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Research Article

Lower Estimates for Certain Harmonic Functions in the Half Space

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We will give the growth properties of harmonic functions of order greater than one in a half space, which generalize the result obtained by B. Levin in a half plane.

1. Introduction and Main Theorem

Let $\mathbf{R}$ and $\mathbf{R}_+$ be the sets of all real numbers and of all positive real numbers, respectively. Let $\mathbf{R}^n$ $(n \geq 3)$ denote the n-dimensional Euclidean space with points $x = (x', x_n)$, where $x' = (x_1, x_2, \ldots, x_{n-1}) \in \mathbf{R}^{n-1}$ and $x_n \in \mathbf{R}$. The boundary and closure of an open set D of $\mathbf{R}^n$ are denoted by ∂D and $\overline{D}$, respectively. The upper half space is the set $H = \{(x', x_n) \in \mathbf{R}^n : x_n > 0\}$, whose boundary is ∂H.

For a set E, $E \subset \mathbf{R}_+ \cup \{0\}$, we denote $\{x \in H : |x| \in E\}$ and $\{x \in \partial H : |x| \in E\}$ by HE and ∂HE, respectively. We identify $\mathbf{R}^n$ with $\mathbf{R}^{n-1} \times \mathbf{R}$ and $\mathbf{R}^{n-1}$ with $\mathbf{R}^{n-1} \times \{0\}$, writing typical points $x, y \in \mathbf{R}^n$ as $x = (x', x_n)$, $y = (y', y_n)$, where $y' = (y_1, y_2, \ldots, y_{n-1}) \in \mathbf{R}^{n-1}$, and putting

$$\begin{gathered} x \cdot y = \sum_{j=1}^{n} x_j y_j = x' \cdot y' + x_n y_n, \\ |x| = \sqrt{x \cdot x}, \qquad |x'| = \sqrt{x' \cdot x'}, \\ |x'| = |x| \cos\theta, \quad x_n = |x| \sin\theta \quad \left(0 < \theta \leq \frac{\pi}{2}\right). \end{gathered} \tag{1}$$

Let B_r denote the open ball with center at the origin and radius r (> 0) in $\mathbf{R}^n$. We use the standard notations $u^+ = \max(u, 0)$ and $u^- = -\min(u, 0)$. In the sense of Lebesgue measure $dy' = dy_1 \cdots dy_{n-1}$ and $dy = dy' dy_n$. Let σ denote $(n-1)$-dimensional surface area measure and let $\partial/\partial n$ denote differentiation along the inward normal into H.

The estimate we deal with has a long history which can be traced back to Levin's estimate of harmonic functions from below (see, e.g., [1, page 209]).

Theorem A. *Let A_1 be a constant and let, $u(z)$ be harmonic in the upper half space $\mathbf{C}_+$ and continuous on $\partial\mathbf{C}_+$. Suppose that*

$$\begin{gathered} u(z) \leq A_1 R^{\rho}, \quad z \in \mathbf{C}_+,\ R = |z| > 1,\ \rho > 1, \\ |u(z)| \leq A_1, \quad |z| \leq 1,\ \operatorname{Im} z \geq 0. \end{gathered} \tag{2}$$

Then

$$u\left(\mathrm{Re}^{i\varphi}\right) \geq -A_2 A_1 \left(1 + R^{\rho}\right) \sin^{-1}\varphi, \qquad \mathrm{Re}^{i\varphi} \in \mathbf{C}_+, \tag{3}$$

where A_2 is a constant independent of A_1, R, φ, and the function $u(z)$.

Further versions and refinements of Theorem 1 may be found in [2, Chapter 1], [3, 4] and in the paper of Krasichkov-Ternovskiĭ [5].

In this paper, we will consider functions $u(x)$ harmonic in H and continuous on $\overline{H}$. In what follows we shall denote by M various values which do not depend on K, R $(= |x|)$, θ, and the function $u(x)$.

We prove in this note analogous estimates for $u(x)$ in H.

Theorem 1. *Suppose that*

$$u(x) \le KR^{\rho(R)}, \quad x \in H,\ R = |x| > 1,\ \rho(R) > 1, \tag{4}$$

$$u(x) \ge -K, \quad |x| \le 1,\ x_n \ge 0. \tag{5}$$

Then

$$u(x) \ge -MK\left(1 + \rho(R)R^{\rho(R)}\right)\sin^{1-n}\theta, \tag{6}$$

where $x \in H$ and $\rho(R)$ is nondecreasing on $[1, +\infty)$.

Remark 2. If $n = 2$ and $\rho(R) \equiv \rho$, Theorem 1 is just the result of Theorem A.

Theorem 3. *If* (4) *and* (5) *hold, then*

$$u(x) \ge -MK\left(1 + \rho\left(\frac{N+1}{N}R\right)R^{\rho(((N+1)/N)R)}\right)\sin^{1-n}\theta, \tag{7}$$

where $x \in H$, N (≥ 1) is a sufficiently large number, and $\rho(R)$ is defined in Theorem 1.

2. Main Lemmas

Carleman's formula [6] connects the modulus and the zeros of a function analytic in $\mathbf{C}_+$ (see, e.g., [7, page 224]). Nevanlinna's formula (see [1, page 193]) refers to a harmonic function in a half disk. Ren obtained a generalized Nevanlinna-type formula in a half space and Poisson integral forumla for half balls, resepctively, which play important roles in our discussions.

Lemma 4 (see [8]). *If $R > 1$, then one has*

$$\int_{\{x\in H:|x|=R\}} u(x)\frac{nx_n}{R^{n+1}}d\sigma(x) + \int_{\partial H(1,R)} u(x')\left(\frac{1}{|x'|^n} - \frac{1}{R^n}\right)dx' = c_1 + \frac{c_2}{R^n}, \tag{8}$$

where

$$c_1 = \int_{\{x\in H:|x|=1\}}\left((n-1)x_n u(x) + x_n\frac{\partial u(x)}{\partial n}\right)d\sigma(x),$$
$$c_2 = \int_{\{x\in H:|x|=1\}}\left(x_n u(x) - x_n\frac{\partial u(x)}{\partial n}\right)d\sigma(x). \tag{9}$$

Lemma 5 (see [8]). *Let $R > 1$ and let $u(x)$ be a function in $B_R^+ = B_R \cap H$ and continuous in $\overline{B}_R^+$. Then*

$$u(x) = \int_{\{y\in H:|y|=R\}}\frac{R^2 - |x|^2}{\omega_n R}\times\left(\frac{1}{|y-x|^n} - \frac{1}{|y-x^*|^n}\right)u(y)\,d\sigma(y) + \frac{2x_n}{\omega_n}\int_{\partial H[0,R)}\left(\frac{1}{|y'-x|^n} - \frac{R^n}{|x|^n}\frac{1}{|y'-\tilde{x}|^n}\right)\times u(y')\,dy', \tag{10}$$

where $x \in B_R^+$, $\tilde{x} = R^2x/|x|^2$, $x^* = (x', -x_n)$, and $\omega_n = \pi^{n/2}/\Gamma(1+(n/2))$ is the volume of the unit n-ball in $\mathbf{R}^n$.

3. Proof of Theorem 1

By applying Lemma 4 to $u(x)$, we have

$$\begin{aligned}\int_{\{x\in H:|x|=R\}} u^+(x)\frac{nx_n}{R^{n+1}}d\sigma(x) &+ \int_{\partial H(1,R)} u^+(x')\left(\frac{1}{|x'|^n} - \frac{1}{R^n}\right)dx' \\ = \int_{\{x\in H:|x|=R\}} u^-(x)\frac{nx_n}{R^{n+1}}d\sigma(x) &+ \int_{\partial H(1,R)} u^-(x')\left(\frac{1}{|x'|^n} - \frac{1}{R^n}\right)dx' + c_1 + \frac{c_2}{R^n}.\end{aligned} \tag{11}$$

It immediately follows from (4) that

$$\int_{\{x\in H:|x|=R\}} u^+(x)\frac{nx_n}{R^{n+1}}d\sigma(x) \le MKR^{\rho(R)-1},$$
$$\int_{\partial H(1,R)}\int u^+(x')\left(\frac{1}{|x'|^n} - \frac{1}{R^n}\right)dx' \le MKR^{\rho(R)-1}. \tag{12}$$

Hence from (11) and (12) we have

$$\int_{\{x\in H:|x|=R\}} u^-(x)\frac{nx_n}{R^{n+1}}d\sigma(x) \le MKR^{\rho(R)-1}, \tag{13}$$

$$\int_{\partial H(1,R)} u^-(x')\left(\frac{1}{|x'|^n} - \frac{1}{R^n}\right)dx' \le MKR^{\rho(R)-1}. \tag{14}$$

And (14) gives

$$\begin{aligned}\int_{\partial H(1,R)}\frac{u^-(x')}{|x'|^n}dx' &\le \frac{2^n}{2^n-1}\int_{\partial H(1,R)} u^-(x')\left(\frac{1}{|x'|^n} - \frac{1}{(2R)^n}\right)dx' \\ &\le MK\rho(R)(2R)^{\rho(2R)-1}.\end{aligned} \tag{15}$$

Since $-u(x) \le u^-(x)$, by applying Lemma 5 to $-u(x)$, we have

$$-u(x) \le I_1(x) + I_2(x), \tag{16}$$

where

$$I_1(x) = \int_{\{y\in H:\ |\mathcal{Y}|=R\}} \frac{R^2-|x|^2}{\omega_n R} \times \left(\frac{1}{|y-x|^n} - \frac{1}{|y-x^*|^n}\right) u^-(y)\, d\sigma(y),$$
$$I_2(x) = \frac{2x_n}{\omega_n} \int_{\partial H[0,R)} \left(\frac{1}{|y'-x|^n} - \frac{R^n}{|x|^n}\frac{1}{|y'-\tilde{x}|^n}\right) \times u^-(y')\, dy'. \quad (17)$$

We remark that

$$\frac{1}{|y-x|^n} - \frac{1}{|y-x^*|^n} \le \frac{2nx_n y_n}{|y-x|^{n+2}},$$
$$|y-x|^n \ge x_n^n = |x|^n \sin^n\theta, \quad x\in H,\ y_n = 0. \quad (18)$$

If we put $|x| = r > 1/2$ and $R = 2r$ in (16), then we finally have from (13) and (18)

$$I_1(x) \le \int_{\{y\in H:|\mathcal{Y}|=R\}} \frac{R^2-r^2}{\omega_n R} \frac{2nx_n y_n}{\omega_n |y-x|^{n+2}} u^-(y)\, d\sigma(y) \le MK\rho(R) R^{\rho(R)},$$
$$I_2(x) \le I_{21}(x) + I_{22}(x), \quad (19)$$

where

$$I_{21}(x) = \frac{2}{\omega_n x_n^{n-1}} \int_{\partial H(1,R)} u^-(y')\, dy',$$
$$I_{22}(x) = \frac{2}{\omega_n x_n^{n-1}} \int_{\partial H[0,1]} u^-(y')\, dy'. \quad (20)$$

We obtain that

$$I_{21}(x) \le \frac{2R^n}{\omega_n x_n^{n-1}} \int_{\partial H(1,R)} \frac{u^-(y')}{|y'|^n} dy' \le MK\rho(R) R^{\rho(R)} \sin^{1-n}\theta,$$
$$I_{22}(x) \le \frac{2K}{\omega_n x_n^{n-1}} \int_{\partial H[0,1]} dy' \le MK\rho(R) \sin^{1-n}\theta, \quad (21)$$

from (15) and (5), respectively.

From (16), (19), and (21), we have for $|x| > 1/2$

$$-u(x) \le MK\rho(R)\left(1+\rho(R) R^{\rho(R)}\right)\sin^{1-n}\theta. \quad (22)$$

For $|x| \le 1/2$, we have from (5)

$$-u(x) \le K \le K\left(1+\rho(R) R^{\rho(R)}\right)\sin^{1-n}\theta. \quad (23)$$

Thus the conclusion immediately follows from (22) and (23).

4. Proof of Theorem 3

By modifying (15), we have

$$\int_{\partial H(1,R)} \frac{u^-(x')}{|x'|^n} dx' \le \frac{(N+1)^n}{(N+1)^n - N^n} \int_{\partial H(1,R)} u^-(x') \times \left(\frac{1}{|x'|^n} - \frac{1}{(((N+1)/N)R)^n}\right) dx' \le MK\rho\left(\frac{N+1}{N}R\right)\left(\frac{N+1}{N}R\right)^{\rho(((N+1)/N)R)-1}. \quad (24)$$

Then (21), (22), and (23) are replaced accordingly by the following estimates:

$$I_{21}(x) \le MK\rho\left(\frac{N+1}{N}R\right)\left(\frac{N+1}{N}R\right)^{\rho(((N+1)/N)R)-1} \sin^{1-n}\theta,$$
$$-u(x) \le MK\left(1+\rho\left(\frac{N+1}{N}R\right) R^{\rho(((N+1)/N)R)}\right)\sin^{1-n}\theta,$$
$$-u(x) \le K \le MK\left(1+\rho\left(\frac{N+1}{N}R\right) R^{\rho(((N+1)/N)R)}\right)\sin^{1-n}\theta. \quad (25)$$

All (16), (19), (25), and (21) give

$$u(x) \ge -MK\left(1+\rho\left(\frac{N+1}{N}R\right) R^{\rho(((N+1)/N)R)}\right)\sin^{1-n}\theta, \quad (26)$$

from which the conclusion immediately follows.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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