convex functions and Schur convex functions is one of the most important research fields in modern analysis and geometry.

Recently, Yang presents the Schur f-convexity in [7] as follows.

Definition 8 (see [7–9]). Let $\Omega \subseteq \mathbb{R}^n$ be a set with nonempty interior and f a strictly monotone function defined on Ω. Let

$$f(\mathbf{x}) = (f(x_1), f(x_2), \ldots, f(x_n)),$$
$$f(\mathbf{y}) = (f(y_1), f(y_2), \ldots, f(y_n)). \tag{12}$$

Then function $\varphi : \Omega \to \mathbb{R}$ is said to be Schur f-convex on Ω if $f(\mathbf{x}) \prec f(\mathbf{y})$ on Ω implies $\psi(\mathbf{x}) \leq \psi(\mathbf{y})$.

ψ is said to be Schur f-concave if $-\psi$ is Schur f-convex.

Take $f(x) = x, \ln x, x^{-1}$ in Definition 8, it yields the Schur convexity, Schur geometrical convexity, and Schur harmonic convexity. It is clear that the Schur f-convexity is a generalization of the Schur convexity mentioned above. In general, we have the following.

Definition 9 (see [7–9]). Let $f : \mathbb{R}_{++} \to \mathbb{R}$ be defined by $f(x) = (x^m - 1)/m$ if $m \neq 0$ and $f(x) = \ln x$ if $m = 0$. Then function $\psi : \Omega \subseteq \mathbb{R}_{++}^n \to \mathbb{R}$ is said to be Schur m-power convex on Ω if $f(\mathbf{x}) \prec f(\mathbf{y})$ on Ω implies $\psi(\mathbf{x}) \leq \psi(\mathbf{y})$.

ψ is said to be Schur m-power concave if $-\psi$ is Schur m-power convex.

Lemma 10 (see [7–9]). *Let $\psi : \Omega \subseteq \mathbb{R}_{++}^n \to \mathbb{R}$ be continuous on Ω and differentiable in Ω^0. Then ψ is schur m-power convex (Schur m-power concave) on Ω if and only if ψ is symmetric on Ω and*

$$\frac{x_1^m - x_2^m}{m} \left(x_1^{1-m} \frac{\partial \varphi}{\partial x_1} - x_2^{1-m} \frac{\partial \varphi}{\partial x_2} \right)$$
$$\geq 0 \ (\leq 0), \quad \text{if } m \neq 0,$$
$$(\ln x_1 - \ln x_2) \left(x_1 \frac{\partial \varphi}{\partial x_1} - x_2 \frac{\partial \varphi}{\partial x_2} \right)$$
$$\geq 0 \ (\leq 0), \quad \text{if } m \neq 0, \tag{13}$$

hold for any $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \Omega^0$ with $x_1 \neq x_2$, where $\Omega \subseteq \mathbb{R}_{++}^n$ is a symmetric set with nonempty interior Ω^0.

The following lemma is clearly due to the monotonicity property of the function x^p on $\mathbb{R}_{++}$.

Lemma 11 (see [10]). *For $x_1, x_2 > 0$ with $x_1 \neq x_2$, let U be defined by*

$$U(p; x_1, x_2) := \begin{cases} \dfrac{x_1^p - x_2^p}{p(x_1 - x_2)}, & \text{if } p \neq 0, \\ \dfrac{\ln x_1 - \ln x_2}{x_1 - x_2}, & \text{if } p = 0. \end{cases} \tag{14}$$

Then sgn $((x_1^p - x_2^p)/p(x_1 - x_2)) = 1$, (i.e., $U(p; x_1, x_2) > 0$.)

Remark 12 (see [10]). By Lemma 11, we see that

$$\operatorname{sgn}\left(\frac{x_1^p - x_2^p}{p} \right) = \operatorname{sgn}(x_1 - x_2) \quad \text{if } p \neq 0,$$
$$\operatorname{sgn}(\ln x_1 - \ln x_2) = \operatorname{sgn}(x_1 - x_2). \tag{15}$$

Then the two discrimination inequalities in Lemma 10 are equivalent to

$$(x_1 - x_2)\left(x_1^{1-m} \frac{\partial \varphi}{\partial x_1} - x_2^{1-m} \frac{\partial \varphi}{\partial x_2} \right) \geq (\leq) 0. \tag{16}$$

Definition 13 (see [3, 17]). Function $f : I \subset R_{++} \to R_{++}$ is said to be multipicatively convex if

$$x, y \in I, \quad \mu \in [0, 1] \implies f\left(x^{1-\mu} y^\mu \right) \leq f(x)^{1-\mu} f(y)^\mu. \tag{17}$$

The following results have been proven, respectively.

Lemma 14 (see [17]). *A continuous function $f : I \subset R_{++} \to R_{++}$ is multiplicatively convex if and only if*

$$x, y \in I \implies f\left(\sqrt{xy} \right) \leq \sqrt{f(x) f(y)}, \tag{18}$$

or

$$x_1, \ldots, x_n \in I \implies f\left(\sqrt[n]{x_1, \ldots, x_n} \right) \leq \sqrt[n]{f(x_1) \cdots f(x_n)}. \tag{19}$$

Lemma 15 (see [17]). *Assume that $f : I \subset R_{++} \to R_{++}$ is a differential function. Then the following assertions are equivalent.*

(i) *f is multiplicatively convex.*

(ii) *The function $x f'(x)/f(x)$ is nondecreasing.*

Moreover, if f is twice differentiable, then f is multiplicatively convex if and only if

$$x \left[f(x) f''(x) - f'^2(x) \right] + f(x) f'(x) \geq 0 \quad \forall x > 0. \tag{20}$$

3. Main Results and Proof

Our main results are stated as follows.

Theorem 16. *If f is increasing and multiplicatively convex, then for $m \leq 0$ and $r > 0$, $F_{n,k}(\mathbf{x}, r)$ defined in (7) are Schur m-power convex on Ω, where $k = 1, 2, \ldots, n$.*

Take $m = 0, -1$ in Theorem 16, we get the following Corollaries.

Corollary 17. *If f is increasing and multiplicatively convex, then for any $k = 1, 2, \ldots, n$ and $r > 0$, $F_{n,k}(\mathbf{x}, r)$ defined in (7) are Schur geometrically convex on Ω.*

Corollary 18. *If f is increasing and multiplicatively convex, then for any $k = 1, 2, \ldots, n$ and $r > 0$, $F_{n,k}(\mathbf{x}, r)$ defined in (7) are Schur harmonically convex function on Ω.*

Theorem 19. *Let $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}_{++}^n$, and $\sum_{i=1}^n x_i = s$. If f is increasing and multiplicatively convex, then for any $k = 1, 2, \ldots, n$ and $r > 0$, one has*

$$F_{n,k}\left(\frac{n-\mu}{s-\mu\mathbf{x}}, r\right) \le F_{n,k}\left(\frac{1}{\mathbf{x}}, r\right), \quad 0 \le \mu \le 1;$$

$$F_{n,k}\left(\frac{n+\mu}{s+\mu\mathbf{x}}, r\right) \le F_{n,k}\left(\frac{1}{\mathbf{x}}, r\right), \quad 0 \le \mu \le 1. \tag{21}$$

To prove the main results, we first establish some lemmas.

Lemma 20. *Let the function $f : I \subset \mathbb{R}_{++} \to \mathbb{R}_{++}$ be continuous on Ω and differentiable in the interior of Ω. For $m \le 0$, if f is increasing and multiplicatively convex, then $x^{1-m} f'(x)/f(x)$ is increasing.*

Proof. Since f is multiplicatively convex, and by using Lemma 15, we can easily see that $xf'(x)/f(x)$ is increasing. Further, by applying $f(x) \ge 0$ and the monotonicity of f, it follows that $x^{1-m} f'(x)/f(x)$ is also increasing for $x > 0$ and $m \le 0$. □

Lemma 21. *If $r > 0$, $c > 0$ and $m \le 0$, the function $g(x) = (x^r - c)^{(r-m)/r}/x^r$ is increasing.*

Proof. We can easily derive that

$$g'(x) = \frac{1}{x^{2r}}\left[\frac{r-m}{r}(x^r - c)^{-m/r}\right.$$

$$\left. \cdot rx^{r-1} \cdot x^r - rx^{r-1}(x^r - c)^{(r-m)/r}\right] \tag{22}$$

$$= \frac{1}{x^{2r}} rx^{r-1}(x^r - c)^{-m/r}\left[-\frac{m}{r}x^r + c\right] \ge 0.$$

So the function $g(x)$ is increasing. □

Lemma 22. *Let $f : \Omega \subset \mathbb{R}_{++}^n \to \mathbb{R}_{++}$ be continuous on Ω and differentiable in the interior of Ω. For $m \le 0$ and $r > 0$, if f is increasing and multiplicatively convex, then*

$$I = F_{n,2}(\mathbf{x}, r)(x_1 - x_2)(x_1^{r-m} - x_2^{r-m})$$

$$\times \frac{f'\left[(x_1^r + x_2^r)^{1/r}\right]}{f\left[(x_1^r + x_2^r)^{1/r}\right]}(x_1^r + x_2^r)^{1/r-1} \ge 0, \tag{23}$$

$$II = F_{n,2}(\mathbf{x}, r)(x_1 - x_2)$$

$$\times \sum_{j=3}^n \left\{ \frac{f'\left[(x_1^r + x_j^r)^{1/r}\right]}{f\left[(x_1^r + x_j^r)^{1/r}\right]} \cdot (x_1^r + x_j^r)^{1/r-1} \cdot x_1^{r-m} \right.$$

$$- \frac{f'\left[(x_2^r + x_j^r)^{1/r}\right]}{f\left[(x_2^r + x_j^r)^{1/r}\right]}$$

$$\left. \cdot (x_2^{r-m} + x_j^r)^{1/r-1} \cdot x_2^r \right\} \ge 0. \tag{24}$$

Proof. Firstly, we prove that (23) holds. Since $m \le 0$ and $r > 0$, then

$$(x_1 - x_2)(x_1^{r-m} - x_2^{r-m}) \ge 0. \tag{25}$$

So, we deduce that $I \ge 0$.

Secondly, we prove that (24) holds. Set $u = (x_1^r + x_j^r)^{1/r}$, $v = (x_2^r + x_j^r)^{1/r}$, obviously, $u, v \in \Omega$. One can easily find that

$$II = F_{n,2}(\mathbf{x}, r)\frac{x_1 - x_2}{u - v}(u - v)$$

$$\times \sum_{j=3}^n \left\{ \frac{uf'(u)}{f(u)} \cdot \frac{(u^r - x_j^r)^{(r-m)/r}}{u^r} \right.$$

$$\left. - \frac{vf'(v)}{f(v)} \cdot \frac{(u^r - x_j^r)^{(r-m)/r}}{v^r} \right\}. \tag{26}$$

Because f is multiplicatively convex, and by Lemma 15, we get

$$(u - v)\left(\frac{uf'(u)}{f(u)} - \frac{vf'(v)}{f(v)}\right) \ge 0. \tag{27}$$

On the other hand, for $r > 0$ and $m \le 0$, we easily know that the functions x^r and $(x^r + x_j^r)^{1/r}$ are increasing about x. By applying Lemma 21, we have

$$(x_1 - x_2)(x_1^r - x_2^r) \ge 0;$$

$$\frac{x_1 - x_2}{u - v} = \frac{(x_1 - x_2)^2}{(x_1 - x_2)(u - v)} \ge 0;$$

$$(u - v)\left\{ \frac{uf'(u)}{f(u)} \cdot \frac{(u^r - x_j^r)^{(r-m)/r}}{u^r} \right. \tag{28}$$

$$\left. - \frac{vf'(v)}{f(v)} \cdot \frac{(u^r - x_j^r)^{(r-m)/r}}{v^r} \right\} \ge 0.$$

Because the function f is increasing and $f \ge 0$, and applying (27)-(28), we obtain $II \ge 0$. □

Lemma 23. *Let $f : \Omega \subset \mathbb{R}_{++}^n \to \mathbb{R}_{++}$ be continuous on Ω and differentiable in the interior of Ω. If f is increasing and multiplicatively convex, then for $m \le 0$ and $r > 0$, one has*

$$III = F_{n,k}(\mathbf{x}, r)(x_1 - x_2)(x_1^{r-m} - x_2^{r-m})$$

$$\times \sum_{2 \le i_1 < i_2 < \cdots < i_{k-2} \le n} \frac{f'\left[\left(x_1^r + \sum_{j=1}^{k-1} x_{i_j}^r\right)^{1/r}\right]}{f\left[\left(x_1^r + \sum_{j=1}^{k-1} x_{i_j}^r\right)^{1/r}\right]} \ge 0,$$

$$IV = F_{n,k}(\mathbf{x}, r)(x_1 - x_2)$$

$$\times \sum_{3 \le i_1 < i_2 < \ldots < i_{k-1} \le n} \left\{ \frac{f'\left[\left(x_1^r + \sum_{j=1}^{k-1} x_{i_j}^r\right)^{1/r}\right]}{f\left[\left(x_1^r + \sum_{j=1}^{k-1} x_{i_j}^r\right)^{1/r}\right]} \right.$$

$$\cdot \left(x_1^r + \sum_{j=1}^{k-1} x_{i_j}^r\right)^{1/r-1}$$

$$\cdot x_1^{r-m} - \frac{f'\left[\left(x_2^r + \sum_{j=1}^{k-1} x_{i_j}^r\right)^{1/r}\right]}{f\left[\left(x_2^r + \sum_{j=1}^{k-1} x_{i_j}^r\right)^{1/r}\right]}$$

$$\left. \cdot \left(x_2^r + \sum_{j=1}^{k-1} x_{i_j}^r\right)^{1/r-1} \cdot x_2^{r-m} \right\} \ge 0.$$

$$(29)$$

Proof. Set $u = (x_1^r + \sum_{j=1}^{k-1} x_{i_j}^r)^{1/r}$, $v = (x_1^r + \sum_{j=1}^{k-1} x_{i_j}^r)^{1/r}$; it is easy that $u, v \in \Omega$. Then

$$IV = F_{n,k}(\mathbf{x}, r) \frac{x_1 - x_2}{u - v}(u - v)$$

$$\times \sum_{3 \le i_1 < i_2 < \cdots < i_{k-1} \le n} \left\{ \frac{uf'(u)}{f(u)} \cdot \frac{(u^r - w)^{(r-m)/r}}{u^r} \right.$$

$$\left. - \frac{vf'(v)}{f(v)} \cdot \frac{(v^r - w)^{(r-m)/r}}{v^r} \right\},$$

$$(30)$$

where $w = \sum_{j=1}^{k-1} x_{i_j}^r$.

By (27)-(28) and the monotonicity property of the non-negative function f, we get that $III \ge 0, IV \ge 0$. $\square$

Proof of Theorem 16. By Lemma 10 and Remark 12, we only need to prove that

$$(x_1 - x_2)\left(x_1^{1-m} \frac{\partial F_{n,k}(\mathbf{x}, r)}{x_1} - x_2^{1-m} \frac{\partial F_{n,k}(\mathbf{x}, r)}{x_2}\right) \ge 0. \quad (31)$$

To prove the above inequality, we consider the following three cases for k.

Case 1. For $k = 1$. It is clear that $F_{n,1}(\mathbf{x}, r) = \prod_{i=1}^n f(x_i)$. From (31), it follows that

$$\Delta_1 = (x_1 - x_2)\left(x_1^{1-m} \frac{\partial F_{n,1}(\mathbf{x}, r)}{x_1} - x_2^{1-m} \frac{\partial F_{n,1}(\mathbf{x}, r)}{x_2}\right)$$

$$= (x_1 - x_2)\left(x_1^{1-m} \frac{f'(x_1)}{f(x_1)} - x_2^{1-m} \frac{f'(x_2)}{f(x_2)}\right) \prod_{i=1}^n f(x_i).$$

$$(32)$$

By Lemma 20, and $f(x) \ge 0$, it follows that $\Delta_1 \ge 0$.

Case 2. For $k = 2, r > 0$, we have $F_{n,2}(\mathbf{x}, r) = \prod_{1 \le i < j \le n} f[(x_i^r + x_j^r)^{1/r}]$. We can easily derive that

$$\ln F_{n,2}(\mathbf{x}, r) = \sum_{1 \le i < j \le n} \ln f\left[(x_i^r + x_j^r)^{1/r}\right]$$

$$= \sum_{j=2}^n \ln f\left[\left(x_1^r + x_j^r\right)^{1/r}\right] \quad (33)$$

$$+ \sum_{2 \le i < j \le n} \ln f\left[\left(x_i^r + x_j^r\right)^{1/r}\right].$$

By differentiating the above equation with respect to x_1, we obtain

$$\frac{\partial F_{n,2}(\mathbf{x}, r)}{x_1} = F_{n,2}(\mathbf{x}, r) \cdot \frac{\partial \ln F_{n,2}(\mathbf{x}, r)}{x_1}$$

$$= F_{n,2}(\mathbf{x}, r) \cdot \left\{ \sum_{j=2}^n \frac{f'\left[\left(x_1^r + x_j^r\right)^{1/r}\right]}{f\left[\left(x_1^r + x_j^r\right)^{1/r}\right]} \right.$$

$$\left. \cdot \left(x_1^r + x_j^r\right)^{1/r-1} \cdot x_1^{r-1} \right\}$$

$$= F_{n,2}(\mathbf{x}, r) \left\{ \frac{f'\left[\left(x_1^r + x_2^r\right)^{1/r}\right]}{f\left[\left(x_1^r + x_2^r\right)^{1/r}\right]} \right. \quad (34)$$

$$\cdot \left(x_1^r + x_2^r\right)^{1/r-1} \cdot x_1^{r-1}$$

$$+ \sum_{j=3}^n \frac{f'\left[\left(x_1^r + x_j^r\right)^{1/r}\right]}{f\left[\left(x_1^r + x_j^r\right)^{1/r}\right]}$$

$$\left. \cdot \left(x_1^r + x_j^r\right)^{1/r-1} \cdot x_1^{r-1} \right\}.$$

Similarly, we have

$$\frac{\partial F_{n,2}(\mathbf{x},r)}{x_2} = F_{n,2}(\mathbf{x},r)$$

$$\times \left\{ \frac{f'\left[\left(x_1^r + x_2^r\right)^{1/r}\right]}{f\left[\left(x_1^r + x_2^r\right)^{1/r}\right]} \cdot \left(x_1^r + x_2^r\right)^{1/r-1} \cdot x_2^{r-1} \right.$$

$$+ \sum_{j=3}^{n} \frac{f'\left[\left(x_2^r + x_j^r\right)^{1/r}\right]}{f\left[\left(x_2^r + x_j^r\right)^{1/r}\right]}$$

$$\left. \cdot \left(x_2^r + x_j^r\right)^{1/r-1} \cdot x_2^{r-1} \right\} .$$

$$(35)$$

So, from (34) and (35), and by applying Lemma 22, we have

$$\Delta_2 = (x_1 - x_2)\left(x_1^{1-m}\frac{\partial F_{n,2}(\mathbf{x},r)}{x_1} - x_2^{1-m}\frac{\partial F_{n,2}(\mathbf{x},r)}{x_2}\right)$$

$$= F_{n,2}(\mathbf{x},r)(x_1 - x_2)\left(x_1^{r-m} - x_2^{r-m}\right)\frac{f'\left[\left(x_1^r + x_2^r\right)^{1/r}\right]}{f\left[\left(x_1^r + x_2^r\right)^{1/r}\right]}$$

$$\times \left(x_1^r + x_2^r\right)^{1/r-1} + F_{n,2}(\mathbf{x},r)(x_1 - x_2)$$

$$\cdot \sum_{j=3}^{n} \left\{ \frac{f'\left[\left(x_1^r + x_j^r\right)^{1/r}\right]}{f\left[\left(x_1^r + x_j^r\right)^{1/r}\right]} \cdot \left(x_1^r + x_j^r\right)^{1/r-1} \cdot x_1^{r-m} \right.$$

$$\left. - \frac{f'\left[\left(x_2^r + x_j^r\right)^{1/r}\right]}{f\left[\left(x_2^r + x_j^r\right)^{1/r}\right]} \cdot \left(x_2^r + x_j^r\right)^{1/r-1} \cdot x_2^{r-m} \right\}$$

$$= I + II \geq 0.$$

$$(36)$$

So we get that $\Delta_2 \geq 0$.

Case 3. For $3 \leq k \leq n, r > 0$, similarly to the discussion of Case 2, we have

$$\ln F_{n,2}(\mathbf{x},r) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \ln f\left[\left(\sum_{j=1}^{k} x_{i_j}^r\right)^{1/r}\right]$$

$$= \sum_{2 \leq i_1 < i_2 < \cdots < i_{k-1} \leq n} \ln f\left[\left(x_1^r + \sum_{j=1}^{k-1} x_{i_j}^r\right)^{1/r}\right] \quad (37)$$

$$+ \sum_{2 \leq i_1 < i_2 < \cdots < i_k \leq n} \ln f\left[\left(\sum_{j=1}^{k} x_{i_j}^r\right)^{1/r}\right].$$

By differentiating to the above with respect to x_1, we have

$$\frac{\partial F_{n,k}(\mathbf{x},r)}{x_1}$$

$$= F_{n,k}(\mathbf{x},r) \cdot \frac{\partial \ln F_{n,k}(\mathbf{x},r)}{x_1}$$

$$= F_{n,k}(\mathbf{x},r) \cdot \sum_{2 \leq i_1 < i_2 < \cdots < i_{k-1} \leq n} \frac{f'\left[\left(x_1^r + \sum_{j=1}^{k-1} x_{i_j}^r\right)^{1/r}\right]}{f\left[\left(x_1^r + \sum_{j=1}^{k-1} x_{i_j}^r\right)^{1/r}\right]}$$

$$\cdot \left(x_1^r + \sum_{j=1}^{k-1} x_{i_j}^r\right)^{1/r-1} \cdot x_1^{r-1}$$

$$= F_{n,k}(\mathbf{x},r)\left\{ \sum_{3 \leq i_1 < i_2 < \cdots < i_{k-2} \leq n} \frac{f'\left[\left(x_1^r + x_2^r + \sum_{j=1}^{k-2} x_{i_j}^r\right)^{1/r}\right]}{f\left[\left(x_1^r + x_2^r + \sum_{j=1}^{k-2} x_{i_j}^r\right)^{1/r}\right]} \right.$$

$$\cdot \left(x_1^r + x_2^r + \sum_{j=1}^{k-2} x_{i_j}^r\right)^{1/r-1} \cdot x_1^{r-1}$$

$$+ \sum_{3 \leq i_1 < i_2 < \cdots < i_{k-1} \leq n} \frac{f'\left[\left(x_1^r + \sum_{j=1}^{k-1} x_{i_j}^r\right)^{1/r}\right]}{f\left[\left(x_1^r + \sum_{j=1}^{k-1} x_{i_j}^r\right)^{1/r}\right]}$$

$$\left. \cdot \left(x_1^r + \sum_{j=1}^{k-1} x_{i_j}^r\right)^{1/r-1} \cdot x_1^{r-1} \right\} .$$

$$(38)$$

Similarly, we can have

$$\frac{\partial F_{n,k}(\mathbf{x},r)}{x_2}$$

$$= F_{n,k}(\mathbf{x},r) \cdot \frac{\partial \ln F_{n,k}(\mathbf{x},r)}{x_2}$$

$$= F_{n,k}(\mathbf{x},r)\left\{ \sum_{3 \leq i_1 < i_2 < \cdots < i_{k-2} \leq n} \frac{f'\left[\left(x_1^r + x_2^r + \sum_{j=1}^{k-2} x_{i_j}^r\right)^{1/r}\right]}{f\left[\left(x_1^r + x_2^r + \sum_{j=1}^{k-2} x_{i_j}^r\right)^{1/r}\right]} \right.$$

$$\cdot \left(x_1^r + x_2^r + \sum_{j=1}^{k-2} x_{i_j}^r\right)^{1/r-1} \cdot x_1^{r-1}$$

$$+ \sum_{3 \leq i_1 < i_2 < \cdots < i_{k-1} \leq n} \frac{f'\left[\left(x_2^r + \sum_{j=1}^{k-1} x_{i_j}^r\right)^{1/r}\right]}{f\left[\left(x_2^r + \sum_{j=1}^{k-1} x_{i_j}^r\right)^{1/r}\right]}$$

$$\left. \cdot \left(x_2^r + \sum_{j=1}^{k-1} x_{i_j}^r\right)^{1/r-1} \cdot x_2^{r-1} \right\} .$$

$$(39)$$

From (38) and (39), we have

$$\Delta_3 = (x_1 - x_2)\left(x_1^{1-m}\frac{\partial F_{n,k}(\mathbf{x},r)}{x_1^{1-m}} - x_2^{1-m}\frac{\partial F_{n,k}(\mathbf{x},r)}{x_2}\right)$$

$$= F_{n,k}(\mathbf{x},r)(x_1 - x_2)(x_1^{r-m} - x_2^{r-m})$$

$$\times \sum_{2 \leq i_1 < i_2 < \cdots < i_{k-2} \leq n} \frac{f'\left[\left(x_1^r + \sum_{j=1}^{k-1} x_{i_j}^r\right)^{1/r}\right]}{f\left[\left(x_1^r + \sum_{j=1}^{k-1} x_{i_j}^r\right)^{1/r}\right]}$$

$$+ F_{n,k}(\mathbf{x},r)(x_1 - x_2)$$

$$\times \sum_{3 \leq i_1 < i_2 < \cdots < i_{k-1} \leq n} \left\{ \frac{f'\left[\left(x_1^r + \sum_{j=1}^{k-1} x_{i_j}^r\right)^{1/r}\right]}{f\left[\left(x_1^r + \sum_{j=1}^{k-1} x_{i_j}^r\right)^{1/r}\right]} \right.$$

$$\cdot \left(x_1^r + \sum_{j=1}^{k-1} x_{i_j}^r\right)^{1/r-1} \cdot x_1^{r-m}$$

$$- \frac{f'\left[\left(x_2^r + \sum_{j=1}^{k-1} x_{i_j}^r\right)^{1/r}\right]}{f\left[\left(x_2^r + \sum_{j=1}^{k-1} x_{i_j}^r\right)^{1/r}\right]}$$

$$\left. \cdot \left(x_2^r + \sum_{j=1}^{k-1} x_{i_j}^r\right)^{1/r-1} \cdot x_2^{r-m} \right\} \qquad (40)$$

$$= III + IV \geq 0.$$

So we get that $\Delta_2 \geq 0$. So the proof of Theorem 16 is complete. □

Proof of Theorem 19. Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}_{++}^n$, and $\sum_{i=1}^n x_i = s$. From [13, 14], we have

$$\frac{s - \mu x}{n - \mu} = \left(\frac{s - \mu x_1}{n - \mu}, \frac{s - \mu x_2}{n - \mu}, \ldots, \frac{s - \mu x_n}{n - \mu}\right)$$

$$\prec (x_1, x_2, \ldots, x_n) = x,$$

$$\frac{s + \mu x}{n + \mu} = \left(\frac{s + \mu x_1}{n + \mu}, \frac{s + \mu x_2}{n + \mu}, \ldots, \frac{s + \mu x_n}{n + \mu}\right)$$

$$\prec (x_1, x_2, \ldots, x_n) = x. \qquad (41)$$

From (41), and contacting Corollaries 17 and 18, we get (21). □

4. Applications

Let $x_i > 0$, $i = 1, 2, \ldots, n$ be n positive real numbers and set $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$. The pth power mean of of order $p \in \mathbb{R}$ of x_i is defined by

$$M_p(\mathbf{x}) = \begin{cases} \left(\dfrac{x_1^p + x_2^p + \cdots + x_n^p}{n}\right)^{1/p}, & p \neq 0; \\ \sqrt{x_1 x_2 \cdots x_n}, & p = 0. \end{cases} \qquad (42)$$

In particular, for $p = 1$, $p = 0$, and $p = 1$ we, respectively, get the arithmetic, the geometric, and the harmonic means of x_i, and set $A_n(\mathbf{x}) = M_1(\mathbf{x}) = (1/n)\sum_{i=1}^n x_i$, $G_n(\mathbf{x}) = M_0(\mathbf{x}) = (\prod_{i=1}^n x_i)^{1/n}$, $H_n(\mathbf{x}) = M_{-1}(\mathbf{x}) = n/\sum_{i=1}^n (1/x_i)$.

In this section, some applications of the results in Section 3 are given. Some analytic inequalities are established. In particular, several inequalities involving the pth power mean and the arithmetic, the geometric, or the harmonic means are presented.

4.1. Analytic Inequalities. To establish some analytic inequalities, we first give a lemma.

Lemma 24 (see [20]). *Let $x_i > 0$, $i = 1, 2, \ldots, n$. Then the following relations is known:*

$$(A_n(\mathbf{x}), A_n(\mathbf{x}), \ldots, A_n(\mathbf{x})) \prec (x_1, x_2, \ldots, x_n);$$

$$\ln(G_n(\mathbf{x}), G_n(\mathbf{x}), \ldots, G_n(\mathbf{x})) \prec \ln(x_1, x_2, \ldots, x_n); \qquad (43)$$

$$\left(\frac{1}{H_n(\mathbf{x})}, \frac{1}{H_n(\mathbf{x})}, \ldots, \frac{1}{H_n(\mathbf{x})}\right) \prec \left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right).$$

By applying Lemma 24 and Theorem 16, Corollaries 17 and 18, we can get the following several theorems and corollaries.

Theorem 25. *For $r > 0$ and $k = 1, 2, \ldots, n$, the symmetric functions*

$$U_k(\mathbf{x}, r) = \prod_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \tan\left(\sum_{j=1}^k x_{i_j}^r\right)^{1/r} \qquad (44)$$

are Schur m-power convex in $\mathbb{R}_{++}^n$ when $m \leq 0$. In particular, $U_k(\mathbf{x}, r)$ are Schur geometrically convex and Schur harmonically convex in $\mathbb{R}_{++}^n$.

Proof. For $x \in \mathbb{R}_{++}$, let $f(x) = \tan x$. Then $f'(x) = 1/\cos^2 x = \sec^2 x \geq 0$ and

$$x\left[f(x)f''(x) - \left(f'(x)\right)^2\right] + f(x)f'(x)$$

$$= \sec^2 x\left(x\sec^2 x - 2x + \tan x\right). \qquad (45)$$

Let $g(x) = x\sec^2 x - 2x + \tan x$; then $g'(x) = 2(1 + x\tan x)\sec^2 x - 2 \geq 0$, and $g(x)$ is increasing. From the monotonicity of $g(x)$, we get that

$$x\left[f(x)f''(x) - \left(f'(x)\right)^2\right] + f(x)f'(x)$$

$$= \sec^2 x g(x) \geq g(0) = 0. \qquad (46)$$

Therefore, $f(x)$ is increasing and multiplicatively convex in $\mathbb{R}_{++}^n$. By applying Theorem 16, we obtain the result. □

By using Theorem 25 and Lemma 24, we get the following inequalities.

Corollary 26. *For* $\mathbf{x} \in \mathbb{R}_{++}^n$, *then*

$$\prod_{i=1}^{n}\left(\tan\frac{1}{x_i}\right)^{1/n} \geq \tan\left(\frac{1}{A(\mathbf{x})}\right);$$

$$\prod_{i=1}^{n}\left(\tan x_i\right)^{1/n} \geq \tan G(\mathbf{x}); \tag{47}$$

$$\prod_{i=1}^{n}\left(\tan x_i\right)^{1/n} \geq \tan H(\mathbf{x}).$$

Theorem 27. *For* $r > 0$ *and* $k = 1, 2, \ldots, n$, *the symmetric functions*

$$V_k(\mathbf{x}, r) = \prod_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \arcsin\left(\sum_{j=1}^{k} x_{i_j}^r\right)^{1/r} \tag{48}$$

are Schur m-power convex in $(0, 1)^n$ *when* $m \leq 0$. *In particular,* $V_k(\mathbf{x}, r)$ *are Schur geometrically convex and Schur harmonically convex in* $(0, 1)^n$.

Proof. For $x \in (0, 1)$, let $f(x) = \arcsin x$. Then $f'(x) = 1/\sqrt{1 - x^2} \geq 0$ and

$$x\left[f(x)f''(x) - \left(f'(x)\right)^2\right] + f(x)f'(x)$$

$$= \frac{\arcsin x - x\sqrt{1 - x^2}}{(1 - x^2)\sqrt{1 - x^2}}. \tag{49}$$

Let $h(x) = \arctan x - x^2\arctan x - x$; then $h'(x) = 2x^2/\sqrt{1 - x^2} \geq 0$, and $h(x)$ is increasing. From the monotonicity of $h(x)$, we get that

$$x\left[f(x)f''(x) - \left(f'(x)\right)^2\right] + f(x)f'(x)$$

$$= \frac{h(x)}{(1 - x^2)\sqrt{1 - x^2}} \geq h(0) = 0. \tag{50}$$

Therefore, $f(x)$ is increasing and multiplicatively convex in $\mathbb{R}_{++}^n$. By applying Theorem 16, we obtain the result. □

By using Theorem 27 and Lemma 24, we get the following inequalities.

Corollary 28. *For* $\mathbf{x} \in (0, 1)^n$, *then*

$$\prod_{i=1}^{n}\left(\arcsin\frac{1}{x_i}\right)^{1/n} \geq \arcsin\left(\frac{1}{A(\mathbf{x})}\right);$$

$$\prod_{i=1}^{n}\left(\arcsin x_i\right)^{1/n} \geq \arcsin G(\mathbf{x}); \tag{51}$$

$$\prod_{i=1}^{n}\left(\arcsin x_i\right)^{1/n} \geq \arcsin H(\mathbf{x}).$$

Theorem 29. *For* $r > 0$ *and* $k = 1, 2, \ldots, n$, *the symmetric functions*

$$M_k(\mathbf{x}, r) = \prod_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \left(\sum_{j=1}^{k} x_{i_j}^r\right)^{1/r} \tag{52}$$

are Schur m-power convex in $\mathbb{R}_{++}^n$ *when* $m \leq 0$. *In particular,* $M_k(\mathbf{x}, r)$ *are Schur geometrically convex in* $\mathbb{R}_{++}^n$ *and Schur harmonically convex in* $\mathbb{R}_{++}^n$.

Proof. Let $f(x) = x, x \in \mathbb{R}_{++}$. We can easily see that $f(x)$ is increasing and multiplicatively convex in $\mathbb{R}_{++}$. By applying Theorem 16, we obtain the result. □

By using Theorem 29 and Lemma 24, we obtain the following results.

Corollary 30. *Let* $\mathbf{x} \in \mathbb{R}_{++}$, *for* $r > 0$ *and* $k = 1, 2, \ldots, n$; *one has*

$$\left(k^{1/r}\frac{1}{A_n(\mathbf{x})}\right)^{C_n^k} \leq \prod_{1 \leq i_1 < i_2 < \cdots < i_k \leq n}\left(\sum_{j=1}^{k}\frac{1}{x_{i_j}^r}\right)^{1/r}, \tag{53}$$

$$\left(k^{1/r}G_n(\mathbf{x})\right)^{C_n^k} \leq \prod_{1 \leq i_1 < i_2 < \cdots < i_k \leq n}\left(\sum_{j=1}^{k}x_{i_j}^r\right)^{1/r}, \tag{54}$$

$$\left(k^{1/r}H_n(\mathbf{x})\right)^{C_n^k} \leq \prod_{1 \leq i_1 < i_2 < \cdots < i_k \leq n}\left(\sum_{j=1}^{k}x_{i_j}^r\right)^{1/r}. \tag{55}$$

In particular, take $k = 1$ in (53); we get the known A-G-H inequality. And take $k = n$ in (53); we get the following inequalities:

$$\left(\sum_{i=1}^{n}\frac{1}{x_i^r}\right)^{1/r} \cdot \sum_{i=1}^{n}x_i \geq n^{1/r+1}. \tag{56}$$

Theorem 31. *For* $r > 0$ *and* $k = 1, 2, \ldots, n$, *the symmetric functions*

$$N_k(\mathbf{x}, r) = \prod_{1 \leq i_1 < i_2 < \cdots < i_k \leq n}\frac{\left(\sum_{j=1}^{k}x_{i_j}^r\right)^{1/r}}{1 - \left(\sum_{j=1}^{k}x_{i_j}^k\right)^{1/r}} \tag{57}$$

are Schur m-power convex on $(0, 1)^n$ *when* $m \leq 0$. *In particular,* $N_k(\mathbf{x}, r)$ *are Schur geometrically convex on* $(0, 1)^n$ *and Schur harmonically convex on* $(0, 1)^n$.

Proof. Let $f(x) = x/(1 - x)$, $x \in (0, 1)$. Then $f'(x) = (1/(1 - x)^2) \geq 0$, and

$$x\left[f(x)f''(x) - f'^2(x)\right] + f(x)f'(x) = \frac{x^2}{(1 - x)^4} \geq 0. \tag{58}$$

So $f(x)$ is increasing and multiplicatively convex in $(0, 1)$. By applying Theorem 16, we obtain the result. □

By using Theorem 31 and Lemma 24, we obtain the following results.

Corollary 32. *Let* $\mathbf{x} \in (0,1)^n$, *for* $r > 0$ *and* $k = 1, 2, \ldots, n$, *one has*

$$\left(\frac{k^{1/r} G_n(\mathbf{x})}{1 - k^{1/r} G_n(\mathbf{x})} \right)^{C_n^k} \leq \prod_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \frac{\left(\sum_{j=1}^k x_{i_j}^r \right)^{1/r}}{1 - \left(\sum_{j=1}^k x_{i_j}^r \right)^{1/r}},$$

$$\left(\frac{k^{1/r} H_n(\mathbf{x})}{1 - k^{1/r} H_n(\mathbf{x})} \right)^{C_n^k} \leq \prod_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \frac{\left(\sum_{j=1}^k x_{i_j}^r \right)^{1/r}}{1 - \left(\sum_{j=1}^k x_{i_j}^r \right)^{1/r}}.$$

(59)

In particular, take $k = 1$ or $k = n$ in (59), we get the following inequalities:

$$\left(\frac{G_n(\mathbf{x})}{1 - G_n(\mathbf{x})} \right)^n \leq \prod_{i=1}^n \frac{x_i}{1 - x_i},$$

$$\left(\frac{H_n(\mathbf{x})}{1 - H_n(\mathbf{x})} \right)^n \leq \prod_{i=1}^n \frac{x_i}{1 - x_i},$$

$$\frac{G_n(\mathbf{x})}{1 - n^{1/r} G_n(\mathbf{x})} \leq \frac{M_r(\mathbf{x})}{1 - n^{1/r} M_r(\mathbf{x})},$$

$$\frac{H_n(\mathbf{x})}{1 - n^{1/r} H_n(\mathbf{x})} \leq \frac{M_r(\mathbf{x})}{1 - n^{1/r} M_r(\mathbf{x})}.$$

(60)

Theorem 33. *For* $r > 0$ *and* $k = 1, 2, \ldots, n$, *the symmetric functions*

$$I_k(\mathbf{x}, r) = \prod_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \frac{1 + \left(\sum_{j=1}^k x_{i_j}^r \right)^{1/r}}{1 - \left(\sum_{j=1}^k x_{i_j}^r \right)^{1/r}}$$

(61)

are Schur m-power convex on $(0,1)^n$ *when* $m \leq 0$. *In particular,* $I_k(\mathbf{x}, r)$ *are Schur geometrically convex and Schur harmonically convex on* $(0,1)^n$.

Proof. Let $f(x) = x/(1-x)$, $x \in (0,1)$. Then $f'(x) = (2/(1-x)^2) \geq 0$, and

$$x \left[f(x) f''(x) - f'^2(x) \right] + f(x) f'(x) = \frac{2 - x^2}{(1-x)^4} \geq 0.$$

(62)

So $f(x)$ is increasing and multiplicatively convex in $(0,1)$. By applying Theorem 16, we obtain the result. □

By using Theorem 33 and Lemma 24, we get the following inequalities.

Corollary 34. *Let* $\mathbf{x} \in (0,1)^n$, *for* $r > 0$ *and* $k = 1, 2, \ldots, n$; *one has*

$$\left(\frac{1 + k^{1/r} G_n(\mathbf{x})}{1 - k^{1/r} G_n(\mathbf{x})} \right)^{C_n^k} \leq \prod_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \frac{1 + \left(\sum_{j=1}^k x_{i_j}^r \right)^{1/r}}{1 - \left(\sum_{j=1}^k x_{i_j}^r \right)^{1/r}},$$

$$\left(\frac{1 + k^{1/r} H_n(\mathbf{x})}{1 - k^{1/r} H_n(\mathbf{x})} \right)^{C_n^k} \leq \prod_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \frac{1 + \left(\sum_{j=1}^k x_{i_j}^r \right)^{1/r}}{1 - \left(\sum_{j=1}^k x_{i_j}^r \right)^{1/r}}.$$

(63)

In particular, take $k = 1$ or $k = n$ in (63); we get the following inequalities:

$$\left(\frac{1 + G_n(\mathbf{x})}{1 - G_n(\mathbf{x})} \right)^n \leq \prod_{i=1}^n \frac{1 + x_i}{1 - x_i},$$

$$\left(\frac{1 + H_n(\mathbf{x})}{1 - H_n(\mathbf{x})} \right)^n \leq \prod_{i=1}^n \frac{1 + x_i}{1 - x_i},$$

$$\frac{1 + n^{1/r} G_n(\mathbf{x})}{1 - n^{1/r} G_n(\mathbf{x})} \leq \frac{1 + n^{1/r} M_r(\mathbf{x})}{1 - n^{1/r} M_r(\mathbf{x})},$$

$$\frac{1 + n^{1/r} H_n(\mathbf{x})}{1 - n^{1/r} H_n(\mathbf{x})} \leq \frac{1 + n^{1/r} M_r(\mathbf{x})}{1 - n^{1/r} M_r(\mathbf{x})}.$$

(64)

Theorem 35. *For* $r > 0$ *and* $k = 1, 2, \ldots, n$, *the symmetric functions*

$$T_k(\mathbf{x}, r) = \prod_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \left[\left(\sum_{j=1}^k x_{i_j}^r \right)^{1/r} \right]^{\left(\sum_{j=1}^r x_{i_j}^r \right)^{1/r}}$$

(65)

are Schur m-power convex on $(e^{-1}, +\infty)^n$ *when* $m \leq 0$. *In particular,* $T_k(\mathbf{x}, r)$ *are Schur geometrically convex and Schur harmonically convex on* $(e^{-1}, +\infty)^n$.

Proof. Let $f(x) = x^x$, $x \in (e^{-1}, +\infty)$. Then $f'(x) = x^x(\ln x + 1) \geq 0$, and

$$x \left[f(x) f''(x) - f'^2(x) \right] + f(x) f'(x)$$
$$= f^2(x)(\ln x + 2) \geq 0.$$

(66)

So $f(x)$ is increasing and multiplicatively convex on $(e^{-1}, +\infty)$. By applying Theorem 16, we obtain the result. □

By using Theorem 35 and Lemma 24, we get the following inequalities.

Corollary 36. *Let* $\mathbf{x} \in (e^{-1}, +\infty)^n$, *for* $r > 0$ *and* $k = 1, 2, \ldots, n$; *one has*

$$\left(k^{1/r} \frac{1}{A_n(\mathbf{x})} \right)^{k^{1/r}(1/A_n(\mathbf{x})) C_n^k}$$

$$\leq \prod_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \left(\sum_{j=1}^k \frac{1}{x_{i_j}^r} \right)^{(1/r)\left(\sum_{j=1}^k (1/x_{i_j}^r) \right)^{1/r}},$$

$$\left(k^{1/r} G_n(\mathbf{x}) \right)^{k^{1/r} G_n(\mathbf{x}) C_n^k}$$

$$\leq \prod_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \left(\sum_{j=1}^k x_{i_j}^r \right)^{(1/r)\left(\sum_{j=1}^k x_{i_j}^r \right)^{1/r}},$$

$$\left(k^{1/r}H_n(\mathbf{x})\right)^{k^{1/r}H_n(\mathbf{x})C_n^k}$$

$$\leq \prod_{1\leq i_1<i_2<\cdots<i_k\leq n}\left(\sum_{j=1}^{k}x_{i_j}^r\right)^{(1/r)\left(\sum_{j=1}^{k}x_{i_j}^r\right)^{1/r}}. \tag{67}$$

In particular, take $k = 1$ or $k = n$ in (67); we get the following inequalities:

$$\left(\frac{1}{A_n(\mathbf{x})}\right)^{n/A_n(\mathbf{x})} \leq \prod_{i=1}^{n}\left(\frac{1}{x_i}\right)^{1/x_i},$$

$$\left(G_n(\mathbf{x})\right)^{n\cdot G_n(\mathbf{x})} \leq \prod_{i=1}^{n}x_i^{x_i},$$

$$\left(H_n(\mathbf{x})\right)^{n\cdot H_n(\mathbf{x})} \leq \prod_{i=1}^{n}x_i^{x_i}, \tag{68}$$

$$\left(n^{1/r}\frac{1}{A_n(\mathbf{x})}\right)^{n^{1/r}(1/A_n(\mathbf{x}))} \leq \left(n^{1/r}M_r\left(\frac{1}{\mathbf{x}}\right)\right)^{n^{1/r}M_r(1/\mathbf{x})},$$

$$\left(n^{1/r}G_n(\mathbf{x})\right)^{n^{1/r}G_n(\mathbf{x})} \leq \left(n^{1/r}M_r(\mathbf{x})\right)^{n^{1/r}M_r(\mathbf{x})},$$

$$\left(n^{1/r}H_n(\mathbf{x})\right)^{n^{1/r}H_n(\mathbf{x})} \leq \left(n^{1/r}M_r(\mathbf{x})\right)^{n^{1/r}M_r(\mathbf{x})}.$$

4.2. Geometric Inequalities.

In this section, some geometric inequalities of n-dimensional simplex are established by use of the results of Theorem 16.

Lots of geometric inequalities for an n-dimensional simplex are established (see [37–42]). In this section, applying the above Lemma and the main results in Section 2, we establish some interesting geometric inequalities on n-dimensional simplex in n-dimensional Euclidean space E^n.

In what follows, Let $\Omega = \{A_1, A_2, \ldots, A_{n+1}\}$ be an n-dimensional simplex in n-dimensional Euclidean space E^n ($n \geq 2$) with V the volume. We denote by h_i, r_i, F_i ($i = 1, 2, \ldots, n+1$), and r the altitudes, the radii of excircles, the areas of lateral surfaces, and the inradius of Ω, respectively. For a given point P in Ω, let B_i stand for the intersection point of straight line A_iP and hyperplane $a_i = \{A_1\cdots A_{i-1}A_{i+1}\cdots A_{n+1}\}$.

We first give some lemmas.

Lemma 37. Let $\Omega = \{A_1, A_2, \ldots, A_{n+1}\}$ be an n-dimensional simplex in n-dimensional Euclidean space E^n ($n \geq 2$). The following relation is

(1) $\left(\dfrac{1}{n+1}, \dfrac{1}{n+1}, \ldots, \dfrac{1}{n+1}\right) \prec \left(\dfrac{r}{h_1}, \dfrac{r}{h_2}, \ldots, \dfrac{r}{h_{n+1}}\right);$ (69)

(2) $\left(\dfrac{n-1}{n+1}, \dfrac{n-1}{n+1}, \ldots, \dfrac{n-1}{n+1}\right) \prec \left(\dfrac{r}{r_1}, \dfrac{r}{r_2}, \ldots, \dfrac{r}{r_{n+1}}\right);$ (70)

(3) $\left(\dfrac{1}{n+1}, \dfrac{1}{n+1}, \ldots, \dfrac{1}{n+1}\right)$

$\prec \left(\dfrac{r_1}{h_1+2r_1}, \dfrac{r_2}{h_2+2r_2}, \ldots, \dfrac{r_{n+1}}{h_{n+1}+2r_{n+1}}\right);$ (71)

(4) $\left(\dfrac{1}{n+1}, \dfrac{1}{n+1}, \ldots, \dfrac{1}{n+1}\right)$

$\prec \left(\dfrac{|PB_1|}{|A_1B_1|}, \dfrac{|PB_2|}{|A_2B_2|}, \ldots, \dfrac{|PB_{n+1}|}{|A_{n+1}B_{n+1}|}\right),$ (72)

where $F = \sum_{i=1}^{n+1}F_i$.

Proof. By the formula [37, 38]

$$F_ih_i = nV, \qquad \sum_{i=1}^{n+1}F_ir = nV, \qquad \sum_{i=1}^{n+1}(F-2F_i)r_i = nV, \tag{73}$$

where $F = \sum_{i=1}^{n+1}F_i$. From these we get

$$\sum_{i=1}^{n+1}\frac{r}{h_i} = \sum_{i=1}^{n+1}\frac{F_ir}{F_ih_i} = \frac{\sum_{i=1}^{n+1}F_ir}{nV} = \frac{nV}{nV} = 1, \tag{74}$$

$$\sum_{i=1}^{n+1}\frac{r}{r_i} = \sum_{i=1}^{n+1}\frac{(F-2F_i)r}{(F-2F_i)r_i}$$

$$= \frac{\sum_{i=1}^{n+1}(F-2F_i)r}{nV} = \frac{(n-1)nV}{nV} = n-1, \tag{75}$$

$$\sum_{i=1}^{n+1}\frac{r_i}{h_i+2r_i} = \sum_{i=1}^{n+1}\frac{(F-2F_i)F_ir_i}{(F-2F_i)F_i(h_i+2r_i)}$$

$$= \sum_{i=1}^{n+1}\frac{nVF_i}{nV(F-2F_i)+2nVF_i} = \sum_{i=1}^{n+1}\frac{F_i}{F} = 1. \tag{76}$$

By Definition 3, and using (74), (75), and (76), respectively, we obtain (69), (70), and (71). In addition, (72) follows from Definition 3 and the fact that $\sum_{i=1}^{n+1}|PB_i|/|A_iB_i| = 1$. So the proof is complete. $\square$

Applying Lemma 37 and Theorem 29, we establish some interesting geometric inequalities on n-dimensional simplex in n-dimensional Euclidean space E^n.

Theorem 38. Let $\Omega = \{A_1, A_2, \ldots, A_{n+1}\}$ be an n-dimensional simplex in n-dimensional Euclidean space E^n ($n \geq 2$). For $t > 0$ and $k = 1, 2, \ldots, n$, one has

$$\left(r\cdot k^{1/t}(n+1)\right)^{C_{n+1}^k} \leq \prod_{1\leq i_1<i_2<\cdots<i_k\leq n}\left(\sum_{j=1}^{k}h_{i_j}^t\right)^{1/t};$$

$$\left(r\cdot k^{1/t}\frac{n+1}{n-1}\right)^{C_{n+1}^k} \leq \prod_{1\leq i_1<i_2<\cdots<i_k\leq n}\left(\sum_{j=1}^{k}r_{i_j}^t\right)^{1/t};$$

$$\left(r \cdot k^{1/t}(n+1)\right)^{C_{n+1}^k} \leq \prod_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \left[\sum_{j=1}^{k}\left(\frac{h_{i_j} + 2r_{i_j}}{r_{i_j}}\right)^t\right]^{1/t};$$

$$\left(r \cdot k^{1/t}(n+1)\right)^{C_{n+1}^k} \leq \prod_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \left[\sum_{j=1}^{k}\left(\frac{|PA_{i_j}|}{|PB_{i_j}|}\right)^t\right]^{1/t}. \tag{77}$$

In particular, take $k = 1$ in (77), we get the following inequality

$$r(n+1) \leq \prod_{i=1}^{n+1} h_i^{1/(n+1)}; \qquad r\left(\frac{n+1}{n-1}\right) \leq \prod_{i=1}^{n+1} r_i^{1/(n+1)};$$

$$(n+1) \leq \prod_{i=1}^{n+1}\left(\frac{h_i}{r_i} + 2\right)^{1/(n+1)}; \tag{78}$$

$$(n+1) \leq \prod_{i=1}^{n+1}\left(\frac{|PB_i|}{|PA_i|}\right)^{1/(n+1)}.$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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