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Research Article

Global and Blow-Up Solutions for a Class of Nonlinear Parabolic Problems under Robin Boundary Condition

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We discuss the global and blow-up solutions of the following nonlinear parabolic problems with a gradient term under Robin boundary conditions: $(b(u))_t = \nabla\cdot(h(t)k(x)a(u)\nabla u) + f(x,u,|\nabla u|^2,t)$, in $D\times(0,T)$, $(\partial u/\partial n)+\gamma u = 0$, on $\partial D\times(0,T)$, $u(x,0) = u_0(x) > 0$, in $\overline{D}$, where $D\subset\mathbb{R}^N$ $(N\geq 2)$ is a bounded domain with smooth boundary ∂D. Under some appropriate assumption on the functions f, h, k, b, and a and initial value u_0, we obtain the sufficient conditions for the existence of a global solution, an upper estimate of the global solution, the sufficient conditions for the existence of a blow-up solution, an upper bound for "blow-up time," and an upper estimate of "blow-up rate." Our approach depends heavily on the maximum principles.

1. Introduction

The study of global and blow-up solutions for nonlinear parabolic equations has received a lot of attention in the past several decades (see [1–4]). In most works, so far, a variety of approaches have been developed in dealing with different nonlinear parabolic problems, such as the existence of global solution, blow-up solution, an upper bound for "blow-up time," an upper estimate of "blow-up rate," or global solution. So far, some applications in physics, chemistry, and biology are relevant to blow-up phenomena which can be found in [5–11]. In this paper, we consider the global and blow-up solutions of the following nonlinear parabolic equation with Robin boundary condition:

$$\begin{gathered}(b(u))_t = \nabla\cdot(h(t)k(x)a(u)\nabla u) + f(x,u,q,t), \\ \text{in } D\times(0,T), \\ \frac{\partial u}{\partial n}+\gamma u = 0, \quad \text{on } \partial D\times(0,T), \\ u(x,0) = u_0(x) > 0, \quad \text{in } \overline{D},\end{gathered} \tag{1}$$

where $q = |\nabla u|^2$, $D\subset\mathbb{R}^N$ $(N\geq 2)$ is a bounded domain with smooth boundary ∂D, $\partial u/\partial n$ represents the outward normal derivative on ∂D, γ is positive constant, u_0 is the initial value, T is the maximal existence time of u, and $\overline{D}$ is the closure of D. Set $\mathbb{R}^+ = (0,+\infty)$. We assume, throughout the paper, that $b(s)$ is a positive $C^3(\mathbb{R}^+)$ function, $b'(s) > 0$ for any $s\in\mathbb{R}^+$, $a(s)$ is a positive $C^2(\mathbb{R}^+)$ function, $k(x)$ is a positive $C^1(\overline{D})$ function, $h(t)$ is a positive $C^1(\mathbb{R}^+)$ function, $f(x,s,d,t)$ is a nonnegative $C^1(\overline{D}\times\mathbb{R}^+\times\overline{\mathbb{R}^+}\times\mathbb{R}^+)$ function, and $u_0(x)$ is a positive $C^2(\overline{D})$ function. Under these assumptions, the classical parabolic equation theory [12] ensures that there exists a unique classical solution $u(x,t)$ with some $T>0$ for the problem (1), and the solution is positive over $\overline{D}\times[0,T)$. Moreover, by the regularity theorem [13], $u(x,t)\in C^3(D\times(0,T))\cap C^2(\overline{D}\times[0,T))$.

The problems of the global and blow-up solutions for nonlinear parabolic equations have been investigated extensively by many authors and have got a lot of meaningful results. Some special cases of problem (1) have been treated already. Ding [14] deals with the following problem:

$$\begin{gathered}(b(u))_t = \nabla\cdot(a(u)\nabla u) + f(u), \quad \text{in } D\times(0,T), \\ \frac{\partial u}{\partial n}+\gamma u = 0, \quad \text{on } \partial D\times(0,T), \\ u(x,0) = h(x) > 0, \quad \text{in } \overline{D},\end{gathered} \tag{2}$$

where D is a bounded domain of $\mathbb{R}^N$ ($N \geq 2$) with smooth boundary ∂D. By constructing auxiliary functions and using a first-order differential inequality technique, Ding derives conditions on the data, which guarantee the existence of blow-up or global solution. The following problem is investigated by Enache in [15]:

$$\begin{gathered} u_t = \nabla \cdot (a(u)\nabla u) + f(u), \quad \text{in } D \times (0,T), \\ \frac{\partial u}{\partial n} + \gamma u = 0, \quad \text{on } \partial D \times (0,T), \\ u(x,0) = h(x) > 0, \quad \text{in } \overline{D}, \end{gathered} \tag{3}$$

where D is a bounded domain of $\mathbb{R}^N$ ($N \geq 2$) with smooth boundary ∂D. By constructing auxiliary functions and first-order differential inequality technique, Enache establishes some conditions on nonlinearities and the initial date to guarantee that $u(x,t)$ exists for all times $t > 0$ or blows up at some finite time T. Besides, the following problem is investigated by Zhang in [16]:

$$\begin{gathered} (b(u))_t = \Delta u + f(u), \quad \text{in } D \times (0,T), \\ \frac{\partial u}{\partial n} + \gamma u = 0, \quad \text{on } \partial D \times (0,T), \\ u(x,0) = h(x) > 0, \quad \text{in } \overline{D}, \end{gathered} \tag{4}$$

where D is a bounded domain in $\mathbb{R}^N$ ($N \geq 2$) with smooth boundary. Under appropriate assumptions on the functions f, b, and h, Zhang obtains the conditions under which the solutions may exist globally or blow up in a finite time. Moreover, upper estimates of the "blow-up time," blow-up rate, and global solutions are obtained also.

In this paper, we obtain the existence theorem of global and blow-up solution by constructing completely different auxiliary functions and technically using maximum principles. As a result, the sufficient conditions for the existence of a global solution and an upper estimate of the global solution and the sufficient conditions for the existence of a blow-up solution, an upper bound for "blow-up time," and an upper estimate of "blow-up rate" are specified under some appropriate assumption on the functions f, h, k, b, and a and initial value u_0. Our results extend and supplement those obtained in [14–16].

The content of this paper is organized as follows. In Section 2, we study the existence of the global solution of (1). In Section 3, we investigate the blow-up solution of (1). In Section 4, we will give a few examples to explain our results.

2. Global Solution

Our main result for the global solution is the following Theorem 1.

Theorem 1. *Let u be a solution of* (1). *Suppose that the following conditions* (*i*)–(*iv*) *are satisfied.*

(i) *For any $s \in \mathbb{R}^+$,*

$$\begin{gathered} \left(sb'(s)\right)' \geq 0, \\ sb'(s) - \left(sb'(s)\right)' \leq 0, \\ \left(\frac{a(s)}{b'(s)}\right)' \leq 0, \\ \left[\frac{1}{a(s)}\left(\frac{a(s)}{b'(s)}\right)' + \frac{1}{b'(s)}\right]' + \frac{1}{a(s)}\left(\frac{a(s)}{b'(s)}\right)' + \frac{1}{b'(s)} \leq 0. \end{gathered} \tag{5}$$

(ii) *For any $(x,s,d,t) \in D \times \mathbb{R}^+ \times \overline{\mathbb{R}^+} \times \mathbb{R}^+$,*

$$\begin{gathered} \left(\frac{f(x,s,d,t)}{h(t)}\right)_t \leq 0, \\ f_d(x,s,d,t)\left[\left(\frac{1}{b'(s)}\right)' + \frac{1}{b'(s)}\right] \leq 0, \\ \left(\frac{f(x,s,d,t)\,b'(s)}{a(s)}\right)_s - \frac{f(x,s,d,t)\,b'(s)}{a(s)} \\ + \frac{h'(t)\left(b'(s)\right)^2}{a(s)\,h(t)} \leq 0. \end{gathered} \tag{6}$$

(iii) *Consider the integration*

$$\int_{m_0}^{+\infty} \frac{b'(s)}{e^s}ds = +\infty, \qquad m_0 = \min_{\overline{D}} u_0(x). \tag{7}$$

(iv) *Consider*

$$\alpha = \max_{\overline{D}} \left\{\frac{\nabla \cdot (h(0)k(x)a(u_0)\nabla u_0) + f(x,u_0,q_0,0)}{e^{u_0}}\right\} > 0, \quad q_0 = |\nabla u_0|^2. \tag{8}$$

Then the solution u to problem (1) *must be a global solution and*

$$u(x,t) \leq H^{-1}\left(\alpha t + H\left(u_0(x,t)\right)\right), \quad (x,t) \in \overline{D} \times \overline{\mathbb{R}^+}, \tag{9}$$

where

$$H(z) = \int_{m_0}^{z} \frac{b'(s)}{e^s}ds, \quad z \geq m_0, \tag{10}$$

and H^{-1} is the inverse function of H.

Proof. Consider the auxiliary function

$$P(x,t) = b'(u)\,u_t - \alpha e^u. \tag{11}$$

Then, we have

$$\nabla P = b'' u_t \nabla u + b' \nabla u_t - \alpha e^u \nabla u, \tag{12}$$

$$\begin{aligned}\Delta P &= b''' u_t |\nabla u|^2 + 2b'' \nabla u \cdot \nabla u_t \\ &\quad + b'' u_t \Delta u + b' \Delta u_t - \alpha e^u |\nabla u|^2 - \alpha e^u \Delta u.\end{aligned} \tag{13}$$

By (1),

$$\begin{aligned}(b(u))_t &= b' u_t = \nabla \cdot (h(t) k(x) a(u) \nabla u) + f \\ &= h(t) k(x) a(u) \Delta u + h(t) k(x) a'(u) |\nabla u|^2 \\ &\quad + h(t) a(u) (\nabla k \cdot \nabla u) + f.\end{aligned} \tag{14}$$

We have

$$\begin{aligned}u_t &= \frac{akh}{b'} \Delta u + \frac{a'kh}{b'} |\nabla u|^2 + \frac{ah}{b'} (\nabla k \cdot \nabla u) + \frac{f}{b'}, \\ (u_t)_t &= h' \left(\frac{ak}{b'} \Delta u + \frac{a'k}{b'} |\nabla u|^2 + \frac{a}{b'} (\nabla k \cdot \nabla u) \right) \\ &\quad + h \left(\frac{ak}{b'} \Delta u + \frac{a'k}{b'} |\nabla u|^2 + \frac{a}{b'} (\nabla k \cdot \nabla u) \right)_t \\ &\quad + \left(\frac{f}{b'} \right)_t \\ &= \left(\frac{a''}{b'} - \frac{a'b''}{(b')^2} \right) hku_t |\nabla u|^2 + \frac{2a'kh}{b'} (\nabla u \cdot \nabla u_t) \\ &\quad + \left(\frac{a'}{b'} - \frac{ab''}{(b')^2} \right) hu_t (\nabla k \cdot \nabla u) \\ &\quad + \frac{ah}{b'} (\nabla k \cdot \nabla u_t) + \left(\frac{a'}{b'} - \frac{ab''}{(b')^2} \right) hku_t \Delta u \\ &\quad + \frac{akh}{b'} \Delta u_t + \frac{a'kh'}{b'} |\nabla u|^2 + \frac{ah'}{b'} (\nabla k \cdot \nabla u) \\ &\quad + \frac{akh'}{b'} \Delta u \\ &\quad + \frac{f_u u_t + 2 f_q (\nabla u \cdot \nabla u_t) + f_t}{b'} - \frac{f b'' u_t}{(b')^2}.\end{aligned} \tag{15}$$

Then

$$\begin{aligned}P_t &= b'' (u_t)^2 + b' (u_t)_t - \alpha e^u u_t \\ &= b'' (u_t)^2 + \left(a' - \frac{ab''}{b'} \right) khu_t \Delta u + akh \Delta u_t \\ &\quad + akh' \Delta u + \left(a'' - \frac{a'b''}{b'} \right) khu_t |\nabla u|^2 + a'kh' |\nabla u|^2 \\ &\quad + \left(2a'kh + 2 f_q \right) (\nabla u \cdot \nabla u_t) \\ &\quad + \left(a' - \frac{ab''}{b'} \right) hu_t (\nabla k \cdot \nabla u) + ah (\nabla k \cdot \nabla u_t) \\ &\quad + ah' (\nabla k \cdot \nabla u) \\ &\quad + \left(f_u - \frac{f b''}{b'} - \alpha e^u \right) u_t + f_t.\end{aligned} \tag{16}$$

By (13) and (16), it follows that

$$\begin{aligned}&\frac{akh}{b'} \Delta P - P_t \\ &\quad = \left(\frac{akhb'''}{b'} + \frac{a'khb''}{b'} - a''kh \right) u_t |\nabla u|^2 \\ &\quad + \left(\frac{2akhb''}{b'} - 2a'kh - 2 f_q \right) (\nabla u \cdot \nabla u_t) \\ &\quad + \left(\frac{2akhb''}{b'} - a'kh \right) u_t \Delta u \\ &\quad - \left(\frac{akh}{b'} \alpha e^u + a'h'k \right) |\nabla u|^2 - \left(\frac{akh}{b'} \alpha e^u + akh' \right) \Delta u \\ &\quad - b'' (u_t)^2 + \left(\frac{f b''}{b'} + \alpha e^u - f_u \right) u_t \\ &\quad + \left(\frac{ahb''}{b'} - a'h \right) u_t (\nabla k \cdot \nabla u) \\ &\quad - ah (\nabla k \cdot \nabla u_t) - ah' (\nabla k \cdot \nabla u) - f_t.\end{aligned} \tag{17}$$

By (14), we have

$$\Delta u = \frac{b'}{akh} u_t - \frac{a'}{a} |\nabla u|^2 - \frac{1}{k} (\nabla k \cdot \nabla u) - \frac{f}{akh}. \tag{18}$$

Substitute (18) into (17) to get

$$\begin{aligned}&\frac{akh}{b'} \Delta P - P_t \\ &\quad = \left(\frac{akhb'''}{b'} + \frac{(a')^2 kh}{a} - \frac{a'khb''}{b'} - a''kh \right) u_t |\nabla u|^2 \\ &\quad + \left(\frac{2akhb''}{b'} - 2a'kh - 2 f_q \right) (\nabla u \cdot \nabla u_t) \\ &\quad + \left(b'' - \frac{a'b'}{a} \right) u_t^2 + \left(\frac{a' f}{a} - \frac{f b''}{b'} - f_u - \frac{b'h'}{h} \right) u_t \\ &\quad + \left(\frac{a'kh}{b'} \alpha e^u - \frac{akh}{b'} \alpha e^u \right) |\nabla u|^2\end{aligned}$$

$$
\begin{aligned}
&-\frac{ahb''}{b'}u_t(\nabla k\cdot\nabla u)-ah(\nabla k\cdot\nabla u_t)\\
&+\frac{ah}{b'}\alpha e^u(\nabla k\cdot\nabla u)+\frac{f}{b'}\alpha e^u+\frac{fh'}{h}-f_t.
\end{aligned}
\tag{19}
$$

By (12), we have

$$
\nabla u_t=\frac{1}{b'}\nabla P-\frac{b''}{b'}u_t\nabla u+\alpha\frac{e^u}{b'}\nabla u. \tag{20}
$$

Next, we substitute (20) into (19) to obtain

$$
\begin{aligned}
&\frac{akh}{b'}\Delta P-P_t\\
&=\left(\frac{2akhb''}{(b')^2}-\frac{2a'kh}{b'}-\frac{2f_q}{b'}\right)(\nabla u\cdot\nabla P)-\frac{ah}{b'}(\nabla k\cdot\nabla P)\\
&+\left(\frac{akhb'''}{b'}-\frac{2akh(b'')^2}{(b')^2}+\frac{a'khb''}{b'}-a''kh\right.\\
&\qquad\left.+\frac{(a')^2kh}{a}+\frac{2b''f_q}{b'}\right)u_t|\nabla u|^2\\
&+\left(\frac{2akhb''}{(b')^2}\alpha e^u-\frac{a'kh}{b'}\alpha e^u-\frac{akh}{b'}\alpha e^u-\frac{2f_q}{b'}\alpha e^u\right)|\nabla u|^2\\
&+\left(b''-\frac{a'b'}{a}\right)u_t^2+\left(\frac{a'f}{a}-\frac{fb''}{b'}-f_u-\frac{b'h'}{h}\right)u_t\\
&+\frac{f}{b'}\alpha e^u+\frac{fh'}{h}-f_t.
\end{aligned}
\tag{21}
$$

So we have

$$
\begin{aligned}
&\frac{akh}{b'}\Delta P+\left(\frac{2a'kh}{b'}+\frac{2f_q}{b'}-\frac{2akhb''}{(b')^2}\right)\\
&\quad\times(\nabla u\cdot\nabla P)+\frac{ah}{b'}(\nabla k\cdot\nabla P)-P_t\\
&=\left(\frac{akhb'''}{b'}-\frac{2akh(b'')^2}{(b')^2}+\frac{a'khb''}{b'}-a''kh+\frac{(a')^2kh}{a}\right.\\
&\qquad\left.+\frac{2b''f_q}{b'}\right)u_t|\nabla u|^2\\
&+\left(\frac{2akhb''}{(b')^2}\alpha e^u-\frac{a'kh}{b'}\alpha e^u-\frac{akh}{b'}\alpha e^u-\frac{2f_q}{b'}\alpha e^u\right)|\nabla u|^2\\
&+\left(b''-\frac{a'b'}{a}\right)u_t^2+\left(\frac{a'f}{a}-\frac{fb''}{b'}-f_u-\frac{b'h'}{h}\right)u_t\\
&+\frac{f}{b'}\alpha e^u+\frac{fh'}{h}-f_t.
\end{aligned}
\tag{22}
$$

According to (11), we have

$$
u_t=\frac{1}{b'}P+\alpha\frac{e^u}{b'}. \tag{23}
$$

Substituting (23) into (22), we have

$$
\begin{aligned}
&\frac{akh}{b'}\Delta P+\left[2kh\left(\frac{a}{b'}\right)'+\frac{2f_q}{b'}\right](\nabla u\cdot\nabla P)+\frac{ah}{b'}(\nabla k\cdot\nabla P)\\
&\quad+\left[\frac{a}{(b')^2}\left(\frac{fb'}{a}\right)_u+\frac{h'}{h}\right]P\\
&\quad+\left[akh\left(\frac{1}{a}\left(\frac{a}{b'}\right)'\right)'+2f_q\left(\frac{1}{b'}\right)'\right]|\nabla u|^2P-P_t\\
&\quad=(-\alpha e^u)\left(\frac{2f_q}{b'}-\frac{2b''f_q}{(b')^2}\right)|\nabla u|^2\\
&\quad+(-\alpha e^u)kh\left(\frac{a'}{b'}-\frac{ab''}{(b')^2}\right)|\nabla u|^2\\
&\quad+\alpha e^u\left(\frac{akhb'''}{(b')^2}-\frac{2akh(b'')^2}{(b')^3}+\frac{a'khb''}{(b')^2}-\frac{a''kh}{b'}\right.\\
&\qquad\left.+\frac{(a')^2kh}{ab'}\right)|\nabla u|^2\\
&\quad-\alpha e^u akh\left(\frac{1}{b'}-\frac{b''}{(b')^2}\right)|\nabla u|^2+\left(b''-\frac{a'b'}{a}\right)u_t^2\\
&\quad+\alpha e^u\left(\frac{a'f}{ab'}-\frac{fb''}{(b')^2}-\frac{f_u}{b'}+\frac{f}{b'}-\frac{h'}{h}\right)+\frac{h'}{h}f-f_t.
\end{aligned}
\tag{24}
$$

Namely,

$$
\begin{aligned}
&\frac{akh}{b'}\Delta P+\left[\left(2kh\left(\frac{a}{b'}\right)'+\frac{2f_q}{b'}\right)\nabla u+\frac{ah}{b'}\nabla k\right]\cdot\nabla P\\
&\quad+\left\{\frac{a}{(b')^2}\left(\frac{fb'}{a}\right)_u+\frac{h'}{h}\right.\\
&\qquad\left.+\left[akh\left(\frac{1}{a}\left(\frac{a}{b'}\right)'\right)'+2f_q\left(\frac{1}{b'}\right)'\right]|\nabla u|^2\right\}P-P_t\\
&=-\alpha e^u\left\{2f_q\left[\left(\frac{1}{b'}\right)'+\frac{1}{b'}\right]+akh\right.\\
&\qquad\left.\times\left[\left(\frac{1}{a}\left(\frac{a}{b'}\right)'+\frac{1}{b'}\right)'+\frac{1}{a}\left(\frac{a}{b'}\right)'+\frac{1}{b'}\right]\right\}|\nabla u|^2
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{(b')^2}{a}\left(\frac{a}{b'}\right)' u_t^2 - \alpha e^u \frac{a}{(b')^2} \\
&\times\left[\left(\frac{fb'}{a}\right)_u - \frac{fb'}{a} + \frac{h'(b')^2}{ah}\right] - h\left(\frac{f}{h}\right)_t .
\end{aligned} \quad (25)
$$

The assumptions (5) and (6) guarantee that the right-hand side of (25) is nonnegative; that is,

$$
\begin{aligned}
&\frac{akh}{b'}\Delta P + \left[\left(2kh\left(\frac{a}{b'}\right)' + \frac{2f_q}{b'}\right)\nabla u + \frac{ah}{b'}\nabla k\right]\cdot\nabla P \\
&\quad + \left\{\frac{a}{(b')^2}\left(\frac{fb'}{a}\right)_u + \frac{h'}{h}\right. \\
&\quad\quad \left. + \left[akh\left(\frac{1}{a}\left(\frac{a}{b'}\right)'\right)' + 2f_q\left(\frac{1}{b'}\right)'\right]|\nabla u|^2\right\} P - P_t \\
&\geq 0, \quad \text{in } D\times(0,T).
\end{aligned} \quad (26)
$$

By applying maximum principle (see [17]), it follows from (26) that P can attain its nonnegative maximum only for $\overline{D}\times\{0\}$ or $\partial D\times(0,T)$.

For $\overline{D}\times\{0\}$, by (8), we have

$$
\begin{aligned}
&\max_{\overline{D}} P(x,0) \\
&= \max_{\overline{D}}\left\{b'(u_0)(u_0)_t - \alpha e^{u_0}\right\} \\
&= \max_{\overline{D}}\{[\nabla\cdot(h(0)k(x)a(u_0)\nabla u_0) \\
&\quad\quad + f(x,u_0,q_0,0)] - \alpha e^{u_0}\} \\
&= \max_{\overline{D}}\left\{e^{u_0}\left[\frac{\nabla\cdot(h(0)k(x)a(u_0)\nabla u_0) + f(x,u_0,q_0,0)}{e^{u_0}}\right.\right. \\
&\quad\quad \left.\left. -\alpha\right]\right\} = 0.
\end{aligned} \quad (27)
$$

For $\partial D\times(0,T)$, we claim that P cannot take a positive maximum at any point (x,t). In fact, suppose that P can take a positive maximum at one point $(x_0,t_0)\in\partial D\times(0,T)$; then

$$
P(x_0,t_0) > 0, \qquad \left.\frac{\partial P}{\partial n}\right|_{(x_0,t_0)} > 0. \quad (28)
$$

Combine (1) and (11) with (23); we have

$$
\begin{aligned}
\frac{\partial P}{\partial n} &= b'' u_t \frac{\partial u}{\partial n} + b'\frac{\partial u_t}{\partial n} - \alpha e^u \frac{\partial u}{\partial n} \\
&= -\gamma b'' u u_t + b'\left(\frac{\partial u}{\partial n}\right)_t + \gamma\alpha u e^u \\
&= -\gamma b'' u u_t + b'(-\gamma u)_t + \gamma\alpha u e^u \\
&= -\gamma (ub')' u_t + \gamma\alpha u e^u \\
&= -\gamma (ub')'\left(\frac{1}{b'}P + \alpha\frac{1}{b'}e^u\right) + \gamma\alpha u e^u \\
&= -\gamma\frac{(ub')'}{b'}P + \gamma\alpha e^u \frac{ub' - (ub')'}{b'}, \\
&\quad\quad \text{on } \partial D\times(0,T).
\end{aligned} \quad (29)
$$

Next, by using a part condition of (5) $(sb'(s))' \geq 0$, $sb'(s) - (sb'(s))' \leq 0$ for any $s\in\mathbb{R}^+$, we can obtain

$$
\left.\frac{\partial P}{\partial n}\right|_{(x_0,t_0)} \leq 0, \quad (30)
$$

which contradicts with inequality (28). Thus, we know that the maximum of P in $\overline{D}\times[0,T)$ is zero; that is,

$$
P \leq 0, \quad \text{in } \overline{D}\times[0,T). \quad (31)
$$

With (11), we know

$$
\frac{b'(u)}{e^u}u_t \leq \alpha. \quad (32)
$$

For each fixed $x\in\overline{D}$, we integrate (32) from 0 to t:

$$
\int_0^t \frac{b'(u)}{e^u}u_t dt = \int_{u_0(x)}^{u(x,t)} \frac{b'(s)}{e^s}ds \leq \alpha t, \quad (33)
$$

which implies that u must be a global solution of (1). In fact, suppose that u blows up at finite time T; then

$$
\lim_{t\to T^-} u(x,t) = +\infty. \quad (34)
$$

Passing to the limit as $t \to T^-$ in (33) yields

$$
\begin{aligned}
&\int_{u_0(x)}^{+\infty}\frac{b'(s)}{e^s}ds \leq \alpha T, \\
&\int_{m_0}^{+\infty}\frac{b'(s)}{e^s}ds = \int_{m_0}^{u_0(x)}\frac{b'(s)}{e^s}ds + \int_{u_0(x)}^{+\infty}\frac{b'(s)}{e^s}ds \\
&\quad\quad \leq \int_{m_0}^{u_0(x)}\frac{b'(s)}{e^s}ds + \alpha T < +\infty,
\end{aligned} \quad (35)
$$

which contradicts with the condition (iii). This shows that u is global solution. Moreover, it follows from (33) that

$$
\begin{aligned}
\int_{u_0(x)}^{u(x,t)}\frac{b'(s)}{e^s}ds &= \int_{m_0}^{u(x,t)}\frac{b'(s)}{e^s}ds - \int_{m_0}^{u_0(x)}\frac{b'(s)}{e^s}ds \\
&= H(u(x,t)) - H(u_0(x)) \leq \alpha t.
\end{aligned} \quad (36)
$$

Since H is an increasing function, we have

$$
u(x,t) \leq H^{-1}(\alpha t + H(u_0(x))). \quad (37)
$$

The proof is completed. □

3. Blow-Up Solution

The following theorem is the main result for the blow-up solution of (1).

Theorem 2. *Let u be a solution of problem* (1). *Assume that the following conditions* (*i*)–(*iv*) *are satisfied.*

(i) *For any* $s \in \mathbb{R}^+$,

$$\left(sb'(s)\right)' \geq 0, \quad sb'(s) - \left(sb'(s)\right)' \geq 0, \quad \left(\frac{a(s)}{b'(s)}\right)' \geq 0,$$
$$\left[\frac{1}{a(s)}\left(\frac{a(s)}{b'(s)}\right)' + \frac{1}{b'(s)}\right]' + \frac{1}{a(s)}\left(\frac{a(s)}{b'(s)}\right)' + \frac{1}{b'(s)} \geq 0. \tag{38}$$

(ii) *For any* $(x, s, d, t) \in D \times \mathbb{R}^+ \times \overline{\mathbb{R}^+} \times \mathbb{R}^+$,

$$\begin{gathered}\left(\frac{f(x,s,d,t)}{h(t)}\right)_t \geq 0,\\ f_d(x,s,d,t)\left[\left(\frac{1}{b'(s)}\right)' + \frac{1}{b'(s)}\right] \geq 0,\\ \left(\frac{f(x,s,d,t)b'(s)}{a(s)}\right)_s - \frac{f(x,s,d,t)b'(s)}{a(s)} + \frac{h'(t)\left(b'(s)\right)^2}{a(s)h(t)} \geq 0.\end{gathered} \tag{39}$$

(iii) *Consider the integration*

$$\int_{M_0}^{+\infty} \frac{b'(s)}{e^s} ds < +\infty, \qquad M_0 = \max_{\overline{D}} u_0(x). \tag{40}$$

(iv) *Consider*

$$\beta = \min_{\overline{D}} \left\{\frac{\nabla\cdot\left(h(0)k(x)a(u_0)\nabla u_0\right) + f(x, u_0, q_0, 0)}{e^{u_0}}\right\} > 0,$$
$$q_0 = \left|\nabla u_0\right|^2. \tag{41}$$

Then the solution u of problem (1) *must blow up in finite time T, and*

$$\begin{gathered}T \leq \frac{1}{\beta}\int_{M_0}^{+\infty} \frac{b'(s)}{e^s} ds,\\ u(x,t) \leq G^{-1}\left(\beta(T-t)\right), \quad (x,t) \in \overline{D} \times [0, T),\end{gathered} \tag{42}$$

where

$$G(z) = \int_z^{+\infty} \frac{b'(s)}{e^s} ds, \quad z > 0, \tag{43}$$

and G^{-1} is the inverse function of G.

Proof. Construct the following auxiliary function:

$$Q(x,t) = b'(u)u_t - \beta e^u. \tag{44}$$

So we have

$$\begin{aligned}\nabla Q &= b''u_t\nabla u + b'\nabla u_t - \beta e^u \nabla u,\\ \Delta Q &= b'''u_t|\nabla u|^2 + 2b''\nabla u\cdot\nabla u_t + b''u_t\Delta u + b'\Delta u_t\\ &\quad - \beta e^u|\nabla u|^2 - \beta e^u \Delta u.\end{aligned} \tag{45}$$

As the previous derivation from (14) to (25), we can obtain

$$\begin{aligned}&\frac{akh}{b'}\Delta Q\\ &\quad + \left[\left(2kh\left(\frac{a}{b'}\right)' + \frac{2f_q}{b'}\right)\nabla u + \frac{ah}{b'}\nabla k\right]\cdot\nabla Q\\ &\quad + \left\{\frac{a}{(b')^2}\left(\frac{fb'}{a}\right)_u + \frac{h'}{h}\right.\\ &\quad\quad \left.+\left[akh\left(\frac{1}{a}\left(\frac{a}{b'}\right)'\right)' + 2f_q\left(\frac{1}{b'}\right)'\right]|\nabla u|^2\right\}Q - Q_t\\ &= -\beta e^u\left\{2f_q\left[\left(\frac{1}{b'}\right)' + \frac{1}{b'}\right]\right.\\ &\quad\quad \left.+akh\left[\left(\frac{1}{a}\left(\frac{a}{b'}\right)' + \frac{1}{b'}\right)' + \frac{1}{a}\left(\frac{a}{b'}\right)' + \frac{1}{b'}\right]\right\}|\nabla u|^2\\ &\quad - \frac{(b')^2}{a}\left(\frac{a}{b'}\right)'u_t^2 - \beta e^u\frac{a}{(b')^2}\\ &\quad \times\left[\left(\frac{fb'}{a}\right)_u - \frac{fb'}{a} + \frac{h'(b')^2}{ah}\right] - h\left(\frac{f}{h}\right)_t.\end{aligned} \tag{46}$$

It is seen from (38) and (39) that the right-hand side of (46) is nonpositive; that is,

$$\begin{aligned}&\frac{akh}{b'}\Delta Q\\ &\quad + \left[\left(2kh\left(\frac{a}{b'}\right)' + \frac{2f_q}{b'}\right)\nabla u + \frac{ah}{b'}\nabla k\right]\cdot\nabla Q\\ &\quad + \left\{\frac{a}{(b')^2}\left(\frac{fb'}{a}\right)_u + \frac{h'}{h}\right.\\ &\quad\quad \left.+\left[akh\left(\frac{1}{a}\left(\frac{a}{b'}\right)'\right)' + 2f_q\left(\frac{1}{b'}\right)'\right]|\nabla u|^2\right\}Q - Q_t\\ &\quad \leq 0, \quad \text{in } D\times(0,T).\end{aligned} \tag{47}$$

By applying maximum principle (see [17]), it follows from (47) that Q can attain its nonpositive minimum only for $\overline{D}\times\{0\}$ or $\partial D\times(0,T)$.

For $\overline{D}\times\{0\}$, with (41), we have

$$\begin{aligned}\min_{\overline{D}} Q(x,0) &= \min_{\overline{D}}\left\{b'(u_0)(u_0)_t-\beta e^{u_0}\right\}\\ &= \min_{\overline{D}}\{\nabla\cdot(h(0)k(x)a(u_0)\nabla u_0)\\ &\qquad +f(x,u_0,q_0,0)-\beta e^{u_0}\}\\ &=\min_{\overline{D}}\left\{e^{u_0}\left[\frac{\nabla\cdot(h(0)k(x)a(u_0)\nabla u_0)+f(x,u_0,q_0,0)}{e^{u_0}}\right.\right.\\ &\qquad\left.\left.-\beta\right]\right\}=0.\end{aligned}\tag{48}$$

For $\partial D\times(0,T)$, substituting P and α with Q and β in (29), respectively, we have

$$\frac{\partial Q}{\partial n}=-\gamma\frac{(ub')'}{b'}Q+\gamma\beta e^{u}\frac{ub'-(ub')'}{b'},\quad \text{on } \partial D\times(0,T).\tag{49}$$

Combining (47)–(49) with condition (i), we can apply the maximum principles again to obtain that the minimum of Q in $\overline{D}\times[0,T)$ is zero. Thus,

$$Q\geq 0,\quad \text{in } \overline{D}\times[0,T),\tag{50}$$

$$\frac{b'(u)}{e^{u}}u_t\geq\beta.\tag{51}$$

At the point $x^*\in\overline{D}$, where $u_0(x^*)=M_0$, we can integrate (51) from 0 to t to get

$$\int_0^t\frac{b'(u)}{e^{u}}u_t dt=\int_{M_0}^{u(x^*,t)}\frac{b'(s)}{e^{s}}ds\geq\beta t,\tag{52}$$

which implies that u must blow up in finite time. Actually, if u is a global solution of (1), then, for any $t>0$, (52) shows

$$\int_{M_0}^{+\infty}\frac{b'(s)}{e^{s}}ds\geq\int_{M_0}^{u(x^*,t)}\frac{b'(s)}{e^{s}}ds\geq\beta t.\tag{53}$$

Letting $t\to+\infty$ in (53), we have

$$\int_{M_0}^{+\infty}\frac{b'(s)}{e^{s}}ds=+\infty,\tag{54}$$

which contradicts with assumption (40). This shows that u must blow up in finite time $t=T$. Moreover, letting $t\to T$ in (52), we have

$$T\leq\frac{1}{\beta}\int_{M_0}^{+\infty}\frac{b'(s)}{e^{s}}ds.\tag{55}$$

By integrating inequality (51) over $[t,s]$ $(0<t<s<T)$, for each fixed x, we obtain

$$\begin{aligned}G(u(x,t))&\geq G(u(x,t))-G(u(x,s))\\ &=\int_{u(x,t)}^{+\infty}\frac{b'(s)}{e^{s}}ds-\int_{u(x,s)}^{+\infty}\frac{b'(s)}{e^{s}}ds\\ &=\int_{u(x,t)}^{u(x,s)}\frac{b'(s)}{e^{s}}ds=\int_t^s\frac{b'(u)}{e^{u}}u_t dt\geq\beta(s-t).\end{aligned}\tag{56}$$

Hence, by letting $s\to T$, we have

$$G(u(x,t))\geq\beta(T-t).\tag{57}$$

Since G is a decreasing function, we obtain

$$u(x,t)\leq G^{-1}(\beta(T-t)).\tag{58}$$

The proof is completed. □

4. Applications

When $h(t)\equiv 1$, $k(x)\equiv 1$, $f(x,u,q,t)=f(u)$ or $b(u)=u$, $h(t)\equiv 1$, $k(x)\equiv 1$, $f(x,u,q,t)=f(u)$, or $h(t)\equiv 1$, $k(x)\equiv 1$, $a(u)\equiv 1$, $f(x,u,q,t)=f(u)$, the conclusions of Theorems 1 and 2 still hold true. In this sense, our results extend and supplement the results in [14–16]. In what follows, we present several examples to demonstrate the applications of the abstract results.

Example 1. Let u be a solution of the following problem:

$$\begin{aligned}(ue^{u})_t&=\nabla\cdot\left(\frac{1}{1+t}e^{|x|^2}(1+u)e^{u}\nabla u\right)\\ &\quad+\frac{1}{1+t}\left(e^{-u}+e^{q}\right)\left(e^{-t}+|x|^2\right),\quad \text{in } D\times(0,T),\\ &\frac{\partial u}{\partial n}+2u=0,\quad \text{on } \partial D\times(0,T),\\ &u(x,0)=2-|x|^2,\quad \text{in } \overline{D},\end{aligned}\tag{59}$$

where $q=|\nabla u|^2$, $D=\{x=(x_1,x_2,x_3)\mid |x|^2<1\}$ is the unit ball of $\mathbb{R}^3$. Now we have

$$\begin{gathered}b(u)=ue^{u},\qquad h(t)=\frac{1}{1+t},\\ k(x)=e^{|x|^2},\qquad a(u)=(1+u)e^{u},\qquad \gamma=2,\\ f(x,u,q,t)=\frac{1}{1+t}\left(e^{-u}+e^{q}\right)\left(e^{-t}+|x|^2\right),\\ u_0(x)=2-|x|^2.\end{gathered}\tag{60}$$

In order to determine the constant α, we assume

$$s=|x|^2,\tag{61}$$

and then $0 \le s \le 1$ and

$$\alpha = \max_{\overline{D}} \frac{\nabla \cdot (h(0) k(x) a(u_0) \nabla u_0) + f(x, u_0, q_0, 0)}{e^{u_0}}$$
$$= \max_{\overline{D}} \left\{ e^{|x|^2} \left(10|x|^2 - 18\right) + \left(1 + |x|^2\right) \times \left(e^{-4+2|x|^2} + e^{-2+5|x|^2}\right)\right\}$$
$$= \max_{0\le s\le 1} \left\{ e^{s} (10s - 18) + (1 + s) \times \left[e^{-4+2s} + e^{-2+5s}\right]\right\} = 18.6955. \quad (62)$$

It is easy to check that (5)–(7) hold. By Theorem 1, u must be a global solution, and

$$u(x,t) \le H^{-1}\left(\alpha t + H\left(u_0(x)\right)\right) = -1 + \sqrt[2]{18.6955t + \left(1 + u_0(x)\right)^2} = -1 + \sqrt[2]{18.6955t + \left(3 - |x|^2\right)^2}. \quad (63)$$

Example 2. Let u be a solution of the following problem:

$$(u + \ln u)_t = \nabla \cdot \left((1+t) e^{-|x|^2} \left(1 + \frac{1}{u}\right) \nabla u\right) + (1+t)\left(e^{u} - e^{-q}\right)\left(6 + t|x|^2\right), \quad \text{in } D \times (0,T), \quad (64)$$
$$\frac{\partial u}{\partial n} + 2u = 0, \quad \text{on } \partial D \times (0,T),$$
$$u(x,0) = 2 - |x|^2, \quad \text{in } \overline{D},$$

where $q = |\nabla u|^2$, $D = \{x = (x_1, x_2, x_3) \mid |x|^2 < 1\}$ is the unit ball of $\mathbb{R}^3$. Now we have

$$b(u) = u + \ln u, \qquad h(t) = 1 + t,$$
$$k(x) = e^{-|x|^2}, \qquad a(u) = 1 + \frac{1}{u}, \qquad \gamma = 2,$$
$$f(x,u,q,t) = (1+t)\left(e^{u} - e^{-q}\right)\left(6 + t|x|^2\right),$$
$$u_0(x) = 2 - |x|^2. \quad (65)$$

In order to determine the constant β, we assume

$$s = |x|^2, \quad (66)$$

and then $0 \le s \le 1$ and

$$\beta = \min_{\overline{D}} \frac{\nabla \cdot (h(0) k(x) a(u_0) \nabla u_0) + f(x, u_0, q_0, 0)}{e^{u_0}}$$
$$= \min_{\overline{D}} \left\{ \frac{4|x|^6 - 26|x|^4 + 50|x|^2 - 36}{\left(2 - |x|^2\right)^2 e^2} + 6\left(1 - e^{-3|x|^2 - 2}\right)\right\}$$
$$= \min_{0\le s\le 1} \left\{ \frac{4s^3 - 26s^2 + 50s - 36}{(2-s)^2 e^2} + 6\left(1 - e^{-3s-2}\right)\right\} = 3.96997. \quad (67)$$

It is easy to check that (38)–(40) hold. By Theorem 2, u must blow up in finite time T, and

$$T \le \frac{1}{\beta}\int_{M_0}^{+\infty} \frac{b'(s)}{e^s} ds = \frac{1}{3.96997}\int_{2}^{+\infty} \left(1 + \frac{1}{s}\right)\frac{1}{e^s} ds = 0.04641,$$
$$u(x,t) \le G^{-1}\left(\beta(T - t)\right) = G^{-1}(3.96997(T-t)), \quad (68)$$

where

$$G(z) = \int_{z}^{+\infty} \frac{b'(s)}{e^s} ds = \int_{z}^{+\infty} \left(1 + \frac{1}{s}\right)\frac{1}{e^s} ds, \quad z \ge 0, \quad (69)$$

and G^{-1} is the inverse function of G.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] A. Constantin and J. Escher, "Well-posedness, global existence, and blowup phenomena for a periodic quasi-linear hyperbolic equation," *Communications on Pure and Applied Mathematics*, vol. 51, no. 5, pp. 475–504, 1998.

[2] J. M. Chadam, A. Peirce, and H. Yin, "The blowup property of solutions to some diffusion equations with localized nonlinear reactions," *Journal of Mathematical Analysis and Applications*, vol. 169, no. 2, pp. 313–328, 1992.

[3] J. Wang, Z. J. Wang, and J. X. Yin, "A class of degenerate diffusion equations with mixed boundary conditions," *Journal of Mathematical Analysis and Applications*, vol. 298, no. 2, pp. 589–603, 2004.

[4] N. Wolanski, "Global behavior of positive solutions to nonlinear diffusion problems with nonlinear absorption through the boundary," *SIAM Journal on Mathematical Analysis*, vol. 24, no. 2, pp. 317–326, 1993.

[5] J. Ding, "Blow-up of solutions for a class of semilinear reaction diffusion equations with mixed boundary conditions," *Applied Mathematics Letters*, vol. 15, no. 2, pp. 159–162, 2002.

[6] L. Zhang, "Blow-up of solutions for a class of nonlinear parabolic equations," *Zeitschrift für Analysis und ihre Anwendungen*, vol. 25, no. 4, pp. 479–486, 2006.

[7] L. A. Caffarelli and A. Friedman, "Blowup of solutions of nonlinear heat equations," *Journal of Mathematical Analysis and Applications*, vol. 129, no. 2, pp. 409–419, 1988.

[8] V. A. Galaktionov and J. L. Vázquez, "The problem of blow-up in nonlinear parabolic equations," *Discrete and Continuous Dynamical Systems*, vol. 8, no. 2, pp. 399–433, 2002.

[9] H. Zhang and X. Guo, "Blow-up for nonlinear heat equations with absorptions," *European Journal of Pure and Applied Mathematics*, vol. 1, no. 3, pp. 33–39, 2008.

[10] A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, and A. P. Mikhailov, *Blow-Up in Problems for Quasilinear Parabolic Equations*, Nauka, Moscow, Russia, 1987, (Russian), English Translation: Walter de Gruyter, Berlin, Germany, 1995.

[11] L. E. Payne and P. W. Schaefer, "Lower bounds for blow-up time in parabolic problems under Neumann conditions," *Applicable Analysis*, vol. 85, no. 10, pp. 1301–1311, 2006.

[12] H. Amann, "Quasilinear parabolic systems under nonlinear boundary conditions," *Archive for Rational Mechanics and Analysis*, vol. 92, no. 2, pp. 153–192, 1986.

[13] J. Ding and S. Li, "Blow-up solutions for a class of nonlinear parabolic equations with mixed boundary conditions," *Journal of Systems Science and Complexity*, vol. 18, no. 2, pp. 265–276, 2005.

[14] J. Ding, "Global and blow-up solutions for nonlinear parabolic equations with Robin boundary conditions," *Computers & Mathematics with Applications*, vol. 65, no. 11, pp. 1808–1822, 2013.

[15] C. Enache, "Blow-up phenomena for a class of quasilinear parabolic problems under Robin boundary condition," *Applied Mathematics Letters*, vol. 24, no. 3, pp. 288–292, 2011.

[16] H. Zhang, "Blow-up solutions and global solutions for nonlinear parabolic problems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 12, pp. 4567–4574, 2008.

[17] R. P. Sperb, *Maximum Principles and Their Applications*, vol. 157 of *Mathematics in Science and Engineering*, Academic Press, New York, NY, USA, 1981.