

Research Article

New Integral Inequalities with Weakly Singular Kernel for Discontinuous Functions and Their Applications to Impulsive Fractional Differential Systems

Jing Shao^{1,2}

¹ Department of Mathematics, Jining University, Qufu, Shandong 273155, China

² College of Mathematical Sciences, Qufu Normal University, Qufu, Shandong 273165, China

Correspondence should be addressed to Jing Shao; shaojing99500@163.com

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Some new integral inequalities with weakly singular kernel for discontinuous functions are established using the method of successive iteration and properties of Mittag-Leffler function, which can be used in the qualitative analysis of the solutions to certain impulsive fractional differential systems.

1. Introduction

Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of engineering, physics, and economics. There has been a significant development in the study of fractional differential equations in recent years; see the monographs of Kilbas et al. [1], Lakshmikantham et al. [2], and Podlubny [3] and the survey by Diethelm [4]. Integral inequalities with weakly singular kernels play an important role in the qualitative analysis of the solutions to fractional differential equations. With the development of fractional differential equations, integral inequalities with weakly singular kernels have drawn more and more researchers' attention and lead to inspiring results; see, for example, [5–8].

Impulsive differential equations, that is, differential equations involving impulse effect, appear as a natural description of observed evolution phenomena of several real world problems. Many processes studied in applied sciences are represented by impulsive differential equations. However, the situation is quite different in many physical phenomena that have a sudden change in their states such as mechanical systems with impact, biological systems such as heart beats, blood flow, population dynamics theoretical physics, pharmacokinetics, mathematical economy, chemical technology,

electric technology, metallurgy, ecology, industrial robotics, and biotechnology processes (see the monographs [9–12] for details).

The theory of impulsive differential equations is an important branch of differential equations. In spite of its importance, the development of the theory has been quite slow due to special features possessed by impulsive differential equations in general, such as pulse phenomena, confluence, and loss of autonomy. Among these results, integrosum inequalities for discontinuous functions play increasingly important roles in the study of quantitative properties of solutions of impulsive differential systems. In 2005, Borysenko et al. [13] considered some integrosum inequalities and devoted them to investigate the properties of motion represented by essential nonlinear system of differential equations with impulsive effect. In 2007 and 2009, Gallo and Piccirillo [14, 15] presented some new nonlinear integral inequalities like Gronwall-Bellman-Bihari type with delay for discontinuous functions and applied them to investigate the properties of solutions of impulsive differential systems.

The theory of impulsive fractional differential equations is a new topic of research which involve both the fractional order integral (or differentiation) and the impulsive effect; most of the results related to this topic are the existence of solutions (see [16–19] and the references therein). To our best

knowledge, there is no result on other qualitative properties (such as boundedness and stability), and impulsive fractional differential equations involving the Caputo fractional derivative have not been studied very perfectly, so we set up a new kind of integral inequalities with weakly singular kernel for discontinuous functions and use the new inequalities to study the qualitative properties of the solutions to certain impulsive fractional differential systems.

On the basis of previous studies, in this paper, we consider the following integral inequalities with weakly singular kernel for discontinuous functions:

$$u(t) \leq a + b \int_{t_0}^t (t-s)^{\beta-1} u(s) ds + \sum_{t_0 < t_i < t} \gamma_i u(t_i^-), \quad (1)$$

where a, b , and γ_i are constants, $a \geq 0, b \geq 0, \gamma_i \geq 0$, and $u(t)$ is a nonnegative piecewise-continuous function with the 1st kind of discontinuous points: $t_0 < t_1 < \dots, \lim_{i \rightarrow \infty} t_i = \infty, u(t_i^-) = \lim_{t \rightarrow t_i-0} u(t)$. In general, due to the existence of weak singular integral kernel, the methods of these inequalities for discontinuous functions are quite different to that of classical Gronwall-Bellman-Bihari inequalities. We use the properties of the Mittag-Leffler function $E_\beta(\cdot)$ defined by $E_\beta(z) = \sum_{k=0}^{\infty} (z^k / \Gamma(k\beta + 1))$ and the successive iterative technique to establish the new type of integral inequalities for discontinuous functions. These inequalities are applied to investigate the qualitative analysis of the solutions to certain impulsive fractional differential systems.

2. Preliminary Knowledge

In this section, we give some definitions, symbols, and known inequalities, which will be used in the remainder of this paper.

Definition 1. Given an interval $[a, b]$ of \mathbb{R} , the fractional (arbitrary) order integral of the function $h \in L^1([a, b], \mathbb{R})$ of order $\alpha \in \mathbb{R}_+$ is defined by

$$I_a^\alpha h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds, \quad (2)$$

where $\Gamma(\cdot)$ is the gamma function. When $a = 0$, we write $I^\alpha h(t) = [h * \varphi_\alpha](t)$, where $\varphi_\alpha(t) = t^{\alpha-1} / \Gamma(\alpha)$ for $t > 0, \varphi_\alpha(t) = 0$ for $t \leq 0$, and $\varphi_\alpha \rightarrow \delta(t)$ as $\alpha \rightarrow 0$, where δ is the delta function.

Definition 2. For a given function h on the interval $[a, b]$, the α -order Caputo fractional order derivative of h is defined by

$$({}^c D_{a+}^\alpha h)(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-\alpha-1} h^{(m)}(s) ds, \quad (3)$$

where $m = [\alpha] + 1$.

For calculation simplification, the symbols are defined as follows:

$$S(t, t_i) = E_\beta(b\Gamma(\beta)(t-t_i)^\beta), \quad \text{for } t > t_i, \quad (4)$$

$$S_{i,j} = S(t_i, t_j) = E_\beta(b\Gamma(\beta)(t_i-t_j)^\beta), \quad \text{for } t_i > t_j,$$

where $E_\beta(\cdot)$ is the Mittag-Leffler function.

Lemma 3 (see [6]). Suppose that $\beta > 0, a(t)$ is a nonnegative and nondecreasing function which is locally integrable on $0 \leq t < T$ (for some $T \leq +\infty$), and $g(t)$ is a nonnegative, nondecreasing continuous function defined on $0 \leq t < T, g(t) \leq M$ (constant), and suppose that $u(t)$ is a nonnegative and locally integrable function on $0 \leq t < T$ which satisfies

$$u(t) \leq a(t) + g(t) \int_0^t (t-s)^{\beta-1} u(s) ds, \quad (5)$$

and then

$$u(t) \leq a(t) E_\beta(g(t) \Gamma(\beta) t^\beta). \quad (6)$$

Using Lemma 3, we can easily get the following corollary.

Corollary 4. Let a, b be constants, $a \geq 0$ and $b \geq 0$. And suppose $u(t)$ is a nonnegative and locally integrable function on $t_0 \leq t < T$ with

$$u(t) \leq a + b \int_{t_0}^t (t-s)^{\beta-1} u(s) ds, \quad (7)$$

and then

$$u(t) \leq aS(t, t_0). \quad (8)$$

Lemma 5. Let $\alpha \geq 0$ and $\beta \geq 0$. Then

$$\int_{t_0}^{t_1} (t-s)^{\alpha-1} (s-t_0)^{\beta-1} ds \leq B(\alpha, \beta) (t-t_0)^{\alpha+\beta-1}, \quad (9)$$

$$t \geq t_1 > t_0,$$

where $B(\cdot, \cdot)$ is the Beta function.

Proof. Let $s = t_0 + \xi(t-t_0)$; we obtain for $t \geq t_1$

$$\begin{aligned} & \int_{t_0}^{t_1} (t-s)^{\alpha-1} (s-t_0)^{\beta-1} ds \\ &= \int_0^{(t_1-t_0)/(t-t_0)} (1-\xi)^{\alpha-1} (t-t_0)^{\alpha-1} (t-t_0)^{\beta-1} \xi^{\beta-1} d\xi \\ &\leq (t-t_0)^{\alpha+\beta-1} \int_0^1 (1-\xi)^{\alpha-1} \xi^{\beta-1} d\xi \\ &\leq B(\alpha, \beta) (t-t_0)^{\alpha+\beta-1}, \end{aligned} \quad (10)$$

which is the desired result. \square

Remark 6. If we replace the integration interval $[t_0, t_1]$ by $[t_i, t_{i+1}]$ for $i = 0, 1, 2, \dots$, we can get the following equality:

$$\int_{t_i}^{t_{i+1}} (t-s)^{\alpha-1} (s-t_i)^{\beta-1} ds \leq B(\alpha, \beta) (t-t_i)^{\alpha+\beta-1}, \quad (11)$$

$$t \geq t_{i+1} > t_i, \quad i = 0, 1, 2, \dots$$

Corollary 7. Let $\beta \geq 0$. Then

$$b \int_{t_0}^{t_1} (t-s)^{\beta-1} S(s, t_0) ds \leq S(t, t_0) - 1, \quad t \geq t_1 > t_0. \quad (12)$$

Proof. By the definition of $S(s, t_0)$, we have

$$\begin{aligned} & b \int_{t_0}^{t_1} (t-s)^{\beta-1} S(s, t_0) ds \\ &= b \int_{t_0}^{t_1} (t-s)^{\beta-1} \sum_{k=0}^{\infty} \frac{b^k \Gamma^k(\beta) (s-t_0)^{k\beta}}{\Gamma(k\beta+1)} ds. \end{aligned} \tag{13}$$

Since the series of function is the Mittag-Leffler function $E_{\beta}(b\Gamma(\beta)(s-t_0)^{\beta})$, which is convergent uniformly on $s \in [t_0, t_1]$, we permute the sum and the integral to obtain

$$\begin{aligned} & b \int_{t_0}^{t_1} (t-s)^{\beta-1} S(s, t_0) ds \\ &= \sum_{k=0}^{\infty} \frac{b^{k+1} \Gamma^k(\beta)}{\Gamma(k\beta+1)} \int_{t_0}^{t_1} (t-s)^{\beta-1} (s-t_0)^{k\beta} ds. \end{aligned} \tag{14}$$

Using Lemma 5, we get

$$\begin{aligned} & b \int_{t_0}^{t_1} (t-s)^{\beta-1} S(s, t_0) ds \\ &\leq \sum_{k=0}^{\infty} \frac{b^{k+1} \Gamma^k(\beta)}{\Gamma(k\beta+1)} (t-t_0)^{(k+1)\beta} B(\beta, k\beta+1) \\ &= \sum_{k=0}^{\infty} \frac{b^{k+1} \Gamma^{k+1}(\beta) (t-t_0)^{(k+1)\beta}}{\Gamma((k+1)\beta+1)} \\ &= \sum_{k=1}^{\infty} \frac{b^k \Gamma^k(\beta) (t-t_0)^{k\beta}}{\Gamma(k\beta+1)} \\ &\leq \sum_{k=0}^{\infty} \frac{b^k \Gamma^k(\beta) (t-t_0)^{k\beta}}{\Gamma(k\beta+1)} - 1 = S(t, t_0) - 1. \end{aligned} \tag{15}$$

□

Remark 8. If we replace the integration interval $[t_0, t_1]$ by $[t_i, t_{i+1}]$, we can get the similar conclusion

$$\begin{aligned} & b \int_{t_i}^{t_{i+1}} (t-s)^{\beta-1} S(s, t_0) ds \leq S(t, t_i) - 1, \\ & t \geq t_{i+1} > t_i, \quad i = 0, 1, 2, \dots \end{aligned} \tag{16}$$

3. Main Results

Theorem 9. Let a, b be constants: $a \geq 0, b \geq 0$, and $u(t)$ is a nonnegative piecewise-continuous function with the 1st kind of discontinuous points: $t_0 < t_1 < \dots$ and $\lim_{i \rightarrow \infty} t_i = \infty$. If

$$u(t) \leq a + b \int_{t_0}^t (t-s)^{\beta-1} u(s) ds + \sum_{t_0 < t_i < t} \gamma_i u(t_i^-), \quad t \in I, \tag{17}$$

where $\beta > 0$, then the following assertions hold:

$$\begin{aligned} & u(t) \leq a(1 + \gamma_1) \cdots (1 + \gamma_i) S_{i+1,0} S_{i+1,1} \cdots S_{i+1,i-1} S(t, t_i), \\ & t \in (t_i, t_{i+1}], \\ & u(t) \leq aS(t, t_0), \quad t \in [t_0, t_1]. \end{aligned} \tag{18}$$

Proof. If $t \in [t_0, t_1]$, the inequality (17) is reduced to the following form:

$$u(t) \leq a + b \int_{t_0}^t (t-s)^{\beta-1} u(s) ds. \tag{19}$$

Using Lemma 3, we get

$$\begin{aligned} & u(t) \leq aS(t, t_0), \\ & u(t_1^-) \leq aS(t_1, t_0) = aS_{1,0}. \end{aligned} \tag{20}$$

If $t \in (t_1, t_2]$, then

$$\begin{aligned} & u(t) \leq a + \gamma_1 u(t_1^-) + b \int_{t_0}^t (t-s)^{\beta-1} u(s) ds \\ &\leq a + \gamma_1 u(t_1^-) + b \int_{t_0}^{t_1} (t-s)^{\beta-1} u(s) ds \\ &\quad + b \int_{t_1}^t (t-s)^{\beta-1} u(s) ds \\ &\leq a + a\gamma_1 S_{1,0} + ab \int_{t_0}^{t_1} (t-s)^{\beta-1} S(s, t_0) ds \\ &\quad + b \int_{t_1}^t (t-s)^{\beta-1} u(s) ds. \end{aligned} \tag{21}$$

Using Corollary 7, we get

$$\begin{aligned} & u(t) \leq a + a\gamma_1 S_{1,0} + aS(t, t_0) - a + b \int_{t_1}^t (t-s)^{\beta-1} u(s) ds \\ &\leq a(\gamma_1 S_{1,0} + S_{2,0}) + b \int_{t_1}^t (t-s)^{\beta-1} u(s) ds. \end{aligned} \tag{22}$$

Applying Lemma 3, we obtain that

$$\begin{aligned} & u(t) \leq a(\gamma_1 S_{1,0} + S_{2,0}) S(t, t_1) \leq a(1 + \gamma_1) S_{2,0} S(t, t_1), \\ & u(t_2^-) \leq a(1 + \gamma_1) S_{2,0} S_{2,1}. \end{aligned} \tag{23}$$

Similarly, for $t \in (t_2, t_3]$, using Remark 8, we have

$$\begin{aligned} & u(t) \leq a + \gamma_1 u(t_1^-) + \gamma_2 u(t_2^-) + b \int_{t_0}^t (t-s)^{\beta-1} u(s) ds \\ &\leq a + a\gamma_1 S_{1,0} + a\gamma_2 (1 + \gamma_1) S_{2,0} S_{2,1} \\ &\quad + b \int_{t_0}^{t_1} (t-s)^{\beta-1} u(s) ds \\ &\quad + b \int_{t_1}^{t_2} (t-s)^{\beta-1} u(s) ds + b \int_{t_2}^t (t-s)^{\beta-1} u(s) ds \\ &\leq a(1 + \gamma_1 S_{1,0} + \gamma_2 (1 + \gamma_1) S_{2,0} S_{2,1} + (S_{3,0} - 1) \\ &\quad + (1 + \gamma_1) S_{2,0} (S_{3,1} - 1)) + b \int_{t_2}^t (t-s)^{\beta-1} u(s) ds \end{aligned}$$

$$\begin{aligned} &\leq a(1 + \gamma_1 S_{3,0} + \gamma_2(1 + \gamma_1) S_{3,0} S_{3,1} + (S_{3,0} - 1) \\ &\quad + (1 + \gamma_1) S_{3,0} (S_{3,1} - 1)) + b \int_{t_2}^t (t - s)^{\beta-1} u(s) ds \\ &= a(1 + \gamma_1)(1 + \gamma_2) S_{3,0} S_{3,1} + b \int_{t_2}^t (t - s)^{\beta-1} u(s) ds. \end{aligned} \tag{24}$$

Again, Lemma 3 implies that

$$u(t) \leq a(1 + \gamma_1)(1 + \gamma_2) S_{3,0} S_{3,1} S(t, t_2), \quad t \in (t_2, t_3]. \tag{25}$$

Using the inductive method, suppose that, for $t \in (t_{i-1}, t_i], i = 1, 2, 3, \dots,$

$$\begin{aligned} u(t) &\leq a(1 + \gamma_1)(1 + \gamma_2) \cdots (1 + \gamma_{i-1}) S_{i,0} S_{i,1} S_{i,2} \\ &\quad \cdots S_{i,i-2} S(t, t_{i-1}), \end{aligned} \tag{26}$$

and then for each $t \in (t_i, t_{i+1}]$ and from (26), we have

$$\begin{aligned} &u(t) \\ &\leq a + \gamma_1 u(t_1^-) + \gamma_2 u(t_2^-) + \cdots + \gamma_i u(t_{i-1}^-) \\ &\quad + b \int_{t_0}^t (t - s)^{\beta-1} u(s) ds \\ &= a + a\gamma_1 S_{1,0} + a\gamma_2(1 + \gamma_1) S_{2,0} S_{2,1} + \cdots \\ &\quad + a\gamma_i(1 + \gamma_1) \cdots (1 + \gamma_{i-1}) S_{i,0} S_{i,1} \cdots S_{i,i-1} \\ &\quad + \sum_{j=0}^{i-1} b \int_{t_j}^{t_{j+1}} (t - s)^{\beta-1} u(s) ds \\ &\quad + \int_{t_i}^t (t - s)^{\beta-1} u(s) ds \\ &\leq a + a\gamma_1 S_{1,0} + a\gamma_2(1 + \gamma_1) S_{2,0} S_{2,1} + \cdots \\ &\quad + a\gamma_i(1 + \gamma_1) \cdots (1 + \gamma_{i-1}) S_{i,0} S_{i,1} \cdots S_{i,i-1} \\ &\quad + a(S_{i+1,0} - 1) + a(1 + \gamma_1) S_{2,0} (S_{i+1,1} - 1) + \cdots \\ &\quad + a(1 + \gamma_1) \cdots (1 + \gamma_{i-1}) S_{i,0} S_{i,1} \cdots S_{i,i-2} (S_{i+1,i-1} - 1) \\ &\quad + \int_{t_i}^t (t - s)^{\beta-1} u(s) ds \\ &\leq a(1 + \gamma_1 S_{i+1,0} + \gamma_2(1 + \gamma_1) S_{i+1,0} S_{i+1,1} + \cdots \\ &\quad + \gamma_i(1 + \gamma_1) \cdots (1 + \gamma_{i-1}) S_{i+1,0} S_{i+1,1} \cdots S_{i+1,i-1} \\ &\quad + S_{i+1,0} - 1 + (1 + \gamma_1) S_{i+1,0} (S_{i+1,1} - 1) + \cdots \\ &\quad + (1 + \gamma_1) \cdots (1 + \gamma_{i-1}) S_{i,0} S_{i,1} \cdots S_{i,i-2} (S_{i+1,i-1} - 1)) \\ &\quad + \int_{t_i}^t (t - s)^{\beta-1} u(s) ds \end{aligned}$$

$$\begin{aligned} &\leq a(1 + \gamma_1)(1 + \gamma_2) \cdots (1 + \gamma_{i-1}) S_{i,0} S_{i,1} \cdots S_{i,i-2} \\ &\quad + \int_{t_i}^t (t - s)^{\beta-1} u(s) ds. \end{aligned} \tag{27}$$

Then applying Lemma 3, we obtain the inequality (17). This completes the proof of Theorem 9. \square

Remark 10. The results of Theorem 9 are valid when the function has only finite number of discontinuities points t_0, t_1, \dots, t_i .

4. Application to Impulsive Fractional Differential Systems

Let us consider the following system of Caputo fractional differential equations:

$$\begin{aligned} &{}^C D_{0^+}^\beta x(t) := f(t, x(t)), \quad t \in J : [0, T] \setminus \{t_1, \dots, t_k\}, \\ &\Delta x(t_i) := x(t_i^+) - x(t_i^-) = I_i(x(t_i^-)), \quad i = 1, 2, \dots, k, \\ &x(0) = \chi_0, \end{aligned} \tag{28}$$

where ${}^C D_{0^+}^\beta$ is the Caputo fractional derivatives, $0 = t_0 < t_1 < \dots < t_k < t_{k+1} = T, f : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is Lebesgue measurable with respect to t on $[0, T]$, and $f(t, x)$ is continuous with respect to x on \mathbb{R}^n . $I_i : \mathbb{R}^n \rightarrow \mathbb{R}^n$ are continuous for $i = 1, 2, \dots, k, x(t_i^+) = \lim_{\epsilon \rightarrow 0^+} x(t_i + \epsilon)$ and $x(t_i^-) = \lim_{\epsilon \rightarrow 0^+} x(t_i - \epsilon)$ represent the right and left limits of $x(t)$ at $t = t_i$, and $\chi_0 \in \mathbb{R}^n$ is a constant-valued vector.

Using Lemma 6.2 in [3], $x(t)$ is a solution of Cauchy problem for system (28) if only if $x(t)$ satisfies

$$\begin{aligned} x(t) &= \chi_0 + \sum_{t_0 < t_i < t} I_i(x(t_i^-)) \\ &\quad + \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta-1} f(s, x(s)) ds. \end{aligned} \tag{29}$$

Theorem 11. Suppose that $\|f(t, x)\| \leq b\|x\|$ and $\|I_i(x(t_i^-))\| \leq \gamma_i \|x(t_i^-)\|$, where b, γ_i are constants, $b \geq 0$ and $\gamma_i \geq 0$. If $x(t)$ is any solution of the initial value problem (28), then the following estimations hold:

$$\begin{aligned} &\|x(t)\| \\ &\leq \begin{cases} \|\chi_0\| S^*(t, t_0), & t \in [t_0, t_1], \\ \|\chi_0\| (1 + \gamma_1) \cdots (1 + \gamma_i) S_{i+1,0}^* S_{i+1,1}^* \\ \quad \cdots S_{i+1,i-1}^* S^*(t, t_i), & t \in [t_{i-1}, t_i], \end{cases} \end{aligned} \tag{30}$$

where $S^*(t, t_i) = E_\beta(b(t - t_i)^\beta), S^*(t_i, t_j) = E_\beta(b(t_i - t_j)^\beta)$, and $\|\cdot\|$ is a suitable complete norm in \mathbb{R} .

Proof. From (29), it is easy to obtain that

$$\begin{aligned} \|x(t)\| \leq & \|\chi_0\| + \sum_{t_0 < t_i < t} \gamma_i \|x(t_i^-)\| \\ & + \int_0^t \frac{b}{\Gamma(\beta)} (t-s)^{\beta-1} \|x(s)\| ds, \quad t \geq 0. \end{aligned} \quad (31)$$

Using Theorem 9, we get the desired conclusion. \square

Remark 12. In [13, 15], during considering the qualitative properties of the impulsive differential equations, the functions $f(t, x)$ and $I_i(x)$ are defined in the domain

$$D = \{(t, x) : t \in [t_0, T], T \leq \infty, \|x\| \leq h\} \quad (32)$$

for some $h > 0$. However, we do not add such a restriction since our conclusion in Theorem 11 does not involve $\|x\|$.

Corollary 13. *Let the right-hand side of the initial value problem (28) satisfy the following conditions:*

- (i) $\|f(t, x)\| \leq b\|x\|$, b is a constant and $b \geq 0$;
- (ii) $\|I_i(x(t_i^-))\| \leq \gamma_i \|x(t_i^-)\|$, γ_i are constants and $\gamma_i \geq 0$;
- (iii) *there exists a constant ξ such that $S^*(t, t_i) \leq \xi < \infty$, $t > t_i$, $i = 0, 1, 2, \dots, k$. Then all solutions of the problem (28) are bounded, and the trivial solution of the problem (28) is stable in the sense of Lyapunov stability.*

Corollary 14. *Suppose that*

- (i) $\|f(t, x) - f(t, y)\| \leq b\|x - y\|^l$, $0 < l < 1$, for all $x, y \in PC^1$;
- (ii) $\|I_i(x) - I_i(y)\| \leq \gamma_i \|x - y\|$, where b, γ_i are constants, $b \geq 0, \gamma_i \geq 0, 1 \leq i \leq k$;
- (iii) *there exists a constant ξ such that $S^*(t, t_i) \leq \xi < \infty, t > t_i, i = 0, 1, 2, \dots, k$. Then all solutions of the problem (28) are bounded.*

Conflict of Interests

The author declares that there is no conflict of interests.

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