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Research Article

Generalized Mixed Equilibria, Variational Inclusions, and Fixed Point Problems

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We propose two iterative algorithms for finding a common element of the set of solutions of finite generalized mixed equilibrium problems, the set of solutions of finite variational inclusions for maximal monotone and inverse strong monotone mappings, and the set of common fixed points of infinite nonexpansive mappings and an asymptotically κ-strict pseudocontractive mapping in the intermediate sense in a real Hilbert space. We prove some strong and weak convergence theorems for the proposed iterative algorithms under suitable conditions.

1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, let C be a nonempty closed convex subset of H, and let P_C be the metric projection of H onto C. Let $S : C \to C$ be a self-mapping on C. $\mathrm{Fix}(S)$ is the set of fixed points of S and $\mathbf{R}$ is the set of all real numbers. We recall that a mapping $A : C \to H$ is said to be L-Lipschitz continuous, if there exists a constant $L \geq 0$ such that

$$\|Ax - Ay\| \leq L\|x - y\|, \quad \forall x, y \in C. \tag{1}$$

If $L = 1$, then A is called a nonexpansive mapping and if $L \in [0, 1)$, then A is called a contraction. We also let I denote the identity operator on the Hilbert space H.

Let $\varphi : C \to \mathbf{R}$ be a real-valued function, let $A : C \to H$ be a nonlinear mapping, and let $F : C \times C \to \mathbf{R}$ be a bifunction. The generalized mixed equilibrium problem (GMEP) introduced in [1] is to find $x \in C$ such that

$$F(x, y) + \varphi(y) - \varphi(x) + \langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{2}$$

We denote the set of solutions of GMEP (2) by $\mathrm{GMEP}(F, \varphi, A)$. The GMEP covers many problems [2–6] as special cases and has been extensively studied recently.

Throughout this paper, we assume as in [1] that $F : C \times C \to \mathbf{R}$ is a bifunction satisfying conditions (A1)–(A4) and $\varphi : C \to \mathbf{R}$ is a lower semicontinuous and convex function with restriction (B1) or (B2), where

(A1) $F(x, x) = 0$, for all $x \in C$;

(A2) F is monotone; that is, $F(x, y) + F(y, x) \leq 0$, for any $x, y \in C$;

(A3) F is upper-hemicontinuous; that is, for each $x, y, z \in C$,

$$\limsup_{t \to 0^+} F(tz + (1 - t)x, y) \leq F(x, y); \tag{3}$$

(A4) $F(x, \cdot)$ is convex and lower semicontinuous, for each $x \in C$;

(B1) for each $x \in H$ and $r > 0$, there exists a bounded subset $D_x \subset C$ and $y_x \in C$ such that, for any $z \in C \setminus D_x$,

$$F(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0; \tag{4}$$

(B2) C is a bounded set.

Next we list some elementary consequences for the mixed equilibrium problem studied in [2] where $\mathrm{MEP}(F, \varphi)$ is the solution set.

Proposition 1 (see [2]). *Assume that $F : C \times C \to \mathbf{R}$ satisfies (A1)–(A4) and let $\varphi : C \to \mathbf{R}$ be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. For $r > 0$ and $x \in H$, define a mapping $T_r^{(F,\varphi)} : H \to C$ as follows:*

$$T_r^{(F,\varphi)}(x) = \Big\{z \in C : F(z, y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C\Big\}, \tag{5}$$

for all $x \in H$. Then the following hold:

(i) *for each $x \in H$, $T_r^{(F,\varphi)}(x) \neq \emptyset$;*

(ii) *$T_r^{(F,\varphi)}$ is single-valued;*

(iii) *$T_r^{(F,\varphi)}$ is firmly nonexpansive; that is, for any $x, y \in H$,*

$$\left\|T_r^{(F,\varphi)}x - T_r^{(F,\varphi)}y\right\|^2 \leq \left\langle T_r^{(F,\varphi)}x - T_r^{(F,\varphi)}y, x - y\right\rangle; \tag{6}$$

(iv) *$\mathrm{Fix}(T_r^{(F,\varphi)}) = \mathrm{MEP}(F, \varphi)$;*

(v) *$\mathrm{MEP}(F, \varphi)$ is closed and convex.*

Next, recall some concepts.

Definition 2. Let C be a nonempty subset of a normed space X and let $S : C \to C$ be a self-mapping on C.

(i) S is asymptotically nonexpansive (see [7]), if there exists a sequence $\{k_n\}$ of positive numbers satisfying the property $\lim_{n\to\infty} k_n = 1$ and

$$\|S^n x - S^n y\| \leq k_n \|x - y\|, \quad \forall n \geq 1,\ \forall x, y \in C. \tag{7}$$

(ii) S is asymptotically nonexpansive in the intermediate sense (see [8]) provided that S is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} \left(\|S^n x - S^n y\| - \|x - y\|\right) \leq 0. \tag{8}$$

(iii) S is uniformly Lipschitzian, if there exists a constant $\mathscr{L} > 0$ such that

$$\|S^n x - S^n y\| \leq \mathscr{L} \|x - y\|, \quad \forall n \geq 1,\ \forall x, y \in C. \tag{9}$$

It is clear that every nonexpansive mapping is asymptotically nonexpansive and every asymptotically nonexpansive mapping is uniformly Lipschitzian. Recently, Kim and Xu [9] introduced the concept of asymptotically κ-strict pseudocontractive mappings in a Hilbert space as follows.

Definition 3. Let C be a nonempty subset of a Hilbert space H. A mapping $S : C \to C$ is said to be an asymptotically κ-strict pseudocontractive mapping with sequence $\{\gamma_n\}$, if there exist a constant $\kappa \in [0, 1)$ and a sequence $\{\gamma_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty}\gamma_n = 0$ such that

$$\|S^n x - S^n y\|^2 \leq (1 + \gamma_n)\|x - y\|^2 + \kappa\|x - S^n x - (y - S^n y)\|^2, \quad \forall n \geq 1, \quad \forall x, y \in C. \tag{10}$$

They studied weak and strong convergence theorems for this class of mappings. It is important to note that every asymptotically κ-strict pseudocontractive mapping with sequence $\{\gamma_n\}$ is a uniformly $\mathscr{L}$-Lipschitzian mapping with $\mathscr{L} = \sup\{(\kappa + \sqrt{1 + (1 - \kappa)\gamma_n})/(1 + \kappa) : n \geq 1\}$.

Recently, Sahu et al. [10] considered the concept of asymptotically κ-strict pseudocontractive mappings in the intermediate sense, which are not necessarily Lipschitzian.

Definition 4. Let C be a nonempty subset of a Hilbert space H. A mapping $S : C \to C$ is said to be an asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$, if there exist a constant $\kappa \in [0, 1)$ and a sequence $\{\gamma_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty}\gamma_n = 0$ such that

$$\limsup_{n\to\infty} \sup_{x,y\in C} \left(\|S^n x - S^n y\|^2 - (1 + \gamma_n)\|x - y\|^2 - \kappa\|x - S^n x - (y - S^n y)\|^2\right) \leq 0. \tag{11}$$

Put $c_n := \max\{0, \sup_{x,y\in C}(\|S^n x - S^n y\|^2 - (1+\gamma_n)\|x - y\|^2 - \kappa\|x - S^n x - (y - S^n y)\|^2)\}$. Then, $c_n \geq 0$ $(\forall n \geq 1)$, $c_n \to 0$ $(n \to \infty)$, and (11) reduces to the relation

$$\|S^n x - S^n y\|^2 \leq (1 + \gamma_n)\|x - y\|^2 + \kappa\|x - S^n x - (y - S^n y)\|^2 + c_n, \quad \forall n \geq 1,\ \forall x, y \in C. \tag{12}$$

Whenever $c_n = 0$, for all $n \geq 1$ in (12), then S is an asymptotically κ-strict pseudocontractive mapping with sequence $\{\gamma_n\}$.

Let B be a single-valued mapping of C into H and R be a multivalued mapping with domain $D(R) = C$. Consider the following variational inclusion: find a point $x \in C$ such that

$$0 \in Bx + Rx. \tag{13}$$

We denote by $I(B, R)$ the solution set of the variational inclusion (13). In 1998, Huang [11] studied problem (13) in the case where R is maximal monotone and B is strongly monotone and Lipschitz continuous with $D(R) = C = H$. Subsequently, Zeng et al. [12] further studied problem (13) in the case which is more general than that in [11]. Moreover, the authors [12] obtained the same strong convergence conclusion as in [11]. In addition, the authors also gave the geometric convergence rate estimate for approximate solutions.

In this paper, inspired by the research work mentioned above, we introduce two iterative algorithms for finding a common element of the set of solutions of finite generalized mixed equilibrium problems, the set of solutions of finite variational inclusions for maximal monotone and inverse strong monotone mappings, and the set of common fixed points of infinite nonexpansive mappings and an asymptotically κ-strict pseudocontractive mapping in the intermediate sense in a real Hilbert space. We prove some strong and weak convergence theorems for the proposed iterative algorithms under mild conditions.

2. Preliminaries

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. We use the notation $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges strongly to x. Moreover, we use $\omega_w(x_n)$ to denote the weak ω-limit set of the sequence $\{x_n\}$; that is,

$$\omega_w(x_n) := \left\{x \in H : x_{n_i} \rightharpoonup x \text{ for some subsequence } \left\{x_{n_i}\right\} \text{ of } \{x_n\}\right\}. \tag{14}$$

Recall that a mapping $A : C \to H$ is called

(i) Monotone, if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C; \tag{15}$$

(ii) η-strongly monotone, if there exists a constant $\eta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \eta \|x - y\|^2, \quad \forall x, y \in C; \tag{16}$$

(iii) α-inverse strongly monotone, if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{17}$$

When A is an α-inverse strongly monotone mapping of C into H, it is easy to see that A is $(1/\alpha)$-Lipschitz continuous. We also have that, for all $u, v \in C$ and $\lambda > 0$,

$$\begin{aligned} &\|(I - \lambda A)u - (I - \lambda A)v\|^2 \\ &\quad = \|(u - v) - \lambda(Au - Av)\|^2 \\ &\quad = \|u - v\|^2 - 2\lambda\langle Au - Av, u - v\rangle + \lambda^2\|Au - Av\|^2 \\ &\quad \leq \|u - v\|^2 + \lambda(\lambda - 2\alpha)\|Au - Av\|^2. \end{aligned} \tag{18}$$

So, if $\lambda \leq 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping from C to H.

The metric projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \inf_{y \in C}\|x - y\| =: d(x, C). \tag{19}$$

Some important properties of projections are listed in the following proposition.

Proposition 5. *For given $x \in H$ and $z \in C$,*

(i) $z = P_C x \Leftrightarrow \langle x - z, y - z\rangle \leq 0, \forall y \in C$;

(ii) $z = P_C x \Leftrightarrow \|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2, \forall y \in C$;

(iii) $\langle P_C x - P_C y, x - y\rangle \geq \|P_C x - P_C y\|^2, \forall y \in H$.

Consequently, P_C is nonexpansive and monotone. It is easy to see that the projection P_C is 1-ism.

We need some facts and tools in a real Hilbert space H which are listed as lemmas below.

Lemma 6. *Let X be a real inner product space. Then there holds the following inequality:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in X. \tag{20}$$

Lemma 7. *Let H be a real Hilbert space. Then the following hold:*

(a) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle$, *for all* $x, y \in H$;

(b) $\|\lambda x + \mu y + \nu z\|^2 = \lambda\|x\|^2 + \mu\|y\|^2 + \nu\|z\|^2 - \lambda\mu\|x - y\|^2 - \mu\nu\|y - z\|^2 - \lambda\nu\|x - z\|^2$, *for all* $x, y, z \in H$ *and* $\lambda, \mu, \nu \in [0, 1]$ *with* $\lambda + \mu + \nu = 1$;

(c) *if $\{x_n\}$ is a sequence in H such that $x_n \rightharpoonup x$, it follows that*

$$\limsup_{n\to\infty}\|x_n - y\|^2 = \limsup_{n\to\infty}\|x_n - x\|^2 + \|x - y\|^2, \quad \forall y \in H. \tag{21}$$

Lemma 8 (see [10, Lemma 2.5]). *Let H be a real Hilbert space. Given a nonempty closed convex subset C of H and points $x, y, z \in H$ and given also a real number $a \in \mathbf{R}$, the set*

$$\left\{v \in C : \|y - v\|^2 \leq \|x - v\|^2 + \langle z, v\rangle + a\right\} \tag{22}$$

is convex and closed.

Lemma 9 (see [10, Lemma 2.6]). *Let C be a nonempty subset of a Hilbert space H and let $S : C \to C$ be an asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$. Then*

$$\begin{aligned} &\|S^n x - S^n y\| \\ &\leq \frac{1}{1 - \kappa} \\ &\quad \times \left(\kappa\|x - y\| + \sqrt{(1 + (1 - \kappa)\gamma_n)\|x - y\|^2 + (1 - \kappa)c_n}\right), \end{aligned} \tag{23}$$

for all $x, y \in C$ and $n \geq 1$.

Lemma 10 (see [10, Lemma 2.7]). *Let C be a nonempty subset of a Hilbert space H and let $S : C \to C$ be a uniformly continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$. Let $\{x_n\}$ be a sequence in C such that $\|x_n - x_{n+1}\| \to 0$ and $\|x_n - S^n x_n\| \to 0$ as $n \to \infty$. Then $\|x_n - Sx_n\| \to 0$ as $n \to \infty$.*

Lemma 11 (demiclosedness principle [10, Proposition 3.1]). *Let C be a nonempty closed convex subset of a Hilbert space H and let $S : C \to C$ be a continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$. Then $I - S$ is demiclosed at zero in the sense that if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x \in C$ and* $\limsup_{m\to\infty}\limsup_{n\to\infty}\|x_n - S^m x_n\| = 0$, *then* $(I - S)x = 0$.

Lemma 12 (see [10, Proposition 3.2]). *Let C be a nonempty closed convex subset of a Hilbert space H and let $S : C \to C$ be a continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$ such that* $\mathrm{Fix}(S) \neq \emptyset$. *Then,* $\mathrm{Fix}(S)$ *is closed and convex.*

Lemma 13 (see [13, page 80]). *Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, and $\{\delta_n\}_{n=1}^{\infty}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1+\delta_n) a_n + b_n, \quad \forall n \geq 1. \tag{24}$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. If, in addition, $\{a_n\}_{n=1}^{\infty}$ has a subsequence which converges to zero, then $\lim_{n\to\infty} a_n = 0$.

Corollary 14 (see [14, page 303]). *Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq a_n + b_n, \quad \forall n \geq 0. \tag{25}$$

If $\sum_{n=0}^{\infty} b_n$ converges, then $\lim_{n\to\infty} a_n$ exists.

Recall that a Banach space X is said to satisfy the Opial condition [15], if for any given sequence $\{x_n\} \subset X$ which converges weakly to an element $x \in X$, there holds the inequality

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \quad \forall y \in X,\ y \neq x. \tag{26}$$

It is well known in [15] that every Hilbert space H satisfies the Opial condition.

Lemma 15 (see [16, Proposition 3.1]). *Let C be a nonempty closed convex subset of a real Hilbert space H and let $\{x_n\}$ be a sequence in H. Suppose that*

$$\|x_{n+1} - p\|^2 \leq (1+\lambda_n) \|x_n - p\|^2 + \delta_n, \quad \forall p \in C,\ n \geq 1, \tag{27}$$

where $\{\lambda_n\}$ and $\{\delta_n\}$ are sequences of nonnegative real numbers such that $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $\sum_{n=1}^{\infty} \delta_n < \infty$. Then $\{P_C x_n\}$ converges strongly in C.

Lemma 16 (see [17]). *Let C be a closed convex subset of a real Hilbert space H. Let $\{x_n\}$ be a sequence in H and $u \in H$. Let $q = P_C u$. If $\{x_n\}$ is such that $\omega_w(x_n) \subset C$ and satisfies the condition*

$$\|x_n - u\| \leq \|u - q\|, \quad \forall n, \tag{28}$$

then $x_n \to q$ as $n \to \infty$.

Let $\{T_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive self-mappings on C and let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of nonnegative numbers in $[0,1]$. *For any $n \geq 1$, define a self-mapping W_n on C as follows:*

$$\begin{aligned}
U_{n,n+1} &= I, \\
U_{n,n} &= \lambda_n T_n U_{n,n+1} + (1-\lambda_n) I, \\
U_{n,n-1} &= \lambda_{n-1} T_{n-1} U_{n,n} + (1-\lambda_{n-1}) I, \\
&\ \ \vdots \\
U_{n,k} &= \lambda_k T_k U_{n,k+1} + (1-\lambda_k) I, \\
U_{n,k-1} &= \lambda_{k-1} T_{k-1} U_{n,k} + (1-\lambda_{k-1}) I, \\
&\ \ \vdots \\
U_{n,2} &= \lambda_2 T_2 U_{n,3} + (1-\lambda_2) I, \\
W_n &= U_{n,1} = \lambda_1 T_1 U_{n,2} + (1-\lambda_1) I.
\end{aligned} \tag{29}$$

Such a mapping W_n is called the W-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$.

Lemma 17 (see [18, Lemma 3.2]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C such that $\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \neq \emptyset$ and let $\{\lambda_n\}$ be a sequence in $(0,b]$ for some $b \in (0,1)$. Then, for every $x \in C$ and $k \geq 1$, the limit $\lim_{n\to\infty} U_{n,k} x$ exists.*

Utilizing Lemma 17, we can define a mapping $W : H \to H$ as follows:

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1} x, \quad \forall x \in H. \tag{30}$$

Such a W is called the W-mapping generated by $T_1, T_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$. We remark that since W_n is nonexpansive, $W : H \to H$ is also nonexpansive.

Lemma 18 (see [18, Lemma 3.3]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C such that $\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \neq \emptyset$, and let $\{\lambda_n\}$ be a sequence in $(0,b]$, for some $b \in (0,1)$. Then,* $\mathrm{Fix}(W) = \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$.

Lemma 19 (see [19, demiclosedness principle]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let T be a nonexpansive self-mapping on C. Then $I - T$ is demiclosed. That is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I-T)x_n\}$ strongly converges to some y, it follows that $(I-T)x = y$.*

Recall that a set-valued mapping $R : D(R) \subset H \to 2^H$ is called monotone, if, for all $x, y \in D(R)$, $f \in R(x)$, and $g \in R(y)$, we have

$$\langle f - g, x - y \rangle \geq 0. \tag{31}$$

A set-valued mapping R is called maximal monotone, if R is monotone and $(I + \lambda R) D(R) = H$, for each $\lambda > 0$. We denote by $G(R)$ the graph of R. It is known that a monotone mapping R is maximal if and only if, for $(x, f) \in H\times H$, $\langle f-g, x-y\rangle \geq 0$, for every $(y, g) \in G(R)$, we have $f \in R(x)$.

Assume that $R : D(R) \subset H \rightarrow 2^H$ is a maximal monotone mapping. Then, for $\lambda > 0$, associated with R, the resolvent operator $J_{R,\lambda}$ can be defined as

$$J_{R,\lambda}x = (I + \lambda R)^{-1}x, \quad \forall x \in H. \tag{32}$$

We have the following property for the resolvent operator $J_{R,\lambda} : H \rightarrow \overline{D(R)}$.

Lemma 20 (see [11]). *$J_{R,\lambda}$ is single-valued and firmly nonexpansive; that is,*

$$\langle J_{R,\lambda}x - J_{R,\lambda}y, x-y\rangle \geq \left\|J_{R,\lambda}x - J_{R,\lambda}y\right\|^2, \quad \forall x, y \in H. \tag{33}$$

Consequently, $J_{R,\lambda}$ is nonexpansive and monotone.

Lemma 21 (see [20]). *Let R be a maximal monotone mapping with $D(R) = C$. Then, for any given $\lambda > 0$, $u \in C$ is a solution of the variational inclusion: find a point $x \in C$ such that*

$$0 \in Bx + Rx, \tag{34}$$

where B is a single-valued mapping of C into H if and only if $u \in C$ satisfies

$$u = J_{R,\lambda}(u - \lambda Bu). \tag{35}$$

Lemma 22 (see [12]). *Let R be a maximal monotone mapping with $D(R) = C$ and let $B : C \rightarrow H$ be a strongly monotone, continuous, and single-valued mapping. Then, for each $z \in H$, the equation $z \in (B+\lambda R)x$ has a unique solution x_λ, for $\lambda > 0$.*

Lemma 23 (see [20]). *Let R be a maximal monotone mapping with $D(R) = C$ and let $B : C \rightarrow H$ be a monotone, continuous, and single-valued mapping. Then $(I+\lambda(R+B))C = H$, for each $\lambda > 0$. In this case, $R + B$ is maximal monotone.*

3. Strong Convergence Theorem

In this section, we prove a strong convergence theorem for an iterative algorithm for finding a common element of the set of solutions of finite generalized mixed equilibrium problems, the set of solutions of finite variational inclusions for maximal monotone and inverse strong monotone mappings, and the set of common fixed points of infinite nonexpansive mappings and asymptotically κ-strict pseudocontractive mapping $S : C \rightarrow C$ in the intermediate sense in a real Hilbert space. This iterative algorithm is based on the extragradient method [21], Mann-type iterative method, and shrinking projection method. For more recent related results, see [22] and the references therein.

Theorem 24. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let M, N be two integers. Let F_k be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (A1)–(A4) and let $\varphi_k : C \rightarrow \mathbf{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function, where $k \in \{1, 2, \ldots, M\}$. Let $R_i : C \rightarrow 2^H$ be a maximal monotone mapping and let $A_k : H \rightarrow H$ and $B_i : C \rightarrow H$ be μ_k-inverse strongly monotone and η_i-inverse strongly monotone, respectively, where $k \in \{1, 2, \ldots, M\}$ and $i \in \{1, 2, \ldots, N\}$. Let $S : C \rightarrow C$ be a uniformly continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense, for some $0 \leq \kappa < 1$, with sequence $\{\gamma_n\} \subset [0, \infty)$ such that $\lim_{n\rightarrow\infty}\gamma_n = 0$ and $\{c_n\} \subset [0, \infty)$ such that $\lim_{n\rightarrow\infty}c_n = 0$. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C and let $\{\lambda_n\}$ be a sequence in $(0, b]$ for some $b \in (0, 1)$. Assume that $F := \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \cap \bigcap_{k=1}^{M} \mathrm{GMEP}(F_k, \varphi_k, A_k) \cap \bigcap_{i=1}^{N} I(B_i, R_i) \cap \mathrm{Fix}(S)$ is nonempty and bounded and that either (B1) or (B2) holds. Let $\{\alpha_n\}, \{\beta_n\}$, and $\{\delta_n\}$ be sequences in $[0, 1]$ such that $\alpha_n + \beta_n \leq 1$, $0 < a \leq \alpha_n \leq 1$, and $\kappa \leq \delta_n \leq d < 1$. Pick any $x_0 \in H$ and set $C_1 = C, x_1 = P_{C_1}x_0$. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\begin{aligned}
u_n &= T_{r_{M,n}}^{(F_M,\varphi_M)}\left(I - r_{M,n}A_M\right)T_{r_{M-1,n}}^{(F_{M-1},\varphi_{M-1})} \\
&\quad \times \left(I - r_{M-1,n}A_{M-1}\right)\cdots T_{r_{1,n}}^{(F_1,\varphi_1)}\left(I - r_{1,n}A_1\right)x_n, \\
z_n &= J_{R_N,\lambda_{N,n}}\left(I - \lambda_{N,n}B_N\right)J_{R_{N-1},\lambda_{N-1,n}} \\
&\quad \times \left(I - \lambda_{N-1,n}B_{N-1}\right)\cdots J_{R_1,\lambda_{1,n}}\left(I - \lambda_{1,n}B_1\right)u_n, \\
k_n &= \delta_n z_n + \left(1 - \delta_n\right)S^n z_n, \\
y_n &= \left(1 - \alpha_n - \beta_n\right)x_n + \alpha_n k_n + \beta_n W_n z_n, \\
C_{n+1} &= \left\{z \in C_n : \left\|y_n - z\right\|^2 \leq \left\|x_n - z\right\|^2 + \theta_n\right\}, \\
x_{n+1} &= P_{C_{n+1}}x_0, \quad \forall n \geq 0,
\end{aligned} \tag{36}$$

where W_n is the W-mapping generated by (2.2), $\theta_n = \gamma_n\Delta_n^2 + c_n$, and $\Delta_n = \sup\{\|x_n - p\| : p \in F\} < \infty$. Assume that the following conditions hold:

(i) $0 < \liminf_{n\rightarrow\infty}\beta_n \leq \limsup_{n\rightarrow\infty}(\alpha_n + \beta_n) < 1$;

(ii) $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i), \forall i \in \{1, 2, \ldots, N\}$;

(iii) $\{r_{k,n}\} \subset [e_k, f_k] \subset (0, 2\mu_k), \forall k \in \{1, 2, \ldots, M\}$.

Then $\{x_n\}$ converges strongly to $P_F x_0$.

Proof. We divide the proof into several steps.

Step 1. First note that the defining inequality in C_n is equivalent to the inequality

$$\left\langle 2\left(x_n - z_n\right), z\right\rangle \leq \left\|x_n\right\|^2 - \left\|z_n\right\|^2 + \theta_n. \tag{37}$$

So, by Lemma 8, C_n is closed and convex, for every $n \geq 1$.

We next show that $F \subset C_n$, for all $n \geq 1$. Put

$$\begin{aligned}
\Theta_n^k &= T_{r_{k,n}}^{(F_k,\varphi_k)}\left(I - r_{k,n}A_k\right)T_{r_{k-1,n}}^{(F_{k-1},\varphi_{k-1})} \\
&\quad \times \left(I - r_{k-1,n}A_{k-1}\right)\cdots T_{r_{1,n}}^{(F_1,\varphi_1)}\left(I - r_{1,n}A_1\right)x_n,
\end{aligned} \tag{38}$$

for all $k \in \{1, 2, \ldots, M\}$ and $n \geq 1$;

$$\begin{aligned}
\Omega_n^i &= J_{R_i,\lambda_{i,n}}\left(I - \lambda_{i,n}B_i\right)J_{R_{i-1},\lambda_{i-1,n}} \\
&\quad \times \left(I - \lambda_{i-1,n}B_{i-1}\right)\cdots J_{R_1,\lambda_{1,n}}\left(I - \lambda_{1,n}B_1\right),
\end{aligned} \tag{39}$$

for all $i \in \{1, 2, \ldots, N\}$ and $n \geq 1$, and $\Theta_n^0 = \Omega_n^0 = I$, where I is the identity mapping on H. Then we have that $u_n = \Theta_n^M x_n$ and $z_n = \Omega_n^N u_n$. Suppose that $F \subset C_n$ for some $n \geq 1$. Take $p \in F$ arbitrarily. Then, from (18) and Proposition 1(iii), we have

$$\begin{aligned}
\|u_n - p\| &= \Big\| T_{r_{M,n}}^{(F_M,\varphi_M)} \left(I - r_{M,n} A_M\right) \Theta_n^{M-1} x_n \\
&\qquad - T_{r_{M,n}}^{(F_M,\varphi_M)} \left(I - r_{M,n} A_M\right) \Theta_n^{M-1} p \Big\| \\
&\leq \left\| \left(I - r_{M,n} A_M\right) \Theta_n^{M-1} x_n - \left(I - r_{M,n} A_M\right) \Theta_n^{M-1} p \right\| \\
&\leq \left\| \Theta_n^{M-1} x_n - \Theta_n^{M-1} p \right\| \\
&\ \ \vdots \\
&\leq \left\| \Theta_n^0 x_n - \Theta_n^0 p \right\| \\
&= \|x_n - p\|.
\end{aligned} \tag{40}$$

Similarly, we have

$$\begin{aligned}
\|z_n - p\| &= \Big\| J_{R_N,\lambda_{N,n}} \left(I - \lambda_{N,n} B_N\right) \Omega_n^{N-1} u_n \\
&\qquad - J_{R_N,\lambda_{N,n}} \left(I - \lambda_{N,n} B_N\right) \Omega_n^{N-1} p \Big\| \\
&\leq \left\| \left(I - \lambda_{N,n} B_N\right) \Omega_n^{N-1} u_n - \left(I - \lambda_{N,n} B_N\right) \Omega_n^{N-1} p \right\| \\
&\leq \left\| \Omega_n^{N-1} u_n - \Omega_n^{N-1} p \right\| \\
&\ \ \vdots \\
&\leq \left\| \Omega_n^0 x_n - \Omega_n^0 p \right\| \\
&= \|u_n - p\|.
\end{aligned} \tag{41}$$

Combining (40) and (41), we have

$$\|z_n - p\| \leq \|x_n - p\|. \tag{42}$$

By Lemma 7(b), we deduce from (36) and (42) that

$$\begin{aligned}
\|k_n - p\|^2 &= \left\| \delta_n (z_n - p) + (1 - \delta_n)(S^n z_n - p) \right\|^2 \\
&= \delta_n \|z_n - p\|^2 + (1 - \delta_n) \|S^n z_n - p\|^2 \\
&\quad - \delta_n (1 - \delta_n) \|z_n - S^n z_n\|^2 \\
&\leq \delta_n \|z_n - p\|^2 + (1 - \delta_n) \\
&\quad \times \left[(1 + \gamma_n) \|z_n - p\|^2 + \kappa \|z_n - S^n z_n\|^2 + c_n \right] \\
&\quad - \delta_n (1 - \delta_n) \|z_n - S^n z_n\|^2 \\
&= \left[1 + \gamma_n (1 - \delta_n)\right] \|z_n - p\|^2 \\
&\quad + (1 - \delta_n)(\kappa - \delta_n) \|z_n - S^n z_n\|^2 + (1 - \delta_n) c_n \\
&\leq (1 + \gamma_n) \|z_n - p\|^2 \\
&\quad + (1 - \delta_n)(\kappa - \delta_n) \|z_n - S^n z_n\|^2 + c_n \\
&\leq (1 + \gamma_n) \|z_n - p\|^2 + c_n.
\end{aligned} \tag{43}$$

It follows from (42), (43), and the convexity of $\|\cdot\|^2$ that

$$\begin{aligned}
\|y_n - p\|^2 &= \big\| (1 - \alpha_n - \beta_n)(x_n - p) + \alpha_n (k_n - p) \\
&\qquad + \beta_n (W_n z_n - p) \big\|^2 \\
&\leq \big\| (1 - \alpha_n - \beta_n) \|x_n - p\|^2 + \alpha_n \|k_n - p\|^2 \\
&\quad + \beta_n \|W_n z_n - p\|^2 \\
&\leq (1 - \alpha_n - \beta_n) \|x_n - p\|^2 \\
&\quad + \alpha_n \left((1 + \gamma_n) \|z_n - p\|^2 + c_n \right) + \beta_n \|z_n - p\|^2 \\
&\leq (1 - \alpha_n - \beta_n) \|x_n - p\|^2 \\
&\quad + \alpha_n \left((1 + \gamma_n) \|x_n - p\|^2 + c_n \right) + \beta_n \|x_n - p\|^2 \\
&= (1 - \alpha_n) \|x_n - p\|^2 \\
&\quad + \alpha_n \left((1 + \gamma_n) \|x_n - p\|^2 + c_n \right) \\
&= \|x_n - p\|^2 + \alpha_n \gamma_n \|x_n - p\|^2 + \alpha_n c_n \\
&\leq \|x_n - p\|^2 + \gamma_n \|x_n - p\|^2 + c_n \\
&\leq \|x_n - p\|^2 + \theta_n.
\end{aligned} \tag{44}$$

Hence, $p \in C_{n+1}$. This implies that $F \subset C_n$, for all $n \geq 1$.

Step 2. We prove that $\|x_n - k_n\| \to 0$ as $n \to \infty$.

Indeed, let $v = P_F x_0$. From $x_n = P_{C_n} x_0$ and $v \in F \subset C_n$, we obtain

$$\|x_n - x_0\| \leq \|v - x_0\|. \tag{45}$$

This implies that $\{x_n\}$ is bounded and, hence, $\{u_n\}$, $\{z_n\}$, $\{k_n\}$, and $\{y_n\}$ are also bounded. Since $x_{n+1} \in C_{n+1} \subset C_n$ and $x_n = P_{C_n} x_0$, we have

$$\|x_n - x_0\| \leq \|x_{n+1} - x_0\|, \quad \forall n \geq 1. \tag{46}$$

Therefore, $\lim_{n \to \infty} \|x_n - x_0\|$ exists. From $x_n = P_{C_n} x_0$, $x_{n+1} \in C_{n+1} \subset C_n$, by Proposition 5(ii) we obtain

$$\|x_{n+1} - x_n\|^2 \leq \|x_0 - x_{n+1}\|^2 - \|x_0 - x_n\|^2, \tag{47}$$

which implies

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{48}$$

It follows from $x_{n+1} \in C_{n+1}$ that $\|y_n - x_{n+1}\|^2 \le \|x_n - x_{n+1}\|^2 + \theta_n$ and hence

$$\begin{aligned}\|x_n - y_n\|^2 &\le 2\left(\|x_n - x_{n+1}\|^2 + \|x_{n+1} - y_n\|^2\right)\\ &\le 2\left(\|x_n - x_{n+1}\|^2 + \|x_n - x_{n+1}\|^2 + \theta_n\right)\\ &= 2\left(2\|x_n - x_{n+1}\|^2 + \theta_n\right).\end{aligned} \tag{49}$$

From (48) and $\lim_{n\to\infty}\theta_n = 0$, we have

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{50}$$

Also, utilizing Lemma 7(b), we obtain from (36), (42), and (43) that

$$\begin{aligned}\|y_n - p\|^2 = {} & \|(1-\alpha_n-\beta_n)(x_n - p) + \alpha_n(k_n - p)\\ & + \beta_n(W_n z_n - p)\|^2\\ \le {} & (1-\alpha_n-\beta_n)\|x_n - p\|^2 + \alpha_n\|k_n - p\|^2\\ & + \beta_n\|W_n z_n - p\|^2\\ & - \beta_n(1-\alpha_n-\beta_n)\|x_n - W_n z_n\|^2\\ \le {} & (1-\alpha_n-\beta_n)\|x_n - p\|^2\\ & + \alpha_n\left[(1+\gamma_n)\|z_n - p\|^2 + c_n\right] + \beta_n\|z_n - p\|^2\\ & - \beta_n(1-\alpha_n-\beta_n)\|x_n - W_n z_n\|^2\\ \le {} & (1-\alpha_n-\beta_n)\|x_n - p\|^2\\ & + \alpha_n\left[(1+\gamma_n)\|x_n - p\|^2 + c_n\right] + \beta_n\|x_n - p\|^2\\ & - \beta_n(1-\alpha_n-\beta_n)\|x_n - W_n z_n\|^2\\ = {} & (1-\alpha_n)\|x_n - p\|^2\\ & + \alpha_n\left[(1+\gamma_n)\|x_n - p\|^2 + c_n\right]\\ & - \beta_n(1-\alpha_n-\beta_n)\|x_n - W_n z_n\|^2\\ = {} & \|x_n - p\|^2 + \alpha_n\gamma_n\|x_n - p\|^2 + \alpha_n c_n\\ & - \beta_n(1-\alpha_n-\beta_n)\|x_n - W_n z_n\|^2\\ \le {} & \|x_n - p\|^2 + \theta_n - \beta_n(1-\alpha_n-\beta_n)\|x_n - W_n z_n\|^2,\end{aligned} \tag{51}$$

which, hence, yields

$$\begin{aligned}&\beta_n(1-\alpha_n-\beta_n)\|x_n - W_n z_n\|^2\\ &\quad\le \|x_n - p\|^2 - \|y_n - p\|^2 + \theta_n\\ &\quad\le \|x_n - y_n\|\left(\|x_n - p\| + \|y_n - p\|\right) + \theta_n.\end{aligned} \tag{52}$$

Since $\theta_n \to 0$ and $\{x_n\}$, $\{y_n\}$ are bounded, it follows from (50) and condition (i) that

$$\lim_{n\to\infty} \|x_n - W_n z_n\| = 0. \tag{53}$$

Note that

$$y_n - x_n = \alpha_n(k_n - x_n) + \beta_n(W_n z_n - x_n), \tag{54}$$

which leads to

$$\begin{aligned}\alpha_n\|k_n - x_n\| &= \|y_n - x_n - \beta_n(W_n z_n - x_n)\|\\ &\le \|y_n - x_n\| + \beta_n\|W_n z_n - x_n\|\\ &\le \|y_n - x_n\| + \|W_n z_n - x_n\|.\end{aligned} \tag{55}$$

So, from (50), (53), and $0 < a \le \alpha_n \le 1$, we get

$$\lim_{n\to\infty} \|x_n - k_n\| = 0. \tag{56}$$

Step 3. We prove that $\|x_n - u_n\| \to 0$, $\|u_n - z_n\| \to 0$, $\|z_n - W_n z_n\| \to 0$, and $\|z_n - S^n z_n\| \to 0$ as $n \to \infty$.

Indeed, from (42) and (43), it follows that

$$\begin{aligned}\|k_n - p\|^2 \le {} & [1 + \gamma_n(1-\delta_n)]\|z_n - p\|^2\\ & + (1-\delta_n)(\kappa - \delta_n)\|z_n - S^n z_n\|^2 + (1-\delta_n)c_n\\ \le {} & \|z_n - p\|^2 + \gamma_n\|z_n - p\|^2 + c_n\\ \le {} & \|z_n - p\|^2 + \gamma_n\|x_n - p\|^2 + c_n\\ \le {} & \|z_n - p\|^2 + \theta_n.\end{aligned} \tag{57}$$

Next, we prove that

$$\lim_{n\to\infty} \left\|\Theta_n^k x_n - \Theta_n^{k-1} x_n\right\| = 0, \quad k = 1, 2, \ldots, M. \tag{58}$$

For $p \in F$, it follows from (18) that

$$\begin{aligned}\left\|\Theta_n^k x_n - p\right\|^2 = {} & \Big\|T_{r_{k,n}}^{(F_k,\varphi_k)}(I - r_{k,n}A_k)\Theta_n^{k-1}x_n\\ & - T_{r_{k,n}}^{(F_k,\varphi_k)}(I - r_{k,n}A_k)p\Big\|^2\\ \le {} & \left\|(I - r_{k,n}A_k)\Theta_n^{k-1}x_n - (I - r_{k,n}A_k)p\right\|^2\\ \le {} & \left\|\Theta_n^{k-1}x_n - p\right\|^2\\ & + r_{k,n}(r_{k,n} - 2\mu_k)\left\|A_k\Theta_n^{k-1}x_n - A_k p\right\|^2\\ \le {} & \|x_n - p\|^2\\ & + r_{k,n}(r_{k,n} - 2\mu_k)\left\|A_k\Theta_n^{k-1}x_n - A_k p\right\|^2.\end{aligned} \tag{59}$$

By (40), (41), (57), and (59), we obtain

$$\begin{aligned}\|k_n - p\|^2 &\leq \|z_n - p\|^2 + \theta_n \\ &\leq \|u_n - p\|^2 + \theta_n \\ &\leq \left\|\Theta_n^k x_n - p\right\|^2 + \theta_n \\ &\leq \|x_n - p\|^2 + r_{k,n}\left(r_{k,n} - 2\mu_k\right)\left\|A_k\Theta_n^{k-1}x_n - A_k p\right\|^2 \\ &\quad + \theta_n, \end{aligned} \tag{60}$$

which implies that

$$\begin{aligned} r_{k,n}\left(2\mu_k - r_{k,n}\right)&\left\|A_k\Theta_n^{k-1}x_n - A_k p\right\|^2 \\ &\leq \|x_n - p\|^2 - \|k_n - p\|^2 + \theta_n \\ &\leq \|x_n - k_n\|\left(\|x_n - p\| + \|k_n - p\|\right) + \theta_n. \end{aligned} \tag{61}$$

Since $\{r_{k,n}\} \subset [e_k, f_k] \subset (0, 2\mu_k)$, $k \in \{1, 2, \ldots, M\}$, $\lim_{n\to\infty}\theta_n = 0$, and (56), we have

$$\lim_{n\to\infty}\left\|A_k\Theta_n^{k-1}x_n - A_k p\right\| = 0, \quad k = 1, 2, \ldots, M. \tag{62}$$

By Proposition 1(iii) and Lemma 7(a), we have

$$\begin{aligned} &\left\|\Theta_n^k x_n - p\right\|^2 \\ &= \left\|T_{r_{k,n}}^{(F_k,\varphi_k)}(I - r_{k,n}A_k)\Theta_n^{k-1}x_n - T_{r_{k,n}}^{(F_k,\varphi_k)}(I - r_{k,n}A_k)p\right\|^2 \\ &\leq \left\langle (I - r_{k,n}A_k)\Theta_n^{k-1}x_n - (I - r_{k,n}A_k)p, \Theta_n^k x_n - p\right\rangle \\ &= \frac{1}{2}\left(\left\|(I - r_{k,n}A_k)\Theta_n^{k-1}x_n - (I - r_{k,n}A_k)p\right\|^2\right. \\ &\qquad + \left\|\Theta_n^k x_n - p\right\|^2 - \left\|(I - r_{k,n}A_k)\Theta_n^{k-1}x_n\right. \\ &\qquad\qquad \left.\left. - (I - r_{k,n}A_k)p - \left(\Theta_n^k x_n - p\right)\right\|^2\right) \\ &\leq \frac{1}{2}\left(\left\|\Theta_n^{k-1}x_n - p\right\|^2 + \left\|\Theta_n^k x_n - p\right\|^2\right. \\ &\qquad \left. - \left\|\Theta_n^{k-1}x_n - \Theta_n^k x_n - r_{k,n}(A_k\Theta_n^{k-1}x_n - A_k p)\right\|^2\right), \end{aligned} \tag{63}$$

which implies that

$$\begin{aligned} &\left\|\Theta_n^k x_n - p\right\|^2 \\ &\leq \left\|\Theta_n^{k-1}x_n - p\right\|^2 \\ &\quad - \left\|\Theta_n^{k-1}x_n - \Theta_n^k x_n - r_{k,n}(A_k\Theta_n^{k-1}x_n - A_k p)\right\|^2 \\ &= \left\|\Theta_n^{k-1}x_n - p\right\|^2 - \left\|\Theta_n^{k-1}x_n - \Theta_n^k x_n\right\|^2 \\ &\quad - r_{k,n}^2\left\|A_k\Theta_n^{k-1}x_n - A_k p\right\|^2 \\ &\quad + 2r_{k,n}\left\langle \Theta_n^{k-1}x_n - \Theta_n^k x_n, A_k\Theta_n^{k-1}x_n - A_k p\right\rangle \\ &\leq \left\|\Theta_n^{k-1}x_n - p\right\|^2 - \left\|\Theta_n^{k-1}x_n - \Theta_n^k x_n\right\|^2 \\ &\quad + 2r_{k,n}\left\|\Theta_n^{k-1}x_n - \Theta_n^k x_n\right\|\left\|A_k\Theta_n^{k-1}x_n - A_k p\right\| \\ &\leq \|x_n - p\|^2 - \left\|\Theta_n^{k-1}x_n - \Theta_n^k x_n\right\|^2 \\ &\quad + 2r_{k,n}\left\|\Theta_n^{k-1}x_n - \Theta_n^k x_n\right\|\left\|A_k\Theta_n^{k-1}x_n - A_k p\right\|. \end{aligned} \tag{64}$$

Combining (57) and (64), we have

$$\begin{aligned} &\|k_n - p\|^2 \\ &\leq \|z_n - p\|^2 + \theta_n \\ &\leq \|u_n - p\|^2 + \theta_n \\ &\leq \left\|\Theta_n^k x_n - p\right\|^2 + \theta_n \\ &\leq \|x_n - p\|^2 - \left\|\Theta_n^{k-1}x_n - \Theta_n^k x_n\right\|^2 \\ &\quad + 2r_{k,n}\left\|\Theta_n^{k-1}x_n - \Theta_n^k x_n\right\|\left\|A_k\Theta_n^{k-1}x_n - A_k p\right\| + \theta_n, \end{aligned} \tag{65}$$

which implies

$$\begin{aligned} &\left\|\Theta_n^{k-1}x_n - \Theta_n^k x_n\right\|^2 \\ &\leq \|x_n - p\|^2 - \|k_n - p\|^2 \\ &\quad + 2r_{k,n}\left\|\Theta_n^{k-1}x_n - \Theta_n^k x_n\right\|\left\|A_k\Theta_n^{k-1}x_n - A_k p\right\| + \theta_n \\ &\leq \|x_n - k_n\|\left(\|x_n - p\| + \|k_n - p\|\right) \\ &\quad + 2r_{k,n}\left\|\Theta_n^{k-1}x_n - \Theta_n^k x_n\right\|\left\|A_k\Theta_n^{k-1}x_n - A_k p\right\| + \theta_n. \end{aligned} \tag{66}$$

From $\lim_{n\to\infty}\theta_n = 0$, (56), and (62), we know that (58) holds. Hence, we obtain

$$\begin{aligned} \|x_n - u_n\| &= \left\|\Theta_n^0 x_n - \Theta_n^M x_n\right\| \\ &\leq \left\|\Theta_n^0 x_n - \Theta_n^1 x_n\right\| + \left\|\Theta_n^1 x_n - \Theta_n^2 x_n\right\| \\ &\quad + \cdots + \left\|\Theta_n^{M-1}x_n - \Theta_n^M x_n\right\| \\ &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{67}$$

Next, we show that $\lim_{n\to\infty}\|B_i\Omega_n^i u_n - B_i p\| = 0$, $i = 1, 2, \ldots, N$. It follows from Lemma 20 and (18) that

$$\begin{aligned}
&\left\|\Omega_n^i u_n - p\right\|^2 \\
&\quad = \left\|J_{R_i,\lambda_{i,n}}(I-\lambda_{i,n}B_i)\Omega_n^{i-1}u_n - J_{R_i,\lambda_{i,n}}(I-\lambda_{i,n}B_i)p\right\|^2 \\
&\quad \le \left\|(I-\lambda_{i,n}B_i)\Omega_n^{i-1}u_n - (I-\lambda_{i,n}B_i)p\right\|^2 \\
&\quad \le \left\|\Omega_n^{i-1}u_n - p\right\|^2 + \lambda_{i,n}\left(\lambda_{i,n} - 2\eta_i\right)\left\|B_i\Omega_n^{i-1}u_n - B_i p\right\|^2 \\
&\quad \le \left\|u_n - p\right\|^2 + \lambda_{i,n}\left(\lambda_{i,n} - 2\eta_i\right)\left\|B_i\Omega_n^{i-1}u_n - B_i p\right\|^2 \\
&\quad \le \left\|x_n - p\right\|^2 + \lambda_{i,n}\left(\lambda_{i,n} - 2\eta_i\right)\left\|B_i\Omega_n^{i-1}u_n - B_i p\right\|^2.
\end{aligned} \tag{68}$$

Combining (57) and (68), we have

$$\begin{aligned}
&\left\|k_n - p\right\|^2 \\
&\quad \le \left\|z_n - p\right\|^2 + \theta_n \\
&\quad \le \left\|\Omega_n^i u_n - p\right\|^2 + \theta_n \\
&\quad \le \left\|x_n - p\right\|^2 + \lambda_{i,n}\left(\lambda_{i,n} - 2\eta_i\right)\left\|B_i\Omega_n^{i-1}u_n - B_i p\right\|^2 + \theta_n,
\end{aligned} \tag{69}$$

which implies

$$\begin{aligned}
\lambda_{i,n}&\left(2\eta_i - \lambda_{i,n}\right)\left\|B_i\Omega_n^{i-1}u_n - B_i p\right\|^2 \\
&\le \left\|x_n - p\right\|^2 - \left\|k_n - p\right\|^2 + \theta_n \\
&\le \left\|x_n - k_n\right\|\left(\left\|x_n - p\right\| + \left\|k_n - p\right\|\right) + \theta_n.
\end{aligned} \tag{70}$$

From $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$, $i \in \{1, 2, \ldots, N\}$, $\lim_{n\to\infty}\theta_n = 0$, and (56), we obtain

$$\lim_{n\to\infty}\left\|B_i\Omega_n^{i-1}u_n - B_i p\right\| = 0, \quad i = 1, 2, \ldots, N. \tag{71}$$

By Lemmas 20 and 7(a), we obtain

$$\begin{aligned}
&\left\|\Omega_n^i u_n - p\right\|^2 \\
&= \left\|J_{R_i,\lambda_{i,n}}(I-\lambda_{i,n}B_i)\Omega_n^{i-1}u_n - J_{R_i,\lambda_{i,n}}(I-\lambda_{i,n}B_i)p\right\|^2 \\
&\le \left\langle (I-\lambda_{i,n}B_i)\Omega_n^{i-1}u_n - (I-\lambda_{i,n}B_i)p, \Omega_n^i u_n - p\right\rangle \\
&= \frac{1}{2}\left(\left\|(I-\lambda_{i,n}B_i)\Omega_n^{i-1}u_n - (I-\lambda_{i,n}B_i)p\right\|^2 + \left\|\Omega_n^i u_n - p\right\|^2\right. \\
&\qquad \left. - \left\|(I-\lambda_{i,n}B_i)\Omega_n^{i-1}u_n - (I-\lambda_{i,n}B_i)p - (\Omega_n^i u_n - p)\right\|^2\right) \\
&\le \frac{1}{2}\left(\left\|\Omega_n^{i-1}u_n - p\right\|^2 + \left\|\Omega_n^i u_n - p\right\|^2\right. \\
&\qquad \left. - \left\|\Omega_n^{i-1}u_n - \Omega_n^i u_n - \lambda_{i,n}(B_i\Omega_n^{i-1}u_n - B_i p)\right\|^2\right) \\
&\le \frac{1}{2}\left(\left\|u_n - p\right\|^2 + \left\|\Omega_n^i u_n - p\right\|^2\right. \\
&\qquad \left. - \left\|\Omega_n^{i-1}u_n - \Omega_n^i u_n - \lambda_{i,n}(B_i\Omega_n^{i-1}u_n - B_i p)\right\|^2\right) \\
&\le \frac{1}{2}\left(\left\|x_n - p\right\|^2 + \left\|\Omega_n^i u_n - p\right\|^2\right. \\
&\qquad \left. - \left\|\Omega_n^{i-1}u_n - \Omega_n^i u_n - \lambda_{i,n}(B_i\Omega_n^{i-1}u_n - B_i p)\right\|^2\right),
\end{aligned} \tag{72}$$

which implies

$$\begin{aligned}
&\left\|\Omega_n^i u_n - p\right\|^2 \\
&\quad \le \left\|x_n - p\right\|^2 \\
&\qquad - \left\|\Omega_n^{i-1}u_n - \Omega_n^i u_n - \lambda_{i,n}(B_i\Omega_n^{i-1}u_n - B_i p)\right\|^2 \\
&\quad = \left\|x_n - p\right\|^2 - \left\|\Omega_n^{i-1}u_n - \Omega_n^i u_n\right\|^2 \\
&\qquad - \lambda_{i,n}^2\left\|B_i\Omega_n^{i-1}u_n - B_i p\right\|^2 \\
&\qquad + 2\lambda_{i,n}\left\langle \Omega_n^{i-1}u_n - \Omega_n^i u_n, B_i\Omega_n^{i-1}u_n - B_i p\right\rangle \\
&\quad \le \left\|x_n - p\right\|^2 - \left\|\Omega_n^{i-1}u_n - \Omega_n^i u_n\right\|^2 \\
&\qquad + 2\lambda_{i,n}\left\|\Omega_n^{i-1}u_n - \Omega_n^i u_n\right\|\left\|B_i\Omega_n^{i-1}u_n - B_i p\right\|.
\end{aligned} \tag{73}$$

Combining (57) and (73), we get

$$\begin{aligned}
&\left\|k_n - p\right\|^2 \\
&\quad \le \left\|z_n - p\right\|^2 + \theta_n \\
&\quad \le \left\|\Omega_n^i u_n - p\right\|^2 + \theta_n \\
&\quad \le \left\|x_n - p\right\|^2 - \left\|\Omega_n^{i-1}u_n - \Omega_n^i u_n\right\|^2 \\
&\qquad + 2\lambda_{i,n}\left\|\Omega_n^{i-1}u_n - \Omega_n^i u_n\right\|\left\|B_i\Omega_n^{i-1}u_n - B_i p\right\| + \theta_n,
\end{aligned} \tag{74}$$

which implies

$$\begin{aligned}
&\left\|\Omega_n^{i-1}u_n - \Omega_n^i u_n\right\|^2 \\
&\quad \le \left\|x_n - p\right\|^2 - \left\|k_n - p\right\|^2 \\
&\qquad + 2\lambda_{i,n}\left\|\Omega_n^{i-1}u_n - \Omega_n^i u_n\right\|\left\|B_i\Omega_n^{i-1}u_n - B_i p\right\| + \theta_n
\end{aligned}$$

$$\leq \|x_n - k_n\| (\|x_n - p\| + \|k_n - p\|) + 2\lambda_{i,n} \left\|\Omega_n^{i-1}u_n - \Omega_n^i u_n\right\| \left\|B_i\Omega_n^{i-1}u_n - B_i p\right\| + \theta_n. \quad (75)$$

From (56), (71), and $\lim_{n\to\infty}\theta_n = 0$, we have

$$\lim_{n\to\infty} \left\|\Omega_n^{i-1}u_n - \Omega_n^i u_n\right\| = 0, \quad i = 1, 2, \ldots, N. \quad (76)$$

From (76), we get

$$\begin{aligned} \|u_n - z_n\| &= \left\|\Omega_n^0 u_n - \Omega_n^N u_n\right\| \\ &\leq \left\|\Omega_n^0 u_n - \Omega_n^1 u_n\right\| + \left\|\Omega_n^1 u_n - \Omega_n^2 u_n\right\| \\ &\quad + \cdots + \left\|\Omega_n^{N-1}u_n - \Omega_n^N u_n\right\| \\ &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \quad (77)$$

By (67) and (77), we have

$$\begin{aligned} \|x_n - z_n\| &\leq \|x_n - u_n\| + \|u_n - z_n\| \\ &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \quad (78)$$

From (48) and (78), we have

$$\begin{aligned} \|z_{n+1} - z_n\| \leq\ & \|z_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| \\ & + \|x_n - z_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \quad (79)$$

By (56), (67), and (77), we get

$$\begin{aligned} \|k_n - z_n\| &\leq \|k_n - x_n\| + \|x_n - u_n\| + \|u_n - z_n\| \\ &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \quad (80)$$

We observe that

$$k_n - z_n = (1 - \delta_n)(S^n z_n - z_n). \quad (81)$$

From $\delta_n \leq d < 1$ and (80), we have

$$\lim_{n\to\infty} \|S^n z_n - z_n\| = 0. \quad (82)$$

We note that

$$\begin{aligned} \left\|S^n z_n - S^{n+1}z_n\right\| \leq\ & \|S^n z_n - z_n\| + \|z_n - z_{n+1}\| \\ & + \left\|z_{n+1} - S^{n+1}z_{n+1}\right\| \\ & + \left\|S^{n+1}z_{n+1} - S^{n+1}z_n\right\|. \end{aligned} \quad (83)$$

From (79), (82), and Lemma 9, we obtain

$$\lim_{n\to\infty} \left\|S^n z_n - S^{n+1}z_n\right\| = 0. \quad (84)$$

On the other hand, we note that

$$\begin{aligned} \|z_n - Sz_n\| \leq\ & \|z_n - S^n z_n\| + \left\|S^n z_n - S^{n+1}z_n\right\| \\ & + \left\|S^{n+1}z_n - Sz_n\right\|. \end{aligned} \quad (85)$$

From (82), (84), and the uniform continuity of S, we have

$$\lim_{n\to\infty} \|z_n - Sz_n\| = 0. \quad (86)$$

In addition, note that

$$\begin{aligned} \|z_n - Wz_n\| \leq\ & \|z_n - W_n z_n\| + \|W_n z_n - Wz_n\| \\ \leq\ & \|z_n - x_n\| + \|x_n - W_n z_n\| + \|W_n z_n - Wz_n\|. \end{aligned} \quad (87)$$

So, from (53), (78), and [4, Remark 3.2], it follows that

$$\lim_{n\to\infty} \|z_n - Wz_n\| = 0. \quad (88)$$

Step 4. Finally we prove that $x_n \to v = P_F x_0$ as $n \to \infty$.

Indeed, since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ which converges weakly to some w. From (58) and (76)–(78), we have that $\Theta_{n_i}^k x_{n_i} \rightharpoonup w, \Omega_{n_i}^m u_{n_i} \rightharpoonup w$ and $z_{n_i} \rightharpoonup w$, where $k \in \{1, 2, \ldots, M\}$ and $m \in \{1, 2, \ldots, N\}$. Since S is uniformly continuous, by (86), we get $\lim_{n\to\infty}\|z_n - S^m z_n\| = 0$, for any $m \geq 1$. Hence, from Lemma 11, we obtain $w \in \mathrm{Fix}(S)$. In the meantime, utilizing Lemma 19, we deduce from (88) and $z_{n_i} \rightharpoonup w$ that $w \in \mathrm{Fix}(W) = \bigcap_{n=1}^\infty \mathrm{Fix}(T_n)$. Next, we prove that $w \in \bigcap_{m=1}^N I(B_m, R_m)$. As a matter of fact, since B_m is η_m-inverse strongly monotone, B_m is a monotone and Lipschitz continuous mapping. It follows from Lemma 23 that $R_m + B_m$ is maximal monotone. Let $(v, g) \in G(R_m + B_m)$; that is, $g - B_m v \in R_m v$. Again, since $\Omega_n^m u_n = J_{R_m,\lambda_{m,n}}(I - \lambda_{m,n}B_m)\Lambda_n^{m-1}u_n, n \geq 1, m \in \{1, 2, \ldots, N\}$, we have

$$\Omega_n^{m-1}u_n - \lambda_{m,n}B_m\Omega_n^{m-1}u_n \in (I + \lambda_{m,n}R_m)\,\Omega_n^m u_n; \quad (89)$$

that is,

$$\frac{1}{\lambda_{m,n}}\left(\Omega_n^{m-1}u_n - \Omega_n^m u_n - \lambda_{m,n}B_m\Omega_n^{m-1}u_n\right) \in R_m\Omega_n^m u_n. \quad (90)$$

In terms of the monotonicity of R_m, we get

$$\left\langle v - \Omega_n^m u_n, g - B_m v - \frac{1}{\lambda_{m,n}}\left(\Omega_n^{m-1}u_n - \Omega_n^m u_n - \lambda_{m,n}B_m\Omega_n^{m-1}u_n\right)\right\rangle \geq 0 \quad (91)$$

and, hence,

$$\begin{aligned} &\langle v - \Omega_n^m u_n, g\rangle \\ &\geq \left\langle v - \Omega_n^m u_n, B_m v + \frac{1}{\lambda_{m,n}}\left(\Omega_n^{m-1}u_n - \Omega_n^m u_n - \lambda_{m,n}B_m\Omega_n^{m-1}u_n\right)\right\rangle \end{aligned}$$

$$= \left\langle v - \Omega_n^m u_n, B_m v - B_m \Omega_n^m u_n + B_m \Omega_n^m u_n - B_m \Omega_n^{m-1} u_n + \frac{1}{\lambda_{m,n}} \left(\Omega_n^{m-1} u_n - \Omega_n^m u_n\right) \right\rangle$$
$$\geq \left\langle v - \Omega_n^m u_n, B_m \Omega_n^m u_n - B_m \Omega_n^{m-1} u_n \right\rangle + \left\langle v - \Omega_n^m u_n, \frac{1}{\lambda_{m,n}} \left(\Omega_n^{m-1} u_n - \Omega_n^m u_n\right) \right\rangle. \quad (92)$$

In particular,

$$\begin{aligned} &\left\langle v - \Omega_{n_i}^m u_{n_i}, g \right\rangle \\ &\quad\geq \left\langle v - \Omega_{n_i}^m u_{n_i}, B_m \Omega_{n_i}^m u_{n_i} - B_m \Omega_{n_i}^{m-1} u_{n_i} \right\rangle \\ &\qquad + \left\langle v - \Omega_{n_i}^m u_{n_i}, \frac{1}{\lambda_{m,n_i}} \left(\Omega_{n_i}^{m-1} u_{n_i} - \Omega_{n_i}^m u_{n_i}\right) \right\rangle. \end{aligned} \quad (93)$$

Since $\|\Omega_n^m u_n - \Omega_n^{m-1} u_n\| \to 0$ (due to (76)) and $\|B_m \Omega_n^m u_n - B_m \Omega_n^{m-1} u_n\| \to 0$ (due to the Lipschitz continuity of B_m), we conclude from $\Omega_{n_i}^m u_{n_i} \rightharpoonup w$ and condition (ii) that

$$\lim_{i \to \infty} \left\langle v - \Omega_{n_i}^m u_{n_i}, g \right\rangle = \langle v - w, g \rangle \geq 0. \quad (94)$$

It follows from the maximal monotonicity of $B_m + R_m$ that $0 \in (R_m + B_m)w$; that is, $w \in I(B_m, R_m)$. Therefore, $w \in \bigcap_{m=1}^N I(B_m, R_m)$. Next, we prove that $w \in \bigcap_{k=1}^M \mathrm{GMEP}(F_k, \varphi_k, A_k)$. Since $\Theta_n^k x_n = T_{r_{k,n}}^{(F_k,\varphi_k)}(I - r_{k,n} A_k)\Theta_n^{k-1} x_n$, $n \geq 1$, $k \in \{1, 2, \ldots, M\}$, we have

$$F_k\left(\Theta_n^k x_n, y\right) + \varphi_k(y) - \varphi_k\left(\Theta_n^k x_n\right) + \left\langle A_k \Theta_n^{k-1} x_n, y - \Theta_n^k x_n \right\rangle + \frac{1}{r_{k,n}} \left\langle y - \Theta_n^k x_n, \Theta_n^k x_n - \Theta_n^{k-1} x_n \right\rangle \geq 0. \quad (95)$$

By (A2), we have

$$\varphi_k(y) - \varphi_k\left(\Theta_n^k x_n\right) + \left\langle A_k \Theta_n^{k-1} x_n, y - \Theta_n^k x_n \right\rangle + \frac{1}{r_{k,n}} \langle y - \Theta_n^k x_n, \Theta_n^k x_n - \Theta_n^{k-1} x_n \rangle \geq F_k\left(y, \Theta_n^k x_n\right). \quad (96)$$

Let $z_t = ty + (1-t)w$, for all $t \in (0, 1]$ and $y \in C$. This implies that $z_t \in C$. Then, we have

$$\begin{aligned} &\left\langle z_t - \Theta_n^k x_n, A_k z_t \right\rangle \\ &\quad\geq \varphi_k\left(\Theta_n^k x_n\right) - \varphi_k(z_t) + \left\langle z_t - \Theta_n^k x_n, A_k z_t \right\rangle \\ &\qquad - \left\langle z_t - \Theta_n^k x_n, A_k \Theta_n^{k-1} x_n \right\rangle \\ &\qquad - \left\langle z_t - \Theta_n^k x_n, \frac{\Theta_n^k x_n - \Theta_n^{k-1} x_n}{r_{k,n}} \right\rangle + F_k\left(z_t, \Theta_n^k x_n\right) \\ &\quad= \varphi_k\left(\Theta_n^k x_n\right) - \varphi_k(z_t) + \left\langle z_t - \Theta_n^k x_n, A_k z_t - A_k \Theta_n^k x_n \right\rangle \\ &\qquad + \left\langle z_t - \Theta_n^k x_n, A_k \Theta_n^k x_n - A_k \Theta_n^{k-1} x_n \right\rangle \\ &\qquad - \left\langle z_t - \Theta_n^k x_n, \frac{\Theta_n^k x_n - \Theta_n^{k-1} x_n}{r_{k,n}} \right\rangle + F_k\left(z_t, \Theta_n^k x_n\right). \end{aligned} \quad (97)$$

By (58), we have $\|A_k \Theta_n^k x_n - A_k \Theta_n^{k-1} x_n\| \to 0$ as $n \to \infty$. Furthermore, by the monotonicity of A_k, we obtain $\langle z_t - \Theta_n^k x_n, A_k z_t - A_k \Theta_n^k x_n \rangle \geq 0$. Then, by (A4), we obtain

$$\langle z_t - w, A_k z_t \rangle \geq \varphi_k(w) - \varphi_k(z_t) + F_k(z_t, w). \quad (98)$$

Utilizing (A1), (A4), and (98), we obtain

$$\begin{aligned} 0 &= F_k(z_t, z_t) + \varphi_k(z_t) - \varphi_k(z_t) \\ &\leq t F_k(z_t, y) + (1-t) F_k(z_t, w) + t\varphi_k(y) \\ &\quad + (1-t)\varphi_k(w) - \varphi_k(z_t) \\ &\leq t\left[F_k(z_t, y) + \varphi_k(y) - \varphi_k(z_t)\right] + (1-t)\langle z_t - w, A_k z_t \rangle \\ &= t\left[F_k(z_t, y) + \varphi_k(y) - \varphi_k(z_t)\right] + (1-t)t\langle y - w, A_k z_t \rangle, \end{aligned} \quad (99)$$

and, hence,

$$0 \leq F_k(z_t, y) + \varphi_k(y) - \varphi_k(z_t) + (1-t)\langle y - w, A_k z_t \rangle. \quad (100)$$

Letting $t \to 0$, we have, for each $y \in C$,

$$0 \leq F_k(w, y) + \varphi_k(y) - \varphi_k(w) + \langle y - w, A_k w \rangle. \quad (101)$$

This implies that $w \in \mathrm{GMEP}(F_k, \varphi_k, A_k)$ and, hence, $w \in \bigcap_{k=1}^M \mathrm{GMEP}(F_k, \varphi_k, A_k)$. Consequently, $w \in F = \bigcap_{n=1}^\infty \mathrm{Fix}(T_n) \cap \bigcap_{k=1}^M \mathrm{GMEP}(F_k, \varphi_k, A_k) \cap \bigcap_{m=1}^N I(B_m, R_m) \cap \mathrm{Fix}(S)$. This shows that $\omega_w(\{x_n\}) \subset F$. From (45) and Lemma 16, we infer that $x_n \to v = P_F x_0$ as $n \to \infty$. This completes the proof. □

4. Weak Convergence Theorem

In this section, we prove a weak convergence theorem for an iterative algorithm for finding a common element of the set of solutions of the set of solutions of finite generalized mixed equilibrium problems, the set of solutions of finite variational inclusions for maximal monotone and inverse strong monotone mappings, and the set of common fixed points of infinite nonexpansive mappings and asymptotically κ-strict pseudocontractive mapping $S : C \to C$ in the intermediate sense in a real Hilbert space. This iterative algorithm is based on the extragradient method and Mann-type iterative method.

Theorem 25. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let M, N be two integers. Let F_k be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (A1)–(A4) and let*

$\varphi_k : C \to \mathbf{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function, where $k \in \{1,2,\ldots,M\}$. Let $R_i : C \to 2^H$ be a maximal monotone mapping and let $A_k : H \to H$ and $B_i : C \to H$ be μ_k-inverse strongly monotone and η_i-inverse strongly monotone, respectively, where $k \in \{1,2,\ldots,M\}$, $i \in \{1,2,\ldots,N\}$. Let $S : C \to C$ be a uniformly continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense, for some $0 \le \kappa < 1$, with sequences $\{\gamma_n\} \subset [0,\infty)$ and $\{c_n\} \subset [0,\infty)$. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C and $\{\lambda_n\}$ be a sequence in $(0,b]$, for some $b \in (0,1)$. Assume that $F := \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \cap \bigcap_{k=1}^{M} \mathrm{GMEP}(F_k,\varphi_k,A_k) \cap \bigcap_{i=1}^{N} I(B_i,R_i) \cap \mathrm{Fix}(S)$ is nonempty and that either (B1) or (B2) holds. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\delta_n\}$ be sequences in $[0,1]$ such that $\alpha_n + \beta_n \le 1$, $0 < a \le \alpha_n \le 1$ and $\kappa \le \delta_n \le d < 1$. Pick any $x_1 \in H$ and let $\{x_n\}$ be a sequence generated by the following algorithm:

$$
\begin{aligned}
u_n &= T_{r_{M,n}}^{(F_M,\varphi_M)}\left(I - r_{M,n}A_M\right) T_{r_{M-1,n}}^{(F_{M-1},\varphi_{M-1})} \\
&\quad \times \left(I - r_{M-1,n}A_{M-1}\right) \cdots T_{r_{1,n}}^{(F_1,\varphi_1)}\left(I - r_{1,n}A_1\right)x_n, \\
z_n &= J_{R_N,\lambda_{N,n}}\left(I - \lambda_{N,n}B_N\right) J_{R_{N-1},\lambda_{N-1,n}} \\
&\quad \times \left(I - \lambda_{N-1,n}B_{N-1}\right) \cdots J_{R_1,\lambda_{1,n}}\left(I - \lambda_{1,n}B_1\right)u_n, \\
k_n &= \delta_n z_n + \left(1-\delta_n\right)S^n z_n, \\
y_n &= \left(1-\alpha_n-\beta_n\right)x_n + \alpha_n k_n + \beta_n W_n z_n,
\end{aligned}
\tag{102}
$$

where W_n is the W-mapping generated by (2.2). Assume that the following conditions hold:

(i) $\sum_{n=1}^{\infty} \gamma_n < \infty$ *and* $\sum_{n=1}^{\infty} c_n < \infty$;

(ii) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}(\alpha_n+\beta_n) < 1$;

(iii) $\{\lambda_{i,n}\} \subset [a_i,b_i] \subset (0,2\eta_i)$, $\forall i \in \{1,2,\ldots,N\}$;

(iv) $\{r_{k,n}\} \subset [e_k,f_k] \subset (0,2\mu_k)$, $\forall k \in \{1,2,\ldots,M\}$.

Then, $\{x_n\}$ converges weakly to $w = \lim_{n\to\infty} P_F x_n$.

Proof. First, let us show that $\lim_{n\to\infty}\|x_n - p\|$ exists, for any $p \in F$. Put

$$
\begin{aligned}
\Theta_n^k &= T_{r_{k,n}}^{(F_k,\varphi_k)}\left(I - r_{k,n}A_k\right) T_{r_{k-1,n}}^{(F_{k-1},\varphi_{k-1})} \\
&\quad \times \left(I - r_{k-1,n}A_{k-1}\right) \cdots T_{r_{1,n}}^{(F_1,\varphi_1)}\left(I - r_{1,n}A_1\right)x_n,
\end{aligned}
\tag{103}
$$

for all $k \in \{1,2,\ldots,M\}$, $n \ge 1$;

$$
\begin{aligned}
\Omega_n^i &= J_{R_i,\lambda_{i,n}}\left(I - \lambda_{i,n}B_i\right) J_{R_{i-1},\lambda_{i-1,n}} \\
&\quad \times \left(I - \lambda_{i-1,n}B_{i-1}\right) \cdots J_{R_1,\lambda_{1,n}}\left(I - \lambda_{1,n}B_1\right),
\end{aligned}
\tag{104}
$$

for all $i \in \{1,2,\ldots,N\}$, $n \ge 1$, $\Theta_n^0 = \Omega_n^0 = I$, where I is the identity mapping on H. Then, we have that $u_n = \Theta_n^M x_n$ and $z_n = \Omega_n^N u_n$. Take $p \in F$ arbitrarily. Similar to the proof of Theorem 24, we obtain that

$$\|u_n - p\| \le \|x_n - p\|, \tag{105}$$

$$\|z_n - p\| \le \|u_n - p\|, \tag{106}$$

$$
\begin{aligned}
\left\|\Theta_n^k x_n - p\right\|^2 &\le \|x_n - p\|^2 \\
&\quad + r_{k,n}\left(r_{k,n} - 2\mu_k\right)\left\|A_k\Theta_n^{k-1}x_n - A_k p\right\|^2, \\
&\qquad k \in \{1,2,\ldots,M\},
\end{aligned}
\tag{107}
$$

$$
\begin{aligned}
\left\|\Theta_n^k x_n - p\right\|^2 &\le \|x_n - p\|^2 - \left\|\Theta_n^{k-1}x_n - \Theta_n^k x_n\right\|^2 \\
&\quad + 2r_{k,n}\left\|\Theta_n^{k-1}x_n - \Theta_n^k x_n\right\| \\
&\quad \times \left\|A_k\Theta_n^{k-1}x_n - A_k p\right\|, \quad k = 1,2,\ldots,M,
\end{aligned}
\tag{108}
$$

$$
\begin{aligned}
\left\|\Omega_n^i u_n - p\right\|^2 &\le \|x_n - p\|^2 \\
&\quad + \lambda_{i,n}\left(\lambda_{i,n} - 2\eta_i\right)\left\|B_i\Omega_n^{i-1}u_n - B_i p\right\|^2, \\
&\qquad i \in \{1,2,\ldots,N\},
\end{aligned}
\tag{109}
$$

$$
\begin{aligned}
\left\|\Omega_n^i u_n - p\right\|^2 &\le \|x_n - p\|^2 - \left\|\Omega_n^{i-1}u_n - \Omega_n^i u_n\right\|^2 \\
&\quad + 2\lambda_{i,n}\left\|\Omega_n^{i-1}u_n - \Omega_n^i u_n\right\| \\
&\quad \times \left\|B_i\Omega_n^{i-1}u_n - B_i p\right\|, \quad i \in \{1,2,\ldots,N\}.
\end{aligned}
\tag{110}
$$

We observe that

$$
\begin{aligned}
\|k_n - p\|^2 &= \left\|\delta_n(z_n - p) + (1-\delta_n)(S^n z_n - p)\right\|^2 \\
&= \delta_n\|z_n - p\|^2 + \left(1-\delta_n\right)\left\|S^n z_n - p\right\|^2 \\
&\quad - \delta_n\left(1-\delta_n\right)\left\|z_n - S^n z_n\right\|^2 \\
&\le \delta_n\|z_n - p\|^2 + \left(1-\delta_n\right) \\
&\quad \times \left[\left(1+\gamma_n\right)\|z_n - p\|^2 + \kappa\|z_n - S^n z_n\|^2 + c_n\right] \\
&\quad - \delta_n\left(1-\delta_n\right)\left\|z_n - S^n z_n\right\|^2 \\
&= \left[1 + \gamma_n\left(1-\delta_n\right)\right]\|z_n - p\|^2 \\
&\quad + \left(1-\delta_n\right)\left(\kappa - \delta_n\right)\left\|z_n - S^n z_n\right\|^2 + \left(1-\delta_n\right)c_n \\
&\le \left(1+\gamma_n\right)\|z_n - p\|^2 + c_n.
\end{aligned}
\tag{111}
$$

It follows from (105), (106), and (111) that

$$
\begin{aligned}
&\left\|x_{n+1} - p\right\|^2 \\
&\quad = \left\|(1-\alpha_n-\beta_n)(x_n - p) + \alpha_n(k_n - p) + \beta_n(W_n z_n - p)\right\|^2
\end{aligned}
$$

$$\begin{aligned}
&\leq (1-\alpha_n-\beta_n)\|x_n-p\|^2+\alpha_n\|k_n-p\|^2 \\
&\quad +\beta_n\|W_nz_n-p\|^2 \\
&\leq (1-\alpha_n-\beta_n)\|x_n-p\|^2+\alpha_n\left[(1+\gamma_n)\|z_n-p\|^2+c_n\right] \\
&\quad +\beta_n\|z_n-p\|^2 \\
&\leq (1-\alpha_n-\beta_n)\|x_n-p\|^2+\alpha_n\left[(1+\gamma_n)\|x_n-p\|^2+c_n\right] \\
&\quad +\beta_n\|x_n-p\|^2 \\
&= (1-\alpha_n)\|x_n-p\|^2+\alpha_n\left[(1+\gamma_n)\|x_n-p\|^2+c_n\right] \\
&= (1+\alpha_n\gamma_n)\|x_n-p\|^2+\alpha_nc_n \\
&\leq (1+\gamma_n)\|x_n-p\|^2+c_n.
\end{aligned} \tag{112}$$

By Lemma 13 and condition (i), we deduce that $\lim_{n\to\infty}\|x_n-p\|$ exists. Hence, $\{x_n\}$ is bounded and so are $\{u_n\}$, $\{z_n\}$, and $\{k_n\}$.

In addition, by Lemma 7(b), we obtain from (105), (106), and (111) that

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&= \|(1-\alpha_n-\beta_n)(x_n-p)+\alpha_n(k_n-p)+\beta_n(W_nz_n-p)\|^2 \\
&= (1-\alpha_n-\beta_n)\|x_n-p\|^2 \\
&\quad +(\alpha_n+\beta_n)\left\|\frac{\alpha_n(k_n-p)+\beta_n(W_nz_n-p)}{\alpha_n+\beta_n}\right\|^2 \\
&\quad -(\alpha_n+\beta_n)(1-\alpha_n-\beta_n)\left\|\frac{\alpha_n(k_n-x_n)+\beta_n(W_nz_n-x_n)}{\alpha_n+\beta_n}\right\|^2 \\
&\leq (1-\alpha_n-\beta_n)\|x_n-p\|^2 \\
&\quad +(\alpha_n+\beta_n)\left[\frac{\alpha_n}{\alpha_n+\beta_n}\|k_n-p\|^2+\frac{\beta_n}{\alpha_n+\beta_n}\|W_nz_n-p\|^2\right] \\
&\quad -(\alpha_n+\beta_n)(1-\alpha_n-\beta_n)\left\|\frac{x_{n+1}-x_n}{\alpha_n+\beta_n}\right\|^2 \\
&= (1-\alpha_n-\beta_n)\|x_n-p\|^2+\alpha_n\|k_n-p\|^2 \\
&\quad +\beta_n\|W_nz_n-p\|^2-\frac{1-\alpha_n-\beta_n}{\alpha_n+\beta_n}\|x_{n+1}-x_n\|^2 \\
&\leq (1-\alpha_n-\beta_n)\|x_n-p\|^2+\alpha_n\left[(1+\gamma_n)\|z_n-p\|^2+c_n\right] \\
&\quad +\beta_n\|z_n-p\|^2-\frac{1-\alpha_n-\beta_n}{\alpha_n+\beta_n}\|x_{n+1}-x_n\|^2 \\
&\leq (1-\alpha_n-\beta_n)\|x_n-p\|^2+\alpha_n\left[(1+\gamma_n)\|x_n-p\|^2+c_n\right] \\
&\quad +\beta_n\|x_n-p\|^2-\frac{1-\alpha_n-\beta_n}{\alpha_n+\beta_n}\|x_{n+1}-x_n\|^2 \\
&= (1-\alpha_n)\|x_n-p\|^2+\alpha_n\left[(1+\gamma_n)\|x_n-p\|^2+c_n\right] \\
&\quad -\frac{1-\alpha_n-\beta_n}{\alpha_n+\beta_n}\|x_{n+1}-x_n\|^2 \\
&= (1+\alpha_n\gamma_n)\|x_n-p\|^2+\alpha_nc_n-\frac{1-\alpha_n-\beta_n}{\alpha_n+\beta_n}\|x_{n+1}-x_n\|^2 \\
&\leq (1+\gamma_n)\|x_n-p\|^2+c_n-(1-\alpha_n-\beta_n)\|x_{n+1}-x_n\|^2,
\end{aligned} \tag{113}$$

which immediately yields

$$\begin{aligned}
(1-\alpha_n-\beta_n)\|x_{n+1}-x_n\|^2 &\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2 \\
&\quad +\gamma_n\|x_n-p\|^2+c_n.
\end{aligned} \tag{114}$$

From $\lim_{n\to\infty}\gamma_n=0$, $\lim_{n\to\infty}c_n=0$, $\limsup_{n\to\infty}(\alpha_n+\beta_n)<1$, and the existence of $\lim_{n\to\infty}\|x_n-p\|$, it follows that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{115}$$

Again, utilizing Lemma 7(b), we obtain from (105), (106), and (111) that

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&= \|(1-\alpha_n-\beta_n)(x_n-p)+\alpha_n(k_n-p)+\beta_n(W_nz_n-p)\|^2 \\
&\leq (1-\alpha_n-\beta_n)\|x_n-p\|^2+\alpha_n\|k_n-p\|^2 \\
&\quad +\beta_n\|W_nz_n-p\|^2-\beta_n(1-\alpha_n-\beta_n)\|W_nz_n-x_n\|^2 \\
&\leq (1-\alpha_n-\beta_n)\|x_n-p\|^2+\alpha_n\left[(1+\gamma_n)\|z_n-p\|^2+c_n\right] \\
&\quad +\beta_n\|z_n-p\|^2-\beta_n(1-\alpha_n-\beta_n)\|W_nz_n-x_n\|^2 \\
&\leq (1-\alpha_n-\beta_n)\|x_n-p\|^2+\alpha_n\left[(1+\gamma_n)\|x_n-p\|^2+c_n\right] \\
&\quad +\beta_n\|x_n-p\|^2-\beta_n(1-\alpha_n-\beta_n)\|W_nz_n-x_n\|^2 \\
&= (1-\alpha_n)\|x_n-p\|^2+\alpha_n\left[(1+\gamma_n)\|x_n-p\|^2+c_n\right] \\
&\quad -\beta_n(1-\alpha_n-\beta_n)\|W_nz_n-x_n\|^2 \\
&= (1+\alpha_n\gamma_n)\|x_n-p\|^2+\alpha_nc_n \\
&\quad -\beta_n(1-\alpha_n-\beta_n)\|W_nz_n-x_n\|^2 \\
&\leq (1+\gamma_n)\|x_n-p\|^2+c_n-\beta_n(1-\alpha_n-\beta_n)\|W_nz_n-x_n\|^2,
\end{aligned} \tag{116}$$

which leads to

$$\begin{aligned}
\beta_n(1-\alpha_n-\beta_n)\|W_nz_n-x_n\|^2 &\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2 \\
&\quad +\gamma_n\|x_n-p\|^2+c_n.
\end{aligned} \tag{117}$$

From $\lim_{n\to\infty}\gamma_n=0$, $\lim_{n\to\infty}c_n=0$, condition (ii), and the existence of $\lim_{n\to\infty}\|x_n-p\|$, it follows that

$$\lim_{n\to\infty}\|W_nz_n-x_n\|=0. \tag{118}$$

Note that

$$x_{n+1} - x_n = \alpha_n (k_n - x_n) + \beta_n (W_n z_n - x_n). \tag{119}$$

Hence, it is easy to see from $0 < a \le \alpha_n \le 1$ that

$$\begin{aligned} a \|x_n - k_n\| &\le \alpha_n \|k_n - x_n\| \\ &\le \|x_{n+1} - x_n - \beta_n (W_n z_n - x_n)\| \\ &\le \|x_{n+1} - x_n\| + \beta_n \|W_n z_n - x_n\| \\ &\le \|x_{n+1} - x_n\| + \|W_n z_n - x_n\|. \end{aligned} \tag{120}$$

From (115) and (118), it follows that

$$\lim_{n\to\infty} \|k_n - x_n\| = 0. \tag{121}$$

Combining (107) and (111), we have

$$\begin{aligned} \|k_n - p\|^2 &\le \|z_n - p\|^2 + \gamma_n \|z_n - p\|^2 + c_n \\ &\le \|u_n - p\|^2 + \gamma_n \|z_n - p\|^2 + c_n \\ &\le \left\|\Theta_n^k x_n - p\right\|^2 + \gamma_n M_1 + c_n \\ &\le \|x_n - p\|^2 + r_{k,n} (r_{k,n} - 2\mu_k) \left\|A_k \Theta_n^{k-1} x_n - A_k p\right\|^2 \\ &\quad + \gamma_n M_1 + c_n, \end{aligned} \tag{122}$$

where $M_1 = \sup_{n\ge1} \|z_n - p\|^2$, which implies

$$\begin{aligned} & r_{k,n} (2\mu_k - r_{k,n}) \left\|A_k \Theta_n^{k-1} x_n - A_k p\right\|^2 \\ &\quad \le \|x_n - p\|^2 - \|k_n - p\|^2 + \gamma_n M_1 + c_n \\ &\quad \le \|x_n - k_n\| (\|x_n - p\| + \|k_n - p\|) + \gamma_n M_1 + c_n. \end{aligned} \tag{123}$$

From $\{r_{k,n}\} \subset [e_k, f_k] \subset (0, 2\mu_k)$, $k \in \{1, 2, \ldots, M\}$, $\lim_{n\to\infty} \gamma_n = 0$, $\lim_{n\to\infty} c_n = 0$, and (121), we have

$$\lim_{n\to\infty} \left\|A_k \Theta_n^{k-1} x_n - A_k p\right\| = 0, \quad k = 1, 2, \ldots, M. \tag{124}$$

Combining (108) and (111), we have

$$\begin{aligned} \|k_n - p\|^2 &\le \left\|\Theta_n^k x_n - p\right\|^2 + \gamma_n M_1 + c_n \\ &\le \|x_n - p\|^2 - \left\|\Theta_n^{k-1} x_n - \Theta_n^k x_n\right\|^2 \\ &\quad + 2 r_{k,n} \left\|\Theta_n^{k-1} x_n - \Theta_n^k x_n\right\| \left\|A_k \Theta_n^{k-1} x_n - A_k p\right\| \\ &\quad + \gamma_n M_1 + c_n, \end{aligned} \tag{125}$$

which implies

$$\begin{aligned} & \left\|\Theta_n^{k-1} x_n - \Theta_n^k x_n\right\|^2 \\ & \le \|x_n - p\|^2 - \|k_n - p\|^2 \\ &\quad + 2 r_{k,n} \left\|\Theta_n^{k-1} x_n - \Theta_n^k x_n\right\| \left\|A_k \Theta_n^{k-1} x_n - A_k p\right\| + \gamma_n M_1 + c_n \\ & \le \|x_n - k_n\| (\|x_n - p\| + \|k_n - p\|) \\ & + 2 r_{k,n} \left\|\Theta_n^{k-1} x_n - \Theta_n^k x_n\right\| \left\|A_k \Theta_n^{k-1} x_n - A_k p\right\| + \gamma_n M_1 + c_n. \end{aligned} \tag{126}$$

From $\lim_{n\to\infty} \gamma_n = 0$, $\lim_{n\to\infty} c_n = 0$, (121), and (124), we get

$$\lim_{n\to\infty} \left\|\Theta_n^{k-1} x_n - \Theta_n^k x_n\right\| = 0, \quad k = 1, 2, \ldots, M. \tag{127}$$

From (127), we have

$$\begin{aligned} \|x_n - u_n\| &= \left\|\Theta_n^0 x_n - \Theta_n^M x_n\right\| \\ &\le \left\|\Theta_n^0 x_n - \Theta_n^1 x_n\right\| + \left\|\Theta_n^1 x_n - \Theta_n^2 x_n\right\| \\ &\quad + \cdots + \left\|\Theta_n^{M-1} x_n - \Theta_n^M x_n\right\| \longrightarrow \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{128}$$

Combining (109) and (111), we obtain

$$\begin{aligned} \|k_n - p\|^2 &\le \|z_n - p\|^2 + \gamma_n \|z_n - p\|^2 + c_n \\ &\le \left\|\Omega_n^i u_n - p\right\|^2 + \gamma_n M_1 + c_n \\ &\le \|x_n - p\|^2 + \lambda_{i,n} (\lambda_{i,n} - 2\eta_i) \left\|B_i \Omega_n^{i-1} u_n - B_i p\right\|^2 \\ &\quad + \gamma_n M_1 + c_n, \end{aligned} \tag{129}$$

where $i \in \{1, 2, \ldots, N\}$, which implies

$$\begin{aligned} & \lambda_{i,n} (2\eta_i - \lambda_{i,n}) \left\|B_i \Omega_n^{i-1} u_n - B_i p\right\|^2 \\ &\quad \le \|x_n - p\|^2 - \|k_n - p\|^2 + 2\lambda_n \alpha_n \|p\| \|p - t_n\| \\ &\qquad + \gamma_n \|z_n - p\|^2 + c_n \\ &\quad \le \|x_n - k_n\| (\|x_n - p\| + \|k_n - p\|) + \gamma_n M_1 + c_n. \end{aligned} \tag{130}$$

From $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$, $i \in \{1, 2, \ldots, N\}$, $\lim_{n\to\infty} \gamma_n = 0$, $\lim_{n\to\infty} c_n = 0$, and (121), we get

$$\lim_{n\to\infty} \left\|B_i \Omega_n^{i-1} u_n - B_i p\right\| = 0, \quad i \in \{1, 2, \ldots, N\}. \tag{131}$$

Combining (110) and (111), we have

$$\begin{aligned}\|k_n - p\|^2 &\le \|z_n - p\|^2 + \gamma_n\|z_n - p\|^2 + c_n \\ &\le \left\|\Omega_n^i u_n - p\right\|^2 + \gamma_n M_1 + c_n \\ &\le \|x_n - p\|^2 - \left\|\Omega_n^{i-1}u_n - \Omega_n^i u_n\right\|^2 \\ &\quad + 2\lambda_{i,n}\left\|\Omega_n^{i-1}u_n - \Omega_n^i u_n\right\|\left\|B_i\Omega_n^{i-1}u_n - B_i p\right\| \\ &\quad + \gamma_n M_1 + c_n,\end{aligned} \tag{132}$$

which implies

$$\begin{aligned}&\left\|\Omega_n^{i-1}u_n - \Omega_n^i u_n\right\|^2 \\ &\quad\le \|x_n - p\|^2 - \|k_n - p\|^2 \\ &\qquad + 2\lambda_{i,n}\left\|\Omega_n^{i-1}u_n - \Omega_n^i u_n\right\|\left\|B_i\Omega_n^{i-1}u_n - B_i p\right\| + \gamma_n M_1 + c_n \\ &\quad\le \|x_n - k_n\|\left(\|x_n - p\| + \|k_n - p\|\right) \\ &\qquad + 2\lambda_{i,n}\left\|\Omega_n^{i-1}u_n - \Omega_n^i u_n\right\|\left\|B_i\Omega_n^{i-1}u_n - B_i p\right\| + \gamma_n M_1 + c_n.\end{aligned} \tag{133}$$

From $\lim_{n\to\infty}\gamma_n = 0$, $\lim_{n\to\infty}c_n = 0$, (121), and (131), we obtain

$$\lim_{n\to\infty}\left\|\Omega_n^{i-1}u_n - \Omega_n^i u_n\right\| = 0, \quad i \in \{1, 2, \ldots, N\}. \tag{134}$$

By (134), we have

$$\begin{aligned}\|u_n - z_n\| &= \left\|\Omega_n^0 u_n - \Omega_n^N u_n\right\| \\ &\le \left\|\Omega_n^0 u_n - \Omega_n^1 u_n\right\| + \left\|\Omega_n^1 u_n - \Omega_n^2 u_n\right\| \\ &\quad + \cdots + \left\|\Omega_n^{N-1}u_n - \Omega_n^N u_n\right\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{135}$$

From (128) and (135), we have

$$\begin{aligned}\|x_n - z_n\| &\le \|x_n - u_n\| + \|u_n - z_n\| \\ &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{136}$$

By (121) and (136), we obtain

$$\begin{aligned}\|k_n - z_n\| &\le \|k_n - x_n\| + \|x_n - z_n\| \\ &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{137}$$

We note that

$$k_n - z_n = (1 - \delta_n)(S^n z_n - z_n). \tag{138}$$

From $\delta_n \le d < 1$ and (137), we have

$$\lim_{n\to\infty}\|S^n z_n - z_n\| = 0. \tag{139}$$

On the other hand, we observe that

$$\|z_{n+1} - z_n\| \le \|z_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| + \|x_n - z_n\|. \tag{140}$$

By (115) and (136), we have

$$\lim_{n\to\infty}\|z_{n+1} - z_n\| = 0. \tag{141}$$

We note that

$$\begin{aligned}\|z_n - Sz_n\| &\le \|z_n - z_{n+1}\| + \left\|z_{n+1} - S^{n+1}z_{n+1}\right\| \\ &\quad + \left\|S^{n+1}z_{n+1} - S^{n+1}z_n\right\| + \left\|S^{n+1}z_n - Sz_n\right\|.\end{aligned} \tag{142}$$

From (139), (141), Lemma 9, and the uniform continuity of S, we obtain

$$\lim_{n\to\infty}\|z_n - Sz_n\| = 0. \tag{143}$$

In addition, note that

$$\begin{aligned}\|z_n - Wz_n\| &\le \|z_n - W_n z_n\| + \|W_n z_n - Wz_n\| \\ &\le \|z_n - x_n\| + \|x_n - W_n z_n\| + \|W_n z_n - Wz_n\|.\end{aligned} \tag{144}$$

So, from (118), (136), and [4, Remark 3.2], it follows that

$$\lim_{n\to\infty}\|z_n - Wz_n\| = 0. \tag{145}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to w. From (136), we have that $z_{n_i} \rightharpoonup w$. From (143) and the uniform continuity of S, we have $\lim_{n\to\infty}\|z_n - S^m z_n\| = 0$, for any $m \ge 1$. So, from Lemma 11, we have $w \in \operatorname{Fix}(S)$. Utilizing the similar arguments to those in the proof of Theorem 24, we can derive $w \in \bigcap_{k=1}^M \operatorname{GMEP}(F_k, \varphi_k, A_k) \cap \bigcap_{i=1}^N I(B_i, R_i) \cap \operatorname{Fix}(W)$. Consequently, $w \in F$. This shows that $\omega_w(x_n) \subset F$.

Next, let us show that $\omega_w(x_n)$ is a single-point set. As a matter of fact, let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ such that $x_{n_j} \rightharpoonup w'$. Then, we get $w' \in F$. If $w \ne w'$, from the Opial condition, we have

$$\begin{aligned}\lim_{n\to\infty}\|x_n - w\| &= \lim_{i\to\infty}\left\|x_{n_i} - w\right\| < \lim_{i\to\infty}\left\|x_{n_i} - w'\right\| \\ &= \lim_{n\to\infty}\left\|x_n - w'\right\| = \lim_{j\to\infty}\left\|x_{n_j} - w'\right\| \\ &< \lim_{j\to\infty}\left\|x_{n_j} - w\right\| = \lim_{n\to\infty}\|x_n - w\|.\end{aligned} \tag{146}$$

This attains a contradiction. So we have $w = w'$. Put $v_n = P_F(x_n)$. Since $w \in F$, we have $\langle x_n - v_n, v_n - w\rangle \ge 0$. By Lemma 15, we have that $\{v_n\}$ converges strongly to some $w_0 \in F$. Since $\{x_n\}$ converges weakly to w, we have

$$\langle w - w_0, w_0 - w\rangle \ge 0. \tag{147}$$

Therefore, we obtain $w = w_0 = \lim_{n\to\infty}P_F x_n$. This completes the proof. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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