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Research Article

Class $\mathfrak{A}$-$KKM(X, Y, Z)$, General KKM Type Theorems, and Their Applications in Topological Vector Space

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The class $\mathfrak{A}$-$KKM(X, Y, Z)$ and generalized KKM mapping are introduced, and some generalized KKM theorems are proved. As applications, Ky Fan's matching theorem and Fan-Browder fixed-point theorem are extended, and some existence theorems of solutions for the generalized vector equilibrium problems are established under noncompact setting, which improve and generalize some known results.

1. Introduction and Preliminaries

In 1929, Knaster, Kurnatoaski, and Mazurkiewicz proved the well-known KKM theorem on n-simplex. In 1961, Fan [1] generalized the KKM theorem from Euclid space to infinite dimensional topological vector spaces by introducing KKM mapping. In 1989, Park [2] introduced s-KKM mapping which is a generalized form of the KKM mapping and obtained some new KKM theorems. In 1991, Chang and Zhang [3] improved fundamentally Ky Fan's KKM mapping that makes KKM theory have great development. Since then, many results related to KKM principle were obtained and applied universally in the fields of nonlinear analysis (see [4–17]).

Let X and Z be topological spaces, let $\langle X \rangle$ and 2^X denote the nonempty finite subset of X and the set of the nonempty subsets of X, respectively, let $A \subset B \subset X$, $\text{int}_B A$ denote the interior of A in B, and let $\text{cl}_B A$ be the closure of A in B (when $B = X, \text{int}_B A = \text{int}\, A$, and $\text{cl}_B A = \text{cl}\, A$). A is said to be compactly closed (resp., compactly open) in X if, for every nonempty compact subset K of X, $A \cap K$ is closed (resp., open) in K. The compact closure of A and the compact interior of A (see [10]) are defined, respectively, by

$$\text{ccl}\, A = \bigcap \{B \subset X : A \subset B \text{ and } B \text{ is compactly closed in } X\},$$

$$\text{cint}\, A = \bigcup \{B \subset X : B \subset A \text{ and } B \text{ is compactly open in } X\}. \tag{1}$$

It is easy to see that $\text{ccl}(X \setminus A) = X \setminus \text{cint}\, A$, $\text{int}\, A \subset \text{cint}\, A \subset A$, $A \subset \text{ccl}\, A \subset \text{cl}\, A$. For every nonempty subset K of X, the subset $\text{ccl}\, A \cap K$ is closed in K and $\text{cint}\, A \cap K$ is open in K. The multivalued mapping $F : X \to 2^Z$ is said to be transfer compactly open valued (resp., transfer compactly closed valued) on X, if, for every $x \in X$ and for each nonempty compact subset K of $Z, z \in F(x) \cap K$ (resp., $z \notin F(x) \cap K$) implies that there exists $\overline{x} \in X$ such that $z \in \text{int}_K(T(\overline{x}) \cap K)$ (resp., $z \notin \text{cl}_K(T(\overline{x}) \cap K)$) (see [10]). The mappings $F^c : X \to 2^Z$ and $F^{-1} : Z \to 2^X$ are defined as $F^c(x) = Z \setminus F(x) = \{z \in Z : z \notin F(x)\}$ and $F^{-1}(z) = \{x \in X : z \in F(x)\}$, respectively.

Recently, many scholars (see [18–21]) not only studied further the KKM theorem involving KKM mapping, but also established some new KKM theorem, fixed-point theorems, and coincidence theorems and utilized them to research the existence of solution to generalized vector equilibria, which makes the KKM theory more perfect and rich.

In this paper, we first introduce the new generalized class $KKM(X, Y, Z)$ consisting of all multifunctions $T : Y \to 2^Z$

that have the generalized KKM property and prove some KKM theorems for $\mathfrak{A}$-KKM mapping. Applying these KKM theorems, the Ky Fan matching theorems and the Fan-Browder fixed-point theorems are generalized. Finally, we establish some new existence theorems of solutions for generalized vector equilibrium problems under noncompact setting. These theorems improve and generalize many known results in the literature.

Definition 1. Let X be a nonempty set, Y a nonempty convex subset of a linear space, and Z a topological space, and let $S : X \to 2^Y$, $T : Y \to 2^Z$, and $F : X \to 2^Z$ be three multivalued mappings. F is said to be a $\mathfrak{A}$-KKM mapping with respect to T if, for any $\{x_0, \ldots, x_n\} \in \langle X \rangle$, there exists $y_i \in S(x_i)$ $(i = 0, 1, \ldots, n)$, such that, for any $\{y_{i_0}, \ldots, y_{i_k}\} \subset \{y_0, \ldots, y_n\}$, one has

$$T\left(\operatorname{co}\left\{y_{i_0}, \ldots, y_{i_k}\right\}\right) \subset \bigcup_{j=0}^{k} F\left(x_{i_j}\right). \tag{2}$$

The multivalued mapping $T : Y \to 2^Z$ is said to have the $\mathfrak{A}$-KKM property, if, for any $\mathfrak{A}$-KKM mapping F with respect to T, the family $\{\overline{F(x)} : x \in X\}$ has the finite intersection property. Let the set $\{T : T$ has the $\mathfrak{A}$-KKM property$\}$ be denoted by $\mathfrak{A}$-$KKM(X, Y, Z)$.

Remark 2. $\mathfrak{A}$-KKM mapping with respect to T is strictly weaker than the generalized S-KKM mapping with respect to T in [5]. It is easy to see that Definition 1 is not the degenerate form of Definition 1 in [11] and Definition 3 in [6] and not a special case of Definition 1 in [6] and Definition 2 in [7].

Example 3. Let $X = Y = Z = R$, $T, S, F : R \to 2^R$ be three mappings defined as follows:

$$\begin{gathered} T(x) = [x, +\infty), \qquad F(x) = [x-1, +\infty), \\ S(x) = [x-2, x+2]. \end{gathered} \tag{3}$$

Then F is $\mathfrak{A}$-KKM mapping with respect to T. In fact, for any $\{x_0, \ldots, x_n\} \in \langle R \rangle$, take $y_i = x_i \in S(x_i)$. For any $\{y_{i_0}, \ldots, y_{i_k}\} \subset \{y_0, \ldots, y_n\}$, it is easy to know that

$$\operatorname{co}\left\{y_{i_0}, \ldots, y_{i_k}\right\} = \left[\min\left\{y_{i_0}, \ldots, y_{i_k}\right\}, \max\left\{y_{i_0}, \ldots, y_{i_k}\right\}\right]. \tag{4}$$

Therefore, by the definition of T and F, we have

$$\begin{aligned} T\left(\operatorname{co}\left\{y_{i_0}, \ldots, y_{i_k}\right\}\right) &= \left[\min\left\{y_{i_0}, \ldots, y_{i_k}\right\}, +\infty\right) \\ &= \left[\min\left\{x_{i_0}, \ldots, x_{i_k}\right\}, +\infty\right), \\ \bigcup_{j=0}^{k} F\left(x_{i_j}\right) &= \left[\min\left\{x_{i_0}, \ldots, x_{i_k}\right\} - 1, +\infty\right). \end{aligned} \tag{5}$$

It follows that

$$T\left(\operatorname{co}\left\{y_{i_0}, \ldots, y_{i_k}\right\}\right) \subset \bigcup_{j=0}^{k} F\left(x_{i_j}\right) = \bigcup_{j=0}^{k} F\left(x_{i_j}\right). \tag{6}$$

However, for $\{x_0, \ldots, x_n\} \in \langle R \rangle$, $S(\{x_0, \ldots, x_n\}) = \bigcup_{i=0}^{n}[x_i - 2, x_i + 2]$, $T(\operatorname{co}(S(x_0, \ldots, x_n)) = [\min\{x_i - 2\}, +\infty)$. Hence

$$T\left(\operatorname{co}\left(S\left(\left\{x_0, \ldots, x_n\right\}\right)\right)\right) \not\subset \bigcup_{i=0}^{n} F\left(x_i\right). \tag{7}$$

Hence, F is not a generalized S-KKM mapping with respect to T.

Lemma 4 (see [12]). *Let X and Z be topological spaces and $F : X \to 2^Z$ a set-valued mapping with $F(X) = Z$. Then the following conditions are equivalent:*

(i) *F is transfer compactly open valued,*

(ii) *for each compact subset K of Z and for each $z \in K$, there exists $x \in X$ such that $z \in \operatorname{cint} F(x) \bigcap K$ and $\bigcup_{x \in X}(F(x) \cap K) = \bigcup_{x \in X}(\operatorname{cint} F(x) \cap K)$.*

Lemma 5 (see [13]). *Let X and Z be topological spaces and $F : X \to 2^Z$ a set-valued mapping with $X \neq F^{-1}(z)$ for each $z \in Z$. Then the following conditions are equivalent:*

(i) *F is transfer compactly closed valued,*

(ii) *the mapping $F^c : X \to 2^Y$ defined by $F^c(x) = Y \setminus F(x)$ for each $x \in X$ is transfer compactly open valued,*

(iii) *for each compact subset K of Z, $\bigcup_{x \in X}(F^c(x) \cap K) = \bigcup_{x \in X}(\operatorname{cint} F^c(x) \cap K)$,*

(iv) *for each compact subset K of Z, $\bigcap_{x \in X}(F(x) \cap K) = \bigcap_{x \in X}(\operatorname{ccl} F(x) \cap K)$.*

Lemma 6. *Let $T \in \mathfrak{A}$-$KKM(X, Y, Z)$, and let X_1 be a nonempty subset of X and Y_1 a nonempty convex subset of Y, $S_1 = S|_{X_1}$; then $T|_{Y_1} \in \mathfrak{A}_1$-$KKM(X_1, Y_1, Z)$.*

Proof. Suppose that $F_1 : X_1 \to 2^Z$ is a $\mathfrak{A}_1$-KKM mapping with respect to $T|_{Y_1}$. Then, for each $\{x_0, \ldots, x_n\} \in \langle X_1 \rangle$, there exists $y_i \in S_1(x_i)$ $(i = 0, 1, \ldots, n)$, such that, for any $\{y_{i_0}, \ldots, y_{i_k}\} \subset \{y_0, \ldots, y_n\}$, we have $T|_{Y_1}(\operatorname{co}\{y_{i_0}, \ldots, y_{i_k}\}) \subset \bigcup_{j=0}^{k} F_1(x_{i_j})$. We define a set-valued mapping $F : X \to 2^Z$ by

$$F(x) = \begin{cases} F_1(x), & \text{if } x \in X_1; \\ Z, & \text{if } x \in X \setminus X_1. \end{cases} \tag{8}$$

Obviously, F is also a $\mathfrak{A}$-KKM mapping with respect to T. Since $T \in \mathfrak{A}$-$KKM(X, Y, Z)$, the family $\{\overline{F(x)} : x \in X\}$ has the finite intersection property which implies that the family $\{\overline{F_1(x)} : x \in X_1\}$ has the finite intersection property. □

2. General KKM Theorems

Theorem 7. *Let X be a topological space, Y a convex space, and Z a Hausdorff space. Suppose that $: X \to 2^Y$, $T : Y \to 2^Z$, and $F : X \to 2^Z$ are three multifunctions satisfying the following:*

(1) *$T \in \mathfrak{A}$-$KKM(X, Y, Z)$ such that $\overline{T(\operatorname{co} S(X))}$ is compact in Z,*

(2) *F is transfer compactly closed values,*

(3) *F is a* $\mathfrak{A}$*-KKM mapping with respect to T.*

Then $\overline{T(\mathrm{co}\, S(X))} \cap (\bigcap_{x\in X} F(x)) \neq \emptyset$.

Proof. Define $F^* : X \to 2^Z$ by

$$F^*(x) = \overline{T(\mathrm{co}\, S(X))} \bigcap \mathrm{ccl}\, F(x) \quad \forall x \in X. \tag{9}$$

By (3), for any $\{x_0, \ldots, x_n\} \in \langle X\rangle$, there exists $y_i \in S(x_i)$ ($i = 0, 1, \ldots, n$), such that, for any $\{y_{i_0}, \ldots, y_{i_k}\} \subset \{y_0, \ldots, y_n\}$, we have $T(\mathrm{co}\{y_{i_0}, \ldots, y_{i_k}\}) \subset \bigcup_{j=0}^{k} \mathrm{ccl}\, F(x_{i_j})$. It follows from $T(\mathrm{co}\{y_{i_0}, \ldots, y_{i_k}\}) \subset \overline{T(\mathrm{co}\, S(X))}$ that we have

$$\begin{aligned} T\left(\mathrm{co}\left\{y_{i_0}, \ldots, y_{i_k}\right\}\right) &\subset \bigcup_{j=0}^{k}\left(\overline{T(\mathrm{co}\, S(X))} \cap \mathrm{ccl}\, F\left(x_{i_j}\right)\right) \\ &= \bigcup_{j=0}^{k} F^*\left(x_{i_j}\right) \subset \bigcup_{j=0}^{k} \mathrm{ccl}\, F^*\left(x_{i_j}\right). \end{aligned} \tag{10}$$

This shows that F^* is a $\mathfrak{A}$-*KKM* mapping with respect to T, and so $\{F^*(x) : x \in X\}$ has finite intersection property. Since $F(x)$ is transfer compactly closed by (1) and so ccl $F(x)$ is compactly closed and $\overline{T(\mathrm{co}\, S(X))}$ is compact in Z by (1), consequently $F^*(x)$ is closed in compact subset $\overline{T(\mathrm{co}\, S(X))}$ of Hausdorff space Z. Therefore,

$$\overline{T(\mathrm{co}\, S(X))} \cap \left(\bigcap \{\mathrm{ccl}\, F(x) : x \in X\}\right) = \bigcap_{x\in X} F^*(x) \neq \emptyset. \tag{11}$$

By Lemma 5, we have that $\overline{T(\mathrm{co}\, S(x)} \cap (\bigcap_{x\in X} F(x)) = \overline{T(\mathrm{co}\, S(x)} \cap (\bigcap_{x\in X} \mathrm{ccl}\, F(x)) \neq \emptyset$ holds. □

Remark 8. Theorem 7 improves Theorem 4.3 of Chang et al. [5] in the following two aspects: (1) the generalized *S-KMM* mapping is generalized to $\mathfrak{A}$-*KKM* mapping with respect to T; (2) the compactly closed values property is replaced by the transfer compactly closed values property.

Theorem 9. *Let X be a topological space, Y a topological vector space, and Z a Hausdorff space. Suppose that* $S : X \to 2^Y$, $T : Y \to 2^Z$, *and* $F : X \to 2^Z$ *are three multivalued mappings satisfying the following:*

(1) $T \in \mathfrak{A}$-$KKM(X, Y, Z)$,

(2) *F is transfer compactly closed values,*

(3) *F is a* $\mathfrak{A}$*-KKM mapping with respect to T,*

(4) *for each compact subset* M_1 *of X,* $\overline{T(S(M_1))}$ *is compact in Z, and for each convex subset* M_2 *of X,* $S(M_2)$ *is convex,*

(5) *there exists a nonempty compact subset K of Z such that, for each* $N \in \langle X\rangle$*, there exists compact convex subset* L_N^X *of X including N such that* $\overline{T(S(L_N^X))} \cap (\bigcap_{x\in L_N^X} \mathrm{ccl}\, F(x) \subset K$*. Then* $\bigcap_{x\in X} F(x) \neq \emptyset$.

Proof. Suppose that the conclusion is not true; then $\cap_{x\in X} F(x) = \emptyset$. Define $G : X \to 2^Z$ by $G(x) = Z \setminus F(x)$ for each $x \in X$; then $G(x)$ is nonempty for all $x \in X$. From (2) and Lemma 5, it follows that G is transfer compactly open mapping on X. Since K is compact in Z, by Lemma 4, we have $K = \cup_{x\in X} \mathrm{cint}\, G(x) \cap K$. Hence there exists $N = \{x_0, x_1, \ldots, x_n\}$ such that $K \subset \cup_{i=0}^{n} \mathrm{cint}\, G(x_i) \cap K = \cup_{i=0}^{n} \mathrm{cint}\, G(x_i)$. By (5), there exists compact convex subset L_N^X of X including N such that

$$\overline{T\left(S\left(L_N^X\right)\right)} \cap \left(\bigcap_{x\in L_N^X} \mathrm{ccl}\, F(x)\right) K \tag{12}$$

and $\overline{T(S(L_N^X))} \setminus K \subset \bigcup_{x\in L_N^X} \mathrm{cint}\, G(x)$. Since $\{x_0, x_1, \ldots, x_n\} \subset L_N^X$, we have $K \subset \bigcup_{i=0}^{n} \mathrm{cint}\, G(x_i) \subset \bigcup_{x\in L_N^X} \mathrm{cint}\, G(x)$, so $\overline{T(S(L_N^X))} \subset \bigcup_{x\in L_N^X} \mathrm{cint}\, G(x)$. Therefore,

$$\begin{aligned} &\overline{T\left(S\left(L_N^X\right)\right)} \cap \left(\bigcap_{x\in L_N^X} \mathrm{ccl}\, F(x)\right) \\ &\quad = \overline{T\left(S\left(L_N^X\right)\right)} \cap \left(\bigcap_{x\in L_N^X} \mathrm{ccl}\,(Z \setminus G(x))\right) \\ &\quad = \overline{T\left(S\left(L_N^X\right)\right)} \cap \left(\bigcap_{x\in L_N^X} (Z \setminus \mathrm{cint}\, G(x))\right) = \emptyset. \end{aligned} \tag{13}$$

Since $S(L_N^X)$ is a convex subset of Y, by Lemma 6, $T|_{S(L_N^X)} \in \mathfrak{A}$-$KKM(L_N^X, S(L_N^X), Z)$. Define set-valued mapping F^*, $G^* : L_N^X \to 2^{\overline{T(S(L_N^X))}}$ by $F^*(x) = F(x) \cap \overline{T(S(L_N^X))}$ and $G^*(x) = \overline{T(S(L_N^X))} \setminus F^*(x)$ for each $x \in L_N^X$. Then we have $G^*(x) = \overline{T(S(L_N^X))} \setminus (F(x) \cap \overline{T(S(L_N^X))}) = \overline{T(S(L_N^X))} \cap (Z \setminus F(x)) = \overline{T(S(L_N^X))} \cap G(x)$, and

$$\overline{T\left(S\left(L_N^X\right)\right)} = \bigcup_{x\in L_N^X} \left(\mathrm{cint}\, G(x) \cap \overline{T\left(S\left(L_N^X\right)\right)}\right) = \bigcup_{x\in L_N^X} G^*(x). \tag{14}$$

By Lemma 4, we have that G^* is transfer compactly open valued on L_N^X. Hence it follows from Lemma 5 that F^* is transfer compactly closed valued on L_N^X.

We claim that F^* is a $\mathfrak{A}|_{L_N^X}$-*KKM* mapping with respect to $T|_{S(L_N^X)}$. Since F is a $\mathfrak{A}$-*KKM* mapping with respect to T, for any $\{x_0, \ldots, x_n\} \in \langle L_N^X\rangle \subset \langle X\rangle$, there exists $y_i \in S(x_i) = S(x_i) \cap S(L_N^X) = S|_{L_N^X}(x_i)$ ($i \in \{0, 1, \ldots, n\}$) such that, for any $\{y_{i_0}, \ldots, y_{i_k}\} \subset \{y_0, \ldots, y_n\} \in \langle Y\rangle$,

$$T\left(\mathrm{co}\left\{y_{i_0}, \ldots, y_{i_k}\right\}\right) \subset \bigcup_{j=0}^{k} F\left(x_{i_j}\right). \tag{15}$$

Since $S(L_N^X)$ is convex in Y and $y_{i_j} \in S(L_N^X)$ for each $j = 0, 1, \ldots, k$, we have

$$\begin{aligned} &T|_{S(L_N^X)}\left(\mathrm{co}\left\{y_{i_0}, \ldots, y_{i_k}\right\}\right) \\ &\quad = T\left(\mathrm{co}\left\{y_{i_0}, \ldots, y_{i_k}\right\}\right) \subset \bigcup_{j=0}^{k} F\left(x_{i_j}\right). \end{aligned} \tag{16}$$

Hence $T|_{L_N^X}(\operatorname{co}\{y_{i_0},\dots,y_{i_k}\}) \subset \bigcup_{j=0}^{k}(F(x_{i_j}) \cap \overline{T(S(L_N^X))}) = \cup_{j=0}^{k} F^*(x_{i_j})$. Therefore, F^* is a $\mathfrak{A}|_{L_N^X}$-KKM mapping with respect to $T|_{S(L_N^X)}$. By Theorem 7, we have $\overline{T(S(L_N^X))} \cap (\bigcap_{x\in L_N^X} \operatorname{ccl} F(x)) = \overline{T(S(L_N^X))} \cap (\bigcap_{x\in L_N^X} F(x)) = \overline{T(S(L_N^X))} \cap (\bigcap_{x\in L_N^X} F^*(x)) \neq \emptyset$, which is a contradiction. Therefore, $\bigcap_{x\in X} F(x) \neq \emptyset$. □

Remark 10. Theorem 9 generalizes Theorem 3.3 of Lin and Wan [14] in the following two aspects: (1) from transfer closed values to transfer compactly closed values; (2) from generalized KKM mapping to $\mathfrak{A}$-KKM mapping with respect to T.

Theorem 11. *Let X, Y be two convex spaces and Z a Hausdorff space. Suppose that $S : X \to 2^Y$, $T : Y \to 2^Z$, and $F : X \to 2^Z$ are three multivalued mappings and $T \in \mathfrak{A}$-$KKM(X, Y, Z)$ satisfying the following:*

(1) *$\overline{S(C)}$ is a compact convex subset of Y if C is a compact convex subset of X,*

(2) *for any compact subset Q of Y, $\overline{T(Q)}$ is compact in Z,*

(3) *F is a $\mathfrak{A}$-KKM mapping with respect to T such that F is transfer compactly closed,*

(4) *there exist a nonempty compact convex subset L of X and a compact subset K of Z such that*

$$\bigcap_{x\in L} \operatorname{ccl} F(x) \subseteq K. \tag{17}$$

Then $\overline{T(\operatorname{co} S(X))} \cap (\bigcap\{F(x) : x \in X\}) \neq \emptyset$.

Proof. Assume that $\overline{T(\operatorname{co} S(x))} \cap (\bigcap\{F(x) : x \in X\}) = \emptyset$; then we have $Z = \overline{T(\operatorname{co} S(X))}^c \cup (\bigcup\{F^c(x) : x \in X\})$, and $K \subset \overline{T(\operatorname{co} S(X))}^c \cup (\bigcup\{F^c(x) \cap K : x \in X\})$. It follows from Lemma 5 that

$$\bigcup\{F^c(x) \cap K : x \in X\} = \cup\{\operatorname{cint} F^c(x) \cap K : x \in X\}. \tag{18}$$

Since F^c is transfer compactly open (by condition (3)), $K \subset \overline{T(\operatorname{co} S(X))}^c \cup (\bigcup\{\operatorname{cint} F^c(x) \cap K : x \in X\}) = (\overline{T(\operatorname{co} S(X))}^c \cap K) \cup (\bigcup\{\operatorname{cint} F^c(x) \cap K : x \in X\})$, where $\operatorname{cint} F^c(x) \cap K$ is open in K for each $x \in X$ and $(\overline{T(\operatorname{co} S(X))}^c \cap K)$ is open in K. Therefore, there exists a finite subset $\{x_0,\dots,x_n\}$ of X such that

$$\begin{aligned} K &\subseteq \left(\overline{T(\operatorname{co} S(X))}^c \cap K\right) \cup \left(\bigcup_{i=0}^{n} \operatorname{cint} F^c(x_i) \cap K\right) \\ &\subseteq \overline{T(\operatorname{co} S(X))}^c \cup \left(\bigcup_{i=0}^{n} \operatorname{cint} F^c(x_i)\right). \end{aligned} \tag{19}$$

By (4), we have

$$K^c \subseteq \bigcup_{x\in L} \operatorname{cint} F^c(x). \tag{20}$$

Let $M = \operatorname{co}(L \cup \{x_0,\dots,x_n\})$; then M is a compact convex subset of X such that

$$Z = \overline{T(\operatorname{co} S(x))}^c \cup \left(\bigcup_{x\in M} \operatorname{cint} F^c(x)\right); \tag{21}$$

that is,

$$\overline{T(\operatorname{co} S(x))} \cap \left(\bigcap_{x\in M} \operatorname{ccl} F(x)\right) = \emptyset. \tag{$*$}$$

Define $F^* : M \to 2^Z$ by

$$F^*(x) = \overline{T(\overline{S(M)})} \cap \operatorname{ccl} F(x). \tag{22}$$

Since F is a $\mathfrak{A}$-KKM mapping with respect to T, for any $\{x_0, x_1,\dots,x_n\} \in M$, there exists $y_i \in S(x_i)$ such that, for any $\{y_{i_0}, y_{i_1},\dots,y_{i_k}\} \subset \{y_0, y_1,\dots,y_n\}$, $T(\operatorname{co}\{y_{i_0}, y_{i_1},\dots,y_{i_k}\} \subseteq \cup_{j=0}^{k} F(x_j)$. It follows that

$$\begin{aligned} T\left(\operatorname{co}\left\{y_{i_0}, y_{i_1},\dots,y_{i_k}\right\}\right) &\subseteq \overline{T\left(\overline{S(M)}\right)} \bigcap \bigcup_{j=0}^{k} F\left(x_j\right) \\ &\subset \overline{T\left(\overline{S(M)}\right)} \bigcap \bigcup_{j=0}^{k} \operatorname{ccl} F\left(x_j\right) \\ &\subset \bigcup_{j=0}^{k} F^*\left(x_{i_j}\right), \end{aligned} \tag{23}$$

which shows that F^* is a $\mathfrak{A}$-KKM mapping with respect to T for the triple (M, Y, Z). Since $T \in \mathfrak{A}$-$KKM(X, Y, Z)$, it follows easily that $T \in \mathfrak{A}$-$KKM(M, Y, Z)$. Moreover, since $\overline{S(M)}$ is compact convex, we have $\operatorname{co} S(M) \subseteq \overline{S(M)}$, and $T(\operatorname{co} S(M)) \subseteq T(\overline{S(M)})$. Conditions (1) and (2) imply that $\overline{T(\overline{S(M)})}$ is compact in Z. Then the compactness of $\overline{T(\overline{S(M)})}$ implies that $\overline{T(S(M))}$ is compact in Z. Applying Theorem 7 to S, T, and F^*, we obtain that

$$\begin{aligned} &\overline{T(\operatorname{co} S(M))} \cap \left(\bigcap\{F^*(x) : x \in M\}\right) \neq \emptyset, \\ &\overline{T(\operatorname{co} S(X))} \cap \left(\bigcap\{\operatorname{ccl} F(x) : x \in M\}\right) \neq \emptyset, \end{aligned} \tag{24}$$

which contradicts ($*$). This completes the proof. □

Remark 12. Theorem 11 improves Theorem 5.1 of Chang et al. [5] in the following two aspects: (1) the generalized S-KMM mapping is generalized to $\mathfrak{A}$-KKM mapping with respect to T; (2) the compactly closed values property is replaced by the transfer compactly closed values property.

3. Matching Theorems and Fixed-Point Theorems

In order to apply the above theorem to show the fixed-point theorems, we first establish the following generalization of the Ky Fan's matching theorem.

Theorem 13. *Let X be a nonempty set, Y a convex space, and Z a Hausdorff space. Suppose that $S : X \to 2^Y$, $T : Y \to 2^Z$, and $F : X \to 2^Z$ are three multivalued mappings satisfying the following:*

(1) *$T \in \mathfrak{A}$-$KKM(X, Y, Z)$ such that $\overline{T(\operatorname{co} S(X))}$ is compact in Z,*

(2) *F is transfer compactly open in Z,*

(3) $\overline{T(\mathrm{co}\,S(X))} \subseteq F(X)$.

Then there exists $\{x_0, x_1, \ldots, x_n\} \in \langle X\rangle$ *such that*

$$\overline{T(\mathrm{co}\,S(X))} \cap \left(\bigcap_{i=0}^{n} F(x_i)\right) \neq \emptyset. \quad (25)$$

Proof. Assume that, for any $\{x_0, x_1, \ldots, x_n\} \in \langle X\rangle$, $\overline{T(\mathrm{co}\,S(X))} \cap (\bigcap_{i=0}^{n} F(x_i)) = \emptyset$. Then $\overline{T(\mathrm{co}\,S(X))} \subseteq \cup_{i=0}^{n} F^c(x_i)$ for any $\{x_0, x_1, \ldots, x_n\} \in \langle X\rangle$. Noting that F^c is transfer compactly closed, we have that the conditions of Theorem 7 are satisfied for the mappings S, T, and F^c. Thus

$$\overline{T(\mathrm{co}\,S(X))} \cap \left(\bigcap \{F^c(x) : x \in X\}\right) \neq \emptyset, \quad (26)$$

which implies that $\overline{T(\mathrm{co}\,S(X))} \nsubseteq F(X)$; this contradicts with (3). So there exists a nonempty finite subset $M \subset \langle X\rangle$ such that $\overline{T(\mathrm{co}\,S(X))} \cap \{F(x) : x \in M\} \neq \emptyset$. □

Remark 14. Theorem 13 implies that Theorem 7 holds and if not then $\overline{T(\mathrm{co}\,S(X))} \subseteq \cup_{x\in X} F^c(x) = G(X)$, where $G : X \to 2^Z$ is defined by $G(x) = F^c(x)$. Theorem 13 shows that there exists $M \in X$ such that

$$\overline{T(\mathrm{co}\,S(M))} \cap \left(\bigcap \{G(x) : x \in M\}\right) \neq \emptyset, \quad (27)$$

which implies that $T(\mathrm{co}\,S(M) \nsubseteq F(M)$, contradicting the fact that F is a $\mathfrak{A}$-KKM mapping with respect to T.

Theorem 15. *Let X be a nonempty subset of a compact convex space Y and Z a Hausdorff space. Suppose that* $A : X \to 2^Z$ *satisfies the following:*

(1) *A is transfer compactly open in Z,*

(2) $A(X) = Z$.

Then for any $f \in \mathscr{C}(Y, Z)$ *there exist a finite subset* $\{x_0, \ldots, x_n\}$ *of X and an* $\overline{x} \in \mathrm{co}\{x_0, \ldots, x_n\}$ *such that* $f(\overline{x}) \in \cap_{i=0}^{n} Ax_i$.

Proof. Let $S : X \to 2^Y$ be defined by $S(x) = \{x\}$ for $x \in X$. Then $f \in \mathfrak{A}$-$KKM(X, Y, Z)$. Since $\overline{f(\mathrm{co}(S(X))} \subseteq f(Y)$ and $f(Y)$ is compact, $\overline{f(\mathrm{co}(S(X))}$ is compact in Z. Furthermore, by (2) we have that

$$\overline{f(\mathrm{co}\,(S\,(X))} \subseteq f\,(Y) \subseteq A\,(X)\,. \quad (28)$$

So all of the conditions of Theorem 13 are satisfied for the mappings S, f, and A. Thus, there exists a finite subset $\{x_0, \ldots, x_n\}$ of X such that

$$f\,(\mathrm{co}\,\{x_0, \ldots, x_n\}) \cap \left(\bigcap_{i=1}^{n} A\,(x_i)\right) \neq \emptyset. \quad (29)$$

□

Remark 16. If $A(x)$ is compactly open in Z for each $x \in X$, X is a subset of a convex subset Y of a topological vector space E, and f is the inclusion mapping of X into Y, then Theorem 15 reduces to Lemma 1 by Fan in [15].

In the sequel, we give the famous Fan-Browder type fixed-point theorem. We first give the following conclusion.

Theorem 17. *Suppose that X, Y are two convex spaces and Z is a Hausdorff space. Assume that* $S : X \to 2^Y$, $G : Z \to 2^Y$, *and* $T \in \mathfrak{A}$-$KKM(X, Y, Z)$ *are three functions satisfying the following:*

(1) $\overline{S(C)}$ *is a compact convex subset of Y if C is a compact convex subset of X,*

(2) $\overline{T(Q)}$ *is compact in Z if Q is compact in Y,*

(3) *for any* $z \in T(S(X))$, $G(z)$ *is a nonempty convex subset of Y,*

(4) *there exists a transfer compactly open values mapping* $H : X \to 2^Z$ *such that any* $x \in X$, $H(x) \subset G^{-1}(S(x))$, *and* $\cup_{x\in X} H(x) = Z$,

(5) *there exist a nonempty compact convex subset L of X and a compact subset K of Z such that*

$$M = \bigcap_{x\in L} H^c\,(x) \subset K. \quad (30)$$

Then there exists a finite subset $\{x_0, x_1, \ldots, x_n\}$ *of X; for any* $y_i \in S(x_i)$ $(i = 0, 1, \ldots, n)$ *there exist* $\{y_{i_0}, y_{i_1}, \ldots, y_{i_k}\} \subset \{y_0, y_1, \ldots, y_n\}$, $z \in T(\mathrm{co}\{y_{i_0}, y_{i_1}, \ldots, y_{i_k}\})$ *such that, for any* $j = 0, 1, \ldots, k$, *there exists* $y'_{i_j} \in S(x_{i_j})$ *such that* $\mathrm{co}\{y'_{i_0}, y'_{i_1}, \ldots, y'_{i_k}\} \subset G(z)$.

Proof. Define $F : X \to 2^Z$ by $F(x) = H^c(x)$ for $x \in X$. Then $F(x)$ is transfer compactly closed in Z.

(i) Suppose that $M = \emptyset$. In this case, it is easy to know that $\overline{T(\mathrm{co}(S(X)))} \cap (\bigcap\{F(x) : x \in X\}) = \emptyset$. Then it follows from Theorem 11 that F is not a $\mathfrak{A}$-KKM mapping with respect to T; that is, there exists a finite subset $\{x_0, x_1, \ldots, x_n\}$ of X; for any $y_i = S(x_i)$ $(i = 0, 1, \ldots, n)$, there exists $\{y_{i_0}, y_{i_1}, \ldots, y_{i_k}\} \subset \{y_0, y_1, \ldots, y_n\}$ such that $T(\mathrm{co}\{y_{i_0}, y_{i_1}, \ldots, y_{i_k}\}) \nsubseteq \bigcup_{j=0}^{k} F(x_{i_j})$. Choose $z \in T(\mathrm{co}\{y_{i_0}, y_{i_1}, \ldots, y_{i_k}\})$ such that $z \notin \bigcup_{j=0}^{k} F(x_{i_j})$. Then $\in H(x_{i_j}) \subset G^{-}(S(x_{i_j}))$, and $S(x_{i_j}) \bigcap G(z) \neq \emptyset$ for any $j = 0, 1, \ldots, k$. Thus for any $j = 0, 1, \ldots, k$, there is $y'_{i_j} \in S(x_{i_j})$ such that $y'_{i_j} \in G(z)$. Since $G(z)$ is convex, we see that $\mathrm{co}\{y'_{i_0}, y'_{i_1}, \ldots, y'_{i_k}\} \subset G(z)$.

(ii) Suppose that $M \neq \emptyset$. Checking the proof of case (i), it suffices to show that F is not a $\mathfrak{A}$-KKM mapping. On the contrary, assume that, for any $\{x_0, x_1, \ldots, x_n\} \in \langle X\rangle$, there exists $y_i \in S(x_i)$ such that, for any $\{y_{i_0}, y_{i_1}, \ldots, y_{i_k}\} \subset \{y_0, y_1, \ldots, y_n\}$, we have $T(\mathrm{co}(\{y_{i_0}, y_{i_1}, \ldots, y_{i_k}\})) \subset \cup_{j=0}^{k} F(x_{i_j})$. Then F is a $\mathfrak{A}$-KKM mapping with respect to T, and so by Theorem 11, we have

$$\overline{T\,(\mathrm{co}\,(S\,(X)))} \cap \left(\bigcap \{F\,(x) : x \in X\}\right) \neq \emptyset. \quad (31)$$

In particular, $\bigcap_{x\in X} H^c(x) \neq \emptyset$; that is,

$$\bigcup_{x\in X} H\,(x) \neq Z \quad (32)$$

which contradicts the assumption that $\bigcup_{x\in X} H(x) = Z$. This completes the proof. □

Remark 18. Theorem 17 improves Corollary 5.2 of Chang et al. [5] in the following three aspects: (1) from generalized S-KMM mapping to $\mathfrak{A}$-KKM mapping with respect to T; (2) from compactly closed values to transfer compactly closed values; (3) from the single-valued mapping s to the multivalued mapping S.

For Theorem 17, if $S : X \to 2^Y$ reduces to a single-value mapping $s : X \to Y$, we have the following conclusion.

Theorem 19. *Suppose that X, Y are two convex spaces and Z is a Hausdorff space. Assume that $s : X \to Y$, $G : Z \to 2^Y$, and $T \in \mathfrak{A}$-$KKM(X,Y,Z)$ are three mappings satisfying the following conditions:*

(1) *$\overline{s(C)}$ is a compact convex subset of Y if C is a compact convex subset of X,*

(2) *$\overline{T(Q)}$ is compact in Z if Q is compact in Y,*

(3) *for any $z \in T(s(X))$, $G(z)$ is a nonempty convex subset of Y,*

(4) *there exists a transfer compactly open values mapping $H : X \to 2^Z$ such that any $x \in X$, $H(x) \subset G^{-1}(s(x))$, and $\cup_{x\in X} H(x) = Z$,*

(5) *there exist a nonempty compact convex subset L of X and a compact subset K of Z such that*

$$M = \bigcap_{x\in L} H^c(x) \subset K. \tag{33}$$

Then there exists a finite subset $\{x_0, x_1, \ldots, x_n\}$ of X, $y_i = s(x_i)$ $(i = 0, 1, \ldots, n)$, such that, for any $\{y_{i_0}, y_{i_1}, \ldots, y_{i_k}\} \subset \{y_0, y_1, \ldots, y_n\}$, there exists $z \in T(\mathrm{co}\{s(x_{i_0}), s(x_{i_1}), \ldots, s(x_{i_k})\})$ such that $\mathrm{co}\{s(x_{i_0}), s(x_{i_1}), \ldots, s(x_{i_k})\} \subset G(z)$.

Proof. The proof is similar to Theorem 17. □

Corollary 20. *Suppose that X is a compact convex space. Assume that $s : X \to X$, $G : X \to 2^X$, and $T \in \mathfrak{S}$-$KKM(X,X,X)$ are three functions satisfying the following:*

(1) *$\overline{s(C)}$ is a compact convex subset of X if C is a compact convex subset of X,*

(2) *$\overline{T(Q)}$ is compact in X if Q is compact in X,*

(3) *for any $z \in T(s(X))$, $G(z)$ is a nonempty convex subset of X,*

(4) *there exists a transfer compactly open values mapping $H : X \to 2^Z$ such that any $x \in X$, $H(x) \subset G^{-1}(s(x))$, and $\cup_{x\in X} H(x) = Z$.*

Then there exists a finite subset $x_0, x_1, \ldots, x_n$ of X, $y_i = s(x_i)$ $(i = 0, 1, \ldots, n)$, such that, for any $\{y_{i_0}, y_{i_1}, \ldots, y_{i_k}\} \subset \{y_0, y_1, \ldots, y_n\}$, there exists $z \in T(\mathrm{co}\{s(x_{i_0}), s(x_{i_1}), \ldots, s(x_{i_k})\})$ such that $\mathrm{co}\{s(x_{i_0}), s(x_{i_1}), \ldots, s(x_{i_k})\}) \subset G(z)$.

Corollary 21. *Let X be a compact convex space. Suppose that $G : X \to 2^X$ satisfies the following:*

(1) *for any $y \in X$, $G(y)$ is a nonempty convex subset of X,*

(2) *G^- is transfer open in X.*

Then there is an $\overline{x} \in X$ such that $\overline{x} \in G(x)$.

Proof. Let s and T be the identity mapping id_X; it follows from Corollary 20 that there exist a finite subset $\{x_0, x_1, \ldots, x_n\}$ of X and $z \in \mathrm{co}\{x_0, x_1, \ldots, x_n\}$ such that $\mathrm{co}\{x_0, x_1, \ldots, x_n\} \subset G(z)$. Furthermore, $z \in G(z)$. This completes the proof. □

Remark 22. If X is a nonempty compact convex subset of a topological vector space and for any $x \in X$, $G^-(x)$ is open in X, the above corollary is just the Fan-Browder type fixed-point theorem 1 in [16].

4. Generalized Vector Equilibrium Problems

In this section, we will introduce some definitions and conclusions and show the existence of solutions to the generalized vector equilibrium problems.

Definition 23. Let Y, Z be two topological spaces and X nonempty sets. Let $H : Z \times X \to 2^Y$, $C : Z \to 2^Y$, $Q : Z \to 2^W$ be multivalued mapping. A generalized vector equilibrium problem $(X, Y, Z; H, C)$ is to find $\widehat{z} \in Z$ such that $H(\widehat{z}, x) \cap C(\widehat{z}) \neq \emptyset$, for all $x \in X$. A generalized vector equilibrium problem $(X, Y, Z; Q, H, C)$ is to find $\widehat{z} \in Z$ such that, for each $y \in Y$, there exists $\overline{w} \in Q(\overline{z})$ satisfying $\psi(\overline{z}, \overline{w}, y) \cap C(\overline{z}) \neq \emptyset$.

Lemma 24. *Let Y be a topological vector space and X, Z nonempty sets. Let $S : X \to 2^Y$, $T : Y \to 2^Z$, $H : Z \times X \to 2^Y$, $C : Z \to 2^Y$, $G : Z \times Y \to 2^X$, $D : Z \to 2^X$. Suppose that the following conditions are satisfied:*

(1) *for each $y \in Y$, there exists $z \in T(y)$ such that $G(z, y) \cap D(z) = \emptyset$,*

(2) *for each $x \in X$, $\overline{y} \in Y$, there exists $\overline{z} \in T(\overline{y})$ such that $H(\overline{z}, x) \cap C(\overline{z}) = \emptyset$ implies that $G(z, \overline{y}) \cap D(z) \neq \emptyset$ whenever $y \in S(x)$ and $z \in T(y)$,*

(3) *for each $\overline{y} \in Y$, $R(\overline{y}) = \{y \in Y : G(z, \overline{y}) \cap D(z) \neq \emptyset, \forall z \in T(y)\}$ is convex.*

Then $F : X \to 2^Z$ defined by $F(x) = \{z \in Z : H(z, x) \cap C(z) \neq \emptyset\}$ is a $\mathfrak{A}$-KKM mapping with respect to T.

Proof. If the conclusion does not hold, then there exists $\{x_0, x_1, \ldots, x_n\}$, $\forall y_i \in S(x_i)$, $\exists \{y_{i_0}, \ldots, y_{i_k}\}$ such that

$$T\left(\mathrm{co}\left\{y_{i_0}, \ldots, y_{i_k}\right\}\right) \not\subset \bigcup_{j=0}^{k} F\left(x_{i_j}\right). \tag{34}$$

Therefore, there exist $\overline{y} \in \mathrm{co}\{y_{i_0}, \ldots, y_{i_k}\}$ and $\overline{z} \in T(\overline{y})$ such that, for each $j = 0, \ldots, k$, $\overline{z} \notin F_1(x_{i_j})$. By the definition of F_1, we have

$$H\left(\overline{z}, x_{i_j}\right) \cap C(\overline{z}) = \emptyset, \quad \forall j = 0, \ldots, k. \tag{35}$$

By condition (2), for all $z \in T(y_{i_j})$ and $j = 0, \ldots, k$, we have

$$G\left(z, y_{i_j}\right) \cap D(z) \neq \emptyset. \tag{36}$$

Thus $y_{i_j} \in R(\overline{y})$, and $\overline{y} \in \text{co}\{y_{i_0}, \ldots, y_{i_k}\} \subset R(\overline{y})$. Therefore, $G(z, \overline{y}) \cap D(z) \neq \emptyset$, $\forall z \in T(\overline{y})$; this contradicts with (1). The proof is completed. □

Lemma 25. *Let Y be a topological vector space and X, Z nonempty sets. Let $S : X \to 2^Y$, $T : Y \to 2^Z$, $H : Z \times X \to 2^Y$, $C : Z \to 2^Y$, $G : Z \times Y \to 2^X$, $D : Z \to 2^X$. Suppose that the following conditions are satisfied:*

(1) *for each $y \in Y$ and $z \in T(y)$, $G(z, y) \cap D(z) \neq \emptyset$,*

(2) *for each $x \in X$, $\overline{y} \in Y$, there exists $\overline{z} \in T(\overline{y})$ such that $H(\overline{z}, x) \cap C(\overline{z}) = \emptyset$ implies that $G(z, \overline{y}) \cap D(z) = \emptyset$ for all $y \in S(x)$ and some $z \in T(y)$,*

(3) *for each $\overline{y} \in Y$, $R(\overline{y}) = \{y \in Y : \exists z \in T(y)$ such that $G(z, y) \cap D(z) = \emptyset\}$ is convex.*

Then $F : X \to 2^Z$ defined by $F(x) = \{z \in Z : F(z, x) \cap C(z) \neq \emptyset\}$ is a $\mathfrak{A}$-KKM mapping with respect to T.

Proof. There exists $\{x_0, x_1, \ldots, x_n\}$, $\forall y_i \in S(x_i)$, $\exists\{y_{i_0}, \ldots, y_{i_k}\}$ such that

$$T\left(\text{co}\left\{y_{i_0}, \ldots, y_{i_k}\right\}\right) \not\subset \bigcup_{j=0}^{k} F\left(x_{i_j}\right). \tag{37}$$

Therefore, there exist $\overline{y} \in \text{co}\{y_{i_0}, \ldots, y_{i_k}\}$ and $\overline{z} \in T(\overline{y})$ such that, for each $j = 0, \ldots, k$, $\overline{z} \notin F_1(x_{i_j})$. By the Definition of F, we have

$$H\left(\overline{z}, x_{i_j}\right) \cap C(\overline{z}) = \emptyset, \quad \forall j = 0, \ldots, k. \tag{38}$$

By the condition (2), there exists $z \in Z$ such that, for each $j = 0, \ldots, k$, we have that $z \in T(y_{i_j})$ and

$$G\left(z, y_{i_j}\right) \cap D(z) = \emptyset. \tag{39}$$

Thus $y_{i_j} \in R(\overline{y})$, and $\overline{y} \in \text{co}\{y_{i_0}, \ldots, y_{i_k}\} \subset R(\overline{y})$. Therefore, $G(z, \overline{y}) \cap D(z) = \emptyset$ for some $z \in T(y)$; this contradicts with (i). The proof is completed. □

Remark 26. To avoid the structure of the space, Lemmas 24 and 25 generalize Lemmas 2.4 and 2.5 of X. P. Ding and T. M. Ding [13] from the following two aspects: (1) from generalized *KKM* mapping with respect to T to $\mathfrak{A}$-*KKM* mapping with respect to T; (2) condition (2) in our results is obviously weaker than that in [13].

Definition 27 (see [13]). Let Y and Z be topology spaces and X nonempty set. Let $H : Z \times X \to 2^Y$ and $C : Z \to 2^Y$ be set-valued mappings. $H(z, x)$ is said to be a C-transfer compactly continuous mapping of the generalized vector equilibrium problem $(X, Y, Z; H, C)$ in first argument if, for any compact subset K of Z and any $z \in K$, there exists $x \in X$ such that $H(z, x) \cap C(z) = \emptyset$; then there is a point $x' \in X$ such that $z \in \text{cint}\{x \in Z : H(z, x') \cap C(z) = \emptyset\}$.

Proposition 28 (see [13]). *Let Y and Z be topological spaces and X nonempty set. Let $H : Z \times X \to 2^Y$ and $C : Z \to 2^Y$ be set-valued mappings. $H(z, x)$ is said to be a C-transfer compactly continuous mapping of the generalized vector equilibrium problem $(X, Y, Z; H, C)$ in first argument if and only if the mapping $F : X \to 2^Z$ defined by $F(x) = \{z \in Z : H(z, x) \cap C(z)\} \neq \emptyset$ is a transfer compactly closed-valued mapping.*

Proposition 29 (see [13]). *Let X, Y, and Z be topological spaces. Let $H : Z \times X \to 2^Y$ and $C : Z \to 2^Y$ be set-valued mappings such that*

(i) *C has closed (resp., open) graph;*

(ii) *for each $x \in X$, $H(\cdot, x)$ is upper semicontinuous on each compact subset of Z.*

Then the mapping $F : x \to 2^Z$ defined by $F(x) = \{z \in Z : H(z, x) \cap C(z) \neq \emptyset\}$ has compactly closed values.

Theorem 30. *Let X be a topological space, Y a topological vector space, and Z a Hausdorff space; let $S : X \to 2^Y$, $T \in \mathfrak{A}$-$KKM(X, Y, Z)$, $H : Z \times X \to 2^Y$, $C : Z \to 2^Y$, $G : Z \times Y \to 2^X$, $D : Z \to 2^X$ be multivalued mappings. Suppose that the following conditions are satisfied:*

(1) *$H(z, x)$ is a C-transfer compactly continuous mapping of the generalized vector equilibrium problem $(X, Y, Z; H, C)$,*

(2) *for each $y \in Y$, there exists $z \in T(y)$ such that $G(z, y) \cap D(z) = \emptyset$,*

(3) *for each $x, \overline{y}$, there exists $\overline{z} \in T(\overline{y})$ such that $H(\overline{z}, x) \cap C(\overline{z}) = \emptyset$ which implies that $G(z, \overline{y}) \cap D(z) \neq \emptyset$ whenever $y \in S(x)$ and $z \in T(y)$,*

(4) *for each $\overline{y} \in Y$, the set $R(\overline{y}) = \{y \in Y : G(z, y) \cap D(z) \neq \emptyset, \forall z \in T(y)\}$ is a convex subset of Y,*

(5) *for each compact subset M_1 of X, $\overline{T(S(M_1))}$ is compact in Z, and for each convex subset M_2 of X, $S(M_2)$ is convex,*

(6) *setting $F : X \to 2^Z$ by $F(x) = \{z \in X : H(z, x) \cap C(z) \neq \emptyset\}$, there exists a nonempty compact subset K of Z such that, for each $N \in \langle X \rangle$, there exists compact convex subset L_N^X of X including N such that $\overline{T(S(L_N^X))} \cap (\bigcap_{x \in L_N^X} \text{ccl}\, F_1(x)) \subset K$.*

Then there exists $\widehat{z} \in Z$ such that $H(\widehat{z}, x) \cap C(\widehat{z}) \neq \emptyset$; that is, $\widehat{z}$ is a solution to the generalized vector equilibrium problem $(X, Y, Z; H, C)$.

Proof. By condition (1), F is transfer compactly closed-valued mapping from Proposition 28. By conditions (2)–(4) and Lemma 24, we know that F is a $\mathfrak{A}$-KKM mapping with respect to T. Conditions (5) and (6) imply that conditions (4) and (5) of Theorem 9 hold. From Theorem 9, it follows that $\cap_{x \in X} F(x) \neq \emptyset$ and $H(\widehat{z}, x) \cap C(\widehat{z}) \neq \emptyset$ for all $x \in X$. □

Theorem 31. *Let X be a topological space, Y a topological vector space, and Z a Hausdorff space, and let $S : X \to 2^Y$, $T \in \mathfrak{A}$-$KKM(X, Y, Z)$, $H : Z \times X \to 2^Y$, $C : Z \to$*

2^Y, $G : Z \times Y \to 2^X$, $D : Z \to 2^X$ *be multivalued mappings. Suppose that the following conditions are satisfied:*

(1) *$H(z,x)$ is a C-transfer compactly continuous mapping of the generalized vector equilibrium problem $(X, Y, Z; H, C)$,*

(2) *for each $y \in Y$ and $z \in T(y)$, $G(z,y) \cap D(z) \neq \emptyset$,*

(3) *for each $x, \overline{y}$, there exists $\overline{z} \in T(\overline{y})$ such that $H(\overline{z},x) \cap C(\overline{z}) = \emptyset$ implies that $G(z,\overline{y}) \cap D(z) = \emptyset$ for all $y \in S(x)$ and some $z \in T(y)$,*

(4) *for each $\overline{y} \in Y$, $R(\overline{y}) = \{y \in Y : \exists z \in T(y)$ such that $G(z,y) \cap D(z)\} = \emptyset$ is convex,*

(5) *for each compact subset M_1 of X, $\overline{T(S(M_1))}$ is compact in Z, and for each convex subset M_2 of X, $S(M_2)$ is convex,*

(6) *setting $F : X \to 2^Z$ by $F_1(x) = \{z \in X : H(z,x) \cap C(z) \neq \emptyset\}$, there exists a nonempty compact subset K of Z such that, for each $N \in \langle X \rangle$, there exists compact convex subset L_N^X of X including N such that $\overline{T(S(L_N^X)} \cap (\bigcap_{x \in L_N^X} \operatorname{ccl} F(x)) \subset K$.*

Then there exists $\widehat{x} \in X$ such that $H(\widehat{z},x) \cap C(\widehat{z}) \neq \emptyset$; that is, $\widehat{z}$ is a solution to the generalized vector equilibrium problem $(X, Y, Z; H, C)$.

Proof. By condition (1), F is transfer compactly closed-valued mapping from Proposition 29. By conditions (2)–(4) and Lemma 25, we have that F is a generalized $\mathfrak{A}$-*KKM* mapping with respect to T. Conditions (5) and (6) imply that conditions (4) and (5) of Theorem 9 hold. From Theorem 9, it follows that $\cap_{x \in X} F(x) \neq \emptyset$ and $H(\widehat{z},x) \subset C(\widehat{z})$ for all $x \in X$. □

Theorem 32. *Let X be a topological space, Y a topological vector space, and Z a Hausdorff space, and let $S : X \to 2^Y$, $T \in \mathfrak{A}$-$KKM(X,Y,Z)$, $H : Z \times X \to 2^Y$, $C : Z \to 2^Y$, $G : Z \times Y \to 2^X$, $D : Z \to 2^X$ be multivalued mappings. Suppose that the following conditions are satisfied:*

(1) *C has closed graph,*

(2) *for each $x \in X$, $H(\cdot,x)$ is upper semicontinuous on each compact subset of Z with nonempty compact values on Z,*

(3) *for each $y \in Y$, there exists $z \in T(y)$ such that $G(z,y) \cap D(z) = \emptyset$,*

(4) *for each $x, \overline{y}$, there exists $\overline{z} \in T(\overline{y})$ such that $H(\overline{z},x) \cap C(\overline{z}) = \emptyset$ implies that $G(z,\overline{y}) \cap D(z) \neq \emptyset$ whenever $y \in S(x)$ and $z \in T(y)$,*

(5) *for each $\overline{y} \in Y$, $R(\overline{y}) = \{y \in Y : \exists z \in T(y)$ such that $G(z,y) \cap D(z) = \emptyset\}$ is convex,*

(6) *for each compact subset M_1 of X, $\overline{T(S(M_1))}$ is compact in Z, and for each convex subset M_2 of X, $S(M_2)$ is convex,*

(7) *setting $F : X \to 2^Z$ by $F(x) = \{z \in X : H(z,x) \cap C(z) \neq \emptyset\}$, there exists a nonempty compact subset K of Z such that, for each $N \in \langle X \rangle$, there exists compact convex subset L_N^X of X including N such that $\overline{T(S(L_N^X)} \cap (\bigcap_{x \in L_N^X} \operatorname{ccl} F(x)) \subset K$.*

Then there exists $\widehat{x} \in X$ such that $H(\widehat{z},x) \cap C(\widehat{z}) \neq \emptyset$; that is, $\widehat{z}$ is a solution to the generalized vector equilibrium problem $(X, Y, Z; H, C)$.

Proof. By condition (1) and Proposition 29, F has compactly closed values and so it is a transfer compactly closed-valued mapping. Hence the conclusion of Theorem 32 holds from Theorem 30. □

Theorem 33. *Let X be a topological space, Y a topological vector space, and Z a Hausdorff space, and let $S : X \to 2^Y$, $T \in \mathfrak{A}$-$KKM(X,Y,Z)$, $H : Z \times X \to 2^Y$, $C : Z \to 2^Y$, $G : Z \times Y \to 2^X$, $D : Z \to 2^X$ be multivalued mappings. Suppose that the following conditions are satisfied:*

(1) *C has closed graph,*

(2) *for each $x \in X$, $H(\cdot,x)$ is upper semicontinuous on each compact subset of Z with nonempty compact values on Z,*

(3) *for each $y \in Y$, there exists $z \in T(y)$ such that $G(z,y) \cap D(z) \neq \emptyset$,*

(4) *for each $x, \overline{y}$, there exists $\overline{z} \in T(\overline{y})$ such that $H(\overline{z},x) \cap C(\overline{z}) = \emptyset$ implies that $G(z,\overline{y}) \cap D(z) = \emptyset$ for all $y \in S(x)$ and some $z \in T(y)$,*

(5) *for each $\overline{y} \in Y$, $R(\overline{y}) = \{y \in Y : \exists z \in T(y)$ such that $G(z,y) \cap D(z)\} = \emptyset$ is convex,*

(6) *for each compact subset M_1 of X, $\overline{T(S(M_1))}$ is compact in Z, and for each convex subset M_2 of X, $S(M_2)$ is convex,*

(7) *setting $F : X \to 2^Z$ by $F(x) = \{z \in X : H(z,x) \cap C(z) \neq \emptyset\}$, there exists a nonempty compact subset K of Z such that, for each $N \in \langle X \rangle$, there exists compact convex subset L_N^X of X including N such that $\overline{T(S(L_N^X)} \cap (\bigcap_{x \in L_N^X} \operatorname{ccl} F(x)) \subset K$.*

Then there exists $\widehat{x} \in X$ such that $H(\widehat{z},x) \cap C(\widehat{z}) \neq \emptyset$.

Proof. By condition (1) and Proposition 29, F has compactly closed values and so it is a transfer compactly closed-valued mapping. Hence the conclusion of Theorem 33 holds from Theorem 31. □

The following result is a simplicity version of Theorem 1 in [17].

Lemma 34. *Let X, Y, W, and Z be topological spaces. Let $\psi : Z \times W \times Y \to 2^X$ and $Q : Z \to 2^W$ be set-valued mappings.*

(1) *If, for each fixed $y \in Y$, $(z,w) \to \psi(z,w,y)$ and Q are both lower semicontinuous, then the mapping $H : Z \times X \to 2^Y$ defined by $H(z,x) = \bigcup_{w \in Q(x)} \psi(z,w,y)$ satisfies that, for each $x \in X$, $z \to H(z,x)$ is lower semicontinuous on Z.*

(2) *If, for each fixed* $y \in Y$, $(z, w) \to \psi(z, w, y)$ *and* Q *are both upper semicontinuous with compact values, then, for each* $x \in X$, $z \to H(z, x)$ *is upper semicontinuous on* Z *with nonempty compact values.*

Theorem 35. *Let* X *be a topological space,* Y *a topological vector space, and* Z *a Hausdorff space, and let* $S : X \to 2^Y$, $T \in \mathfrak{A}\text{-}KKM(X, Y, Z)$, $C : Z \to 2^Y$, $D : Z \to 2^X$, $\phi : Z \times W \times X \to 2^Y$, $\psi : Z \times W \times Y \to 2^X$, $Q : Z \to 2^W$ *be multivalued mappings. Suppose that the following conditions are satisfied:*

(1) C *has closed graph,*

(2) *for each* $y \in Y$, $(z, w) \to \psi(z, w, y)$ *and* Q *are both upper semicontinuous compact values on* Z,

(3) *for each* $y \in Y$, *there exists* $z \in T(y)$ *such that* $(\bigcup_{w \in Q(z)} \phi(z, w, y)) \cap D(z) \neq \emptyset$,

(4) *for each* x, $\overline{y}$, *there exists* $\overline{z} \in T(\overline{y})$ *such that* $(\bigcup_{w \in Q(z)} \phi(z, w, y)) \cap C(\overline{z}) = \emptyset$ *implies that* $(\bigcup_{w \in Q(z)} \psi(z, w, y)) \cap D(z) = \emptyset$ *whenever* $y \in S(x)$ *and* $z \in T(y)$,

(5) *for each* $\overline{y} \in Y$, $R(\overline{y}) = \{y \in Y : \bigcup_{w \in Q(z)} \phi(z, w, \overline{y}) \cap D(z) \neq \emptyset, \forall z \in T(y)\}$ *is convex,*

(6) *for each compact subset* M_1 *of* X, $\overline{T(S(M_1))}$ *is compact in* Z, *and for each convex subset* M_2 *of* X, $S(M_2)$ *is convex,*

(7) *setting* $F : X \to 2^Z$ *by* $F(x) = \{z \in X : \bigcup_{w \in Q(z)} \psi(z, w, y) \cap C(z) \neq \emptyset\}$, *there exists a nonempty compact subset* K *of* Z *such that, for each* $N \in \langle X \rangle$, *there exists compact convex subset* L_N^X *of* X *including* N *such that* $\overline{T(S(L_N^X))} \cap (\bigcap_{x \in L_N^X} \operatorname{ccl} F(x)) \subset K$.

Then there exists $\overline{z} \in Z$ *such that, for each* $y \in Y$, *there exists* $\overline{w} \in Q(\overline{z})$ *satisfying* $\psi(\overline{z}, \overline{w}, y) \cap C(\overline{z}) \neq \emptyset$; *that is,* $\hat{z}$ *is a solution to the generalized vector equilibrium problem* $(X, Y, Z; Q, H, C)$.

Proof. Define set-valued mappings $H : Z \times X \to 2^Y$, $G : Z \times Y \to 2^X$ by $H(z, x) = \bigcup_{w \in Q(z)} \psi(z, w, y)$ and $G(z, y) = \bigcup_{w \in Q(z)} \phi(z, w, y))$ for each $(z, x) \in Z \times X$ and $(z, y) \in Z \times Y$, respectively. By Lemma 34 and Theorem 32, the rest is similar to the proof of Theorem 4.7 in [13]. □

Remark 36. For the above results, Theorems 30 and 31 generalize Propositions 4.3 and 4.4 of Lin and Wan in [14] in the following two aspects: (1) from transfer closed values to transfer compactly closed values; (2) from generalized *KKM* mapping to $\mathfrak{A}$-*KKM* mapping with respect to T. To avoid the structure of the space, condition (3) in our results is more general than that in Theorems 4.1, 4.2, 4.3, 4.5, and 4.7 of [13].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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