

Research Article

The Dynamics and Synchronization of a Fractional-Order System with Complex Variables

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A fractional-order system with complex variables is proposed. Firstly, the dynamics of the system including symmetry, equilibrium points, chaotic attractors, and bifurcations with variation of system parameters and derivative order are studied. The routes leading to chaos including the period-doubling and tangent bifurcations are obtained. Then, based on the stability theory of fractional-order systems, the scheme of synchronization for the fractional-order complex system is presented. By designing appropriate controllers, the synchronization for the system is realized. Numerical simulations are carried out to demonstrate the effectiveness of the proposed scheme.

1. Introduction

Fractional calculus considered as the generalization of the integer-order calculus can be dated back to the 17th century. Recently, it has attracted many researchers' interest for the ability to describe practical problems. Moreover, rich dynamics such as chaos and bifurcation exist in many fractional-order systems [1–3]. Meanwhile, compared with integer-order chaotic systems, fractional-order chaotic systems with more complexity dynamics characteristic and more system parameters can provide higher security for secure communication [4, 5].

Chaos synchronization has attracted increasing interests in recent years which was proposed in 1990 [6]. Synchronization of integer-order systems in real space and complex space has been studied extensively and several methods are extended to synchronize fractional-order complex systems [7–9]. It should be noted that fractional-order chaotic systems with complex variables (doubling the numbers of variables) can be used to increase the content of transmitting information signals and enhance their security further. Therefore, it is an interesting and meaningful topic for researchers to study the dynamics and synchronization for fractional-order complex nonlinear systems. Meanwhile, the chaos and synchronization for discrete fractional-order systems have been investigated in detail [10–13].

Motivated by the above discussion, a fractional-order system with complex variables is proposed. The dynamics of the system including symmetry, equilibrium points, chaotic attractors, periodic orbits, and bifurcations with variation of system parameters and derivative orders are studied. Furthermore, based on the stability theory of fractional-order systems, the scheme of synchronization for the fractional-order complex system is presented, and numerical simulations are used to demonstrate the effectiveness and feasibility of the proposed scheme.

The paper is organized as follows. In Section 2, the definitions for fractional calculus and numerical algorithms are introduced. In Section 3, the fractional-order system with complex variables is proposed. And the dynamic behaviors of the system are investigated in detail. In Section 4, the synchronization of the system is investigated. Finally, we summarize the results in Section 5.

2. Fractional Operators and Their Approximation

2.1. Definition. There are at least six kinds of definitions for the fractional derivative. Three of which most frequently used ones are the Grunwald-Letnikov definition and the Riemann-Liouville and the Caputo definitions [14].

The Grunwald-Letnikov definition (GL) derivative with fractional-order q is expressed by

$${}^{\text{GL}}_a D_t^q f(t) = \lim_{h \rightarrow 0} \frac{1}{h^q} \sum_{j=0}^{[(t-a)/h]} (-1)^j \binom{q}{j} f(t-jh), \quad (1)$$

where the symbol $[\cdot]$ means the integer part.

The Riemann-Liouville (RL) definition of fractional derivatives is described by

$${}^{\text{RL}}_a D_t^q f(t) = \frac{d^n}{dt^n} \frac{1}{\Gamma(n-q)} \int_a^t \frac{f(\tau)}{(t-\tau)^{q-n+1}} d\tau, \quad n-1 < q < n, \quad (2)$$

where $\Gamma(\cdot)$ is the gamma function.

The Caputo (C) fractional derivative is defined as follows:

$${}^{\text{C}}_a D_t^q f(t) = \frac{1}{\Gamma(n-q)} \int_a^t (t-\tau)^{n-q-1} f^{(n)}(\tau) d\tau, \quad (3)$$

$$n-1 < q < n.$$

It is well known that the initial conditions for the fractional differential equations with Caputo derivatives take on the same form as those for the integer-order ones, which is very suitable for practical problems [15]. Therefore, we will use the Caputo definition for the fractional derivatives in this paper.

2.2. Numerical Algorithms. Due to the nature of the nonlocality, the operator of fractional derivative is more complicated than the classical ones which cause the difficulty of obtaining the analytic solution. However, numerical solution can make us study this kind of derivative better.

Nowadays, there are two approximation methods which can frequently be used to numerical computation fractional differential equations. One is an improved version of Adams-Bashforth-Moulton algorithm [16–18]. The other is a method, known as frequency domain approximation [19], based on numerical analysis of fractional-order systems in the frequency domain. And the reliability of the former has been widely approved. Therefore, we employ the improved predictor-corrector algorithm for fractional-order differential equations in this paper.

In order to get the approximate solution of a fractional-order chaotic system by the improved predictor-corrector algorithm, the following equation is considered:

$$\frac{d^q x}{dt^q} = f(t, x), \quad 0 \leq t \leq T, \quad (4)$$

$$x^{(k)}(0) = x_0^{(k)}, \quad k = 0, 1, \dots, [q] - 1,$$

where $[q]$ is just the value q rounded up to the nearest integer and $x^{(k)}$ is the ordinary k th derivative of x . Formula (4) is equivalent to the Volterra integral equation

$$x(t) = \sum_{k=0}^{[q]-1} x_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f(\tau, x(\tau)) d\tau. \quad (5)$$

Now, for the sake of simplicity, we assume that we are working on a uniform grid $\{t_n = nh : n = 0, 1, \dots, M\}$ with some integer M and set $h = T/M$. Using the standard quadrature techniques for the integral in (5) and denoting $g(\tau) = f(\tau, x(\tau))$ the integral is replaced by the trapezoidal quadrature formula at point t_{n+1} :

$$\int_0^{t_{n+1}} (t_{n+1} - \tau)^{q-1} g(\tau) d\tau \approx \int_0^{t_{n+1}} (t_{n+1} - \tau)^{q-1} \tilde{g}_{n+1}(\tau) d\tau, \quad (6)$$

where \tilde{g}_{n+1} is the piecewise linear interpolant for g with nodes and knots chosen at the t_j , ($j = 0, 1, \dots, n+1$). After some elementary calculations, the right hand side of (6) gives

$$\int_0^{t_{n+1}} (t_{n+1} - \tau)^{q-1} \tilde{g}_{n+1}(\tau) d\tau = \frac{h^q}{q(q+1)} \sum_{j=0}^{n+1} \alpha_{j,n+1} g(t_j). \quad (7)$$

And if we use the product rectangle rule, the right hand side of (6) can be written as

$$\int_0^{t_{n+1}} (t_{n+1} - \tau)^{q-1} \tilde{g}_{n+1}(\tau) d\tau = \sum_{j=0}^{n+1} \beta_{j,n+1} g(t_j), \quad (8)$$

where

$$\alpha_{j,n+1} = \begin{cases} n^{q+1} - (n-q)(n+1)^q, & j=0, \\ (n-j+2)^{q+1} + (n-j)^{q+1} - 2(n-j+1)^{q+1}, & 1 \leq j \leq n, \\ 1, & j=n+1, \end{cases} \quad (9)$$

$$\beta_{j,n+1} = \frac{h^q}{q} ((n-j+1)^q - (n-j)^q), \quad 1 \leq j \leq n.$$

Then the predictor and corrector formulae for solving (5) are given, respectively, by

$$x_h^P(t_{n+1}) = \sum_{k=0}^{[q]-1} x_0^{(k)} \frac{t_{n+1}^k}{k!} + \frac{1}{\Gamma(q)} \sum_{j=0}^n \beta_{j,n+1} f(t_j, x_h(t_j)), \quad (10)$$

$$x_h(t_{n+1}) = \sum_{k=0}^{[q]-1} x_0^{(k)} \frac{t_{n+1}^k}{k!} + \frac{h^q}{\Gamma(q+2)} f(t_{n+1}, x_h^P(t_{n+1})) + \frac{h^q}{\Gamma(q+2)} \sum_{j=0}^n \alpha_{j,n+1} f(t_j, x_h(t_j)). \quad (11)$$

The approximation accuracy of scheme (10)-(11) is $O(h^{\min\{2, 1+q\}})$.

3. Dynamics of a Fractional-Order Complex System

A three-dimensional chaotic system was proposed in [20], and the dynamical behaviors of the system were studied in

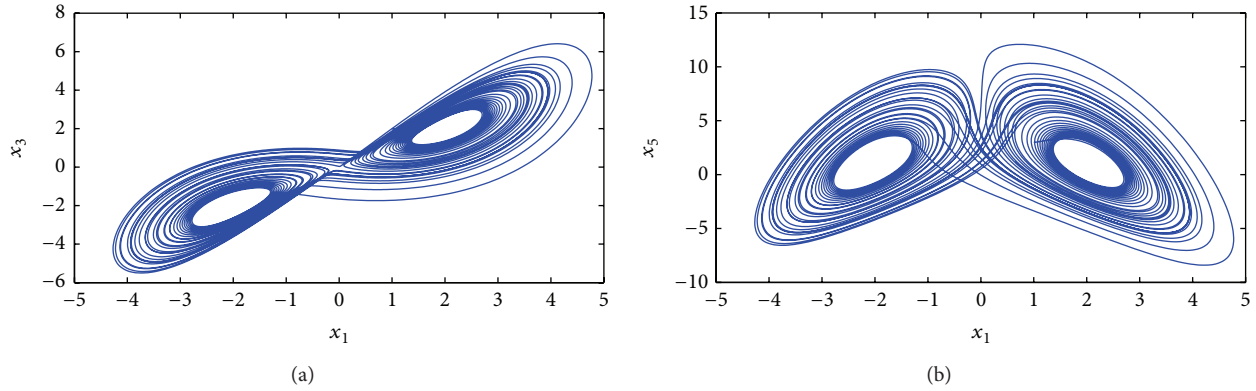


FIGURE 1: The chaotic attractors of the system (14) projected onto different phase plane (a) x_1 - x_3 phase plane and (b) the chaotic attractor projected onto x_1 - x_5 phase plane.

detail. The system can be described by the following differential equations:

$$\begin{aligned} \dot{y}_1 &= a(y_2 - y_1), \\ \dot{y}_2 &= y_1 y_3 - y_2, \\ \dot{y}_3 &= b - y_1 y_2 - c y_3, \end{aligned} \quad (12)$$

where $y = (y_1, y_2, y_3)^T$ is the state variable vector of the system and a, b, c are parameters. The system possesses an attractor when the parameters are as follows: $a = 5, b = 16, c = 1$.

In this paper, we suppose the state variables of system (12) are defined in the complex field, and the derivate orders are fractional. Then the corresponding fractional-order system is defined as

$$\begin{aligned} D^q y_1 &= a(y_2 - y_1), \\ D^q y_2 &= y_1 y_3 - y_2, \\ D^q y_3 &= b - \frac{1}{2}(\bar{y}_1 y_2 + y_1 \bar{y}_2) - c y_3, \end{aligned} \quad (13)$$

where q is the derivative order and $y = (y_1, y_2, y_3)^T$ is the vector of state variables. $y_1 = x_1 + i x_2, y_2 = x_3 + i x_4$ are complex variables, $y_3 = x_5$ is real variable, and $i = \sqrt{-1}$. Then the complex variables are separated into the real and imaginary parts, respectively. Due to the linearity of the Caputo differential operator, the system (13) can be represented as

$$\begin{aligned} D^q x_1 &= a(x_3 - x_1), \\ D^q x_2 &= a(x_4 - x_2), \\ D^q x_3 &= x_1 x_5 - x_3, \\ D^q x_4 &= x_2 x_5 - x_4, \\ D^q x_5 &= b - (x_1 x_3 + x_2 x_4) - c x_5. \end{aligned} \quad (14)$$

The system (14) is more convenient than system (13) for analysis and numerical simulation. When the system parameters

are taken as $a = 5, b = 16, c = 1$ and order of derivative $q = 0.99$, the maximum Lyapunov exponent by numerical computation is $\lambda_1 = 0.071$, which means that the system (14) is chaotic. The chaotic attractors in different phase space projections are shown in Figure 1.

3.1. Symmetry and Equilibrium Points. Firstly, the system (14) is symmetric because the transformation

$$S : (x_1, x_2, x_3, x_4, x_5) \longrightarrow (-x_1, -x_2, -x_3, -x_4, x_5), \quad (15)$$

which permits the system (14), is invariant for all values of parameters with the transformation. And the transformation S also implies that the system is symmetric about x_5 -axis; that is, $S(\psi)$ is one solution of the system (14) if ψ is one of it.

Now, we turn to investigate the equilibrium points of the system (14), which can be obtained by solving the equations $D^q x_j = 0, (j = 1, \dots, 5)$. Therefore, the equilibriums are

$$\begin{aligned} E_1 &(0, 0, 0, 0, 0) \\ E_\theta &(r \cos \theta, r \sin \theta, r \cos \theta, r \sin \theta, 1), \end{aligned} \quad (16)$$

where $r = \sqrt{b - c}$ and $\theta \in [0, 2\pi]$. The equilibrium E_θ exists when the condition $b - c > 0$ is satisfied.

3.2. Bifurcations. Firstly, the order of derivative is taken as $q = 0.99$ and fix the parameters $b = 16$ and $c = -0.2$. The bifurcation of system (14) as the parameter a is varied is depicted in Figure 2. From the bifurcation diagram, it is clear that with the decrease of a from 7.5 there is one limit cycle for a long parameter region until period-doubling bifurcation occurs when $a = 6.45$. For the system, the route leading to chaos is a series of period-doubling bifurcations. The period-doubling bifurcations can be observed when a belongs to the intervals $[1, 1.5]$ and $[4, 4.5]$. Phase portraits are shown in Figure 3, from which we can see that the system exists period-1, period-2 for different values of the parameter a .

Secondly, the dynamics of system (14) varying the parameter b will be investigated when $a = 5, c = 1$, and $q = 0.99$. The bifurcation of the system (14) when the parameter b is

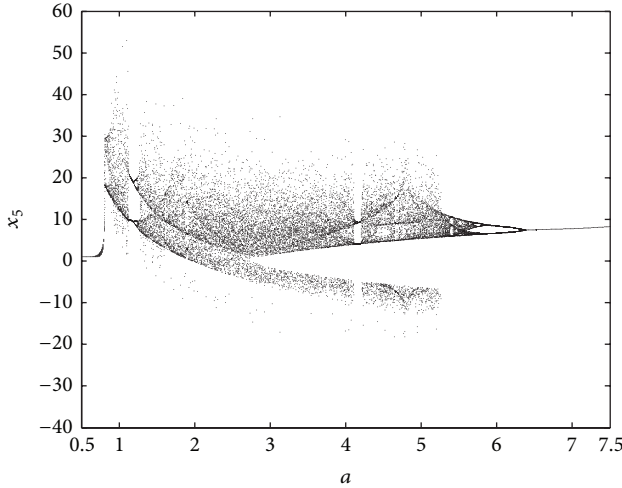


FIGURE 2: The bifurcation diagram of the system (14) with parameter $a \in [0.5, 7.5]$.

varied from 10 to 125 is obtained by numerical computation, which is depicted in Figure 4. It is clear that with the decrease of the parameter b the route out of chaos for the system is through tangent bifurcation when $b = 110.5$. Then two limit cycles coexist until period-doubling bifurcation occurs when $b = 105$. By a series of period-doubling bifurcations, the system enters into chaos again.

Thirdly, the dynamic behaviors of the system as the parameter c varies will be studied when other system parameters are taken as $a = 5$, $b = 16$ and the derivative order $q = 0.99$. The bifurcation diagram of the system (14) with parameter c is depicted in Figure 5. The tangent bifurcation occurs when $c = 0.05$. And the period-doubling bifurcations can be seen with $c \in [0.04, 0.09]$.

It is well known that the derivative order is one of important parameters in dynamics analysis for fractional-order systems. Therefore, the bifurcation diagram versus the derivative order with parameters $a = 5$, $b = 16$, and $c = -0.2$ is studied in the following.

The bifurcation diagram versus the order q varying on the closed interval $[0.9, 1]$ for the system (14) is depicted in Figure 6(a). The evolution of the period-doubling scenario and tangent bifurcation can be observed easily which is shown in the figure. In order to see clearly, the local of Figure 6(a) is zoomed in, which is shown in Figure 6(b). Meanwhile, to verify the chaotic behaviors for the system when $q \in [0.9, 1]$, the corresponding largest Lyapunov exponent (LLE for short) diagram with the step size of 0.1 by the algorithm of small data sets [21, 22] is shown in Figure 6(c).

4. Synchronization

In this section, the synchronization for the fractional-order complex system (14) will be studied.

4.1. The Scheme of Synchronization. The synchronization scheme for fractional-order complex systems will be

introduced. The drive and response systems are given, respectively, as follows:

$$D^\alpha y = f(y), \quad (17)$$

$$D^\beta Z = g(z) + u(y, z), \quad (18)$$

where $\alpha \in (0, 1)$ and $\beta \in (0, 1)$ are the derivative orders of systems (17) and (18), respectively. $y = (y_1, y_2, \dots, y_n)^T \in R^n$ and $z = (z_1, z_2, \dots, z_n)^T \in R^n$ are state complex vectors of drive and response systems, and the complex variables can be defined as $y_j = x_{2j-1} + ix_{2j}$ and $z_j = x'_{2j-1} + ix'_{2j}$, ($j = 1, 2, \dots, n$). Nonlinear vector functions f and g are continuous. $u(y, z) : R^n \times R^n \rightarrow R^n$ is a synchronization controller which will be designed later.

The error vector is defined as $e = z - y$, and the synchronization of drive and response systems is achieved if the following condition is satisfied:

$$\lim_{t \rightarrow +\infty} \|e\| = \lim_{t \rightarrow +\infty} \|z - y\| = 0, \quad (19)$$

where $\|\cdot\|$ is the Euclidean norm. For convenience, the error vector is rewritten as $e = e^{\text{real}} + ie^{\text{image}}$, $e^{\text{real}} = z^{\text{real}} - y^{\text{real}}$, and $e^{\text{image}} = z^{\text{image}} - y^{\text{image}}$.

Then, for the response system (18), we need to define a compensation controller

$$\theta(y) = D^\beta(y) - g(y), \quad (20)$$

and a synchronization controller

$$u(y, z) = \theta(y) + \tau(y, z), \quad (21)$$

where $\tau(y, z) : R^n \times R^n \rightarrow R^n$ is a vector function that will be designed later for the realization of synchronization.

Substituting the controllers (20) and (21) into the response system (18), we can get the error dynamical system as follows:

$$D^\beta e = g(z) - g(y) + \tau(y, z) = A(y, z)e + \tau(y, z). \quad (22)$$

For simplicity, $\tau(y, z)$ is chosen as $B(y, z)e$, and system (22) is rewritten as

$$D^\beta e = A(y, z)e + B(y, z)e = [A(y, z) + B(y, z)]e; \quad (23)$$

then, the system (23) asymptotically converges to zero when error vector tends to zero for $t \rightarrow +\infty$.

Theorem 1. *Given a fractional-order system (17), by designing a control vector $\tau(y, z) = B(y, z)e$ such that the synchronization between (17) and (18) can be realized, if the following condition is satisfied*

$$P[A(y, z) + B(y, z)] + [A(y, z) + B(y, z)]^H P = -Q, \quad (24)$$

where P and Q are real symmetric positive definite matrices, $B(y, z) \in R^{n \times n}$, and H stands for conjugate transpose of a matrix.

Proof. For the error dynamical system (23), assume that λ is one of the eigenvalues of matrix $[A(y, z) + B(y, z)]$, and the corresponding nonzero eigenvector is ξ ; that is,

$$[A(y, z) + B(y, z)]\xi = \lambda\xi. \quad (25)$$

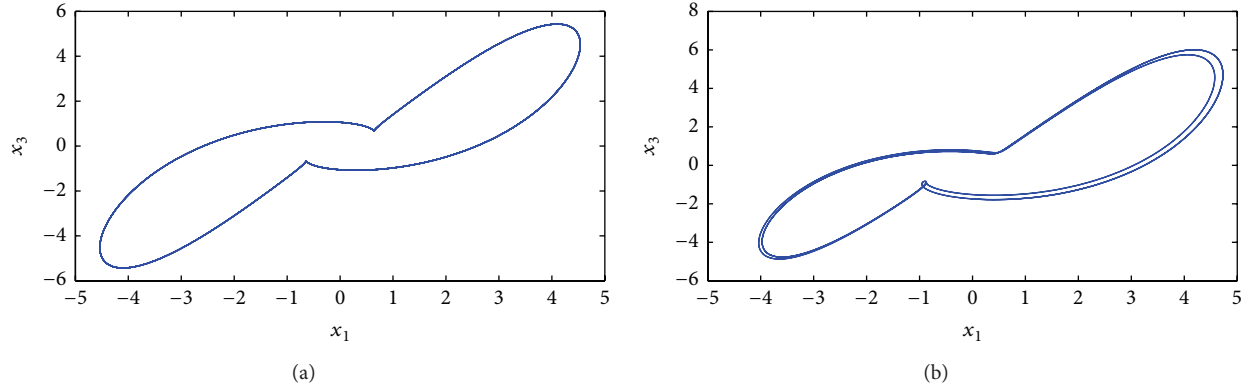


FIGURE 3: The phase portraits of the system for different values of the parameter a : (a) the phase trajectory for $a = 7$ and (b) the phase trajectory for $a = 5.8$.

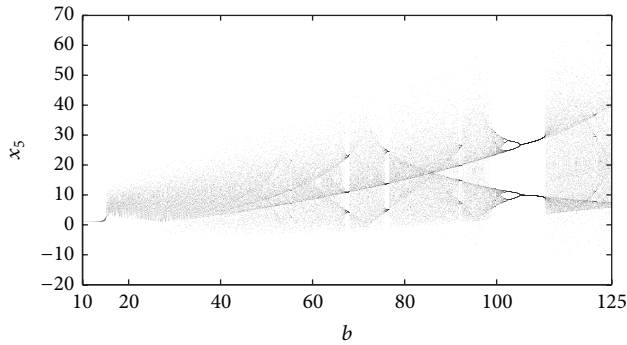


FIGURE 4: The bifurcation diagram of the system (14) with parameter $b \in [10, 125]$.

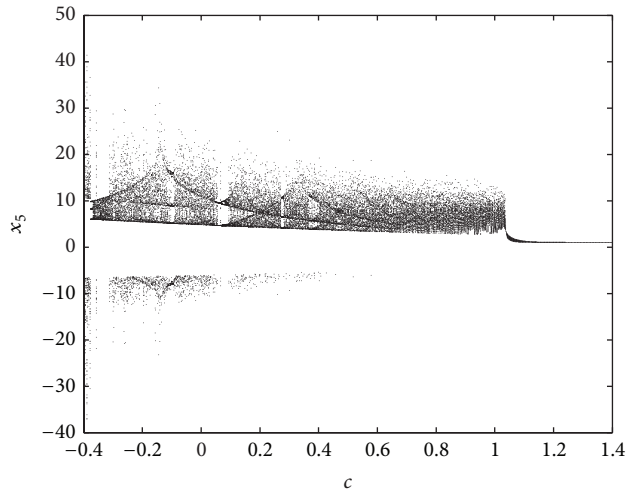


FIGURE 5: The bifurcation diagram of the system (18) with parameter $c \in [-0.4, 1.4]$.

Multiplying the above equation left by $\xi^H \mathbf{P}$, we obtain that

$$\xi^H \mathbf{P} [A(y, z) + B(y, z)] \xi = \xi^H \mathbf{P} (\lambda \xi). \quad (26)$$

Then, by a similar argument, we also can get that

$$\{\xi^H [A(y, z) + B(y, z)]^H\} \mathbf{P} \xi = (\bar{\lambda} \xi^H) \mathbf{P} \xi. \quad (27)$$

From (26) and (27), we derive

$$\begin{aligned} \lambda + \bar{\lambda} = & \{\xi^H \{\mathbf{P} [A(y, z) + B(y, z)] \\ & + [A(y, z) + B(y, z)]^H \mathbf{P}\} \xi\} (\xi^H \mathbf{P} \xi)^{-1}. \end{aligned} \quad (28)$$

Since $\mathbf{P}[A(y, z) + B(y, z)] + [A(y, z) + B(y, z)]^H \mathbf{P} = -\mathbf{Q}$ and \mathbf{P} and \mathbf{Q} are real symmetric positive definite matrices, then

$$\beta^H \mathbf{Q} \beta > 0, \quad \beta^H \mathbf{P} \beta > 0, \quad \lambda + \bar{\lambda} = -\frac{\beta^H \mathbf{Q} \beta}{\beta^H \mathbf{P} \beta} < 0. \quad (29)$$

From the above formula, we obtain

$$|\arg \lambda| \geq \frac{\pi}{2} > \frac{\pi}{2} q, \quad (0 < q < 1). \quad (30)$$

According to the stability theory of fractional-order systems, the equilibrium point of error dynamical system (23) is asymptotically stable. Therefore, the synchronization condition $\lim_{t \rightarrow \infty} \|e\| = \lim_{t \rightarrow \infty} \|z - y\| = 0$ is satisfied, which implies that the synchronization between drive and response systems is realized. The proof is completed. \square

4.2. Numerical Simulation. The system (14) is taken as the drive system, and the corresponding response system with controller is

$$\begin{bmatrix} D^\beta z_1 \\ D^\beta z_2 \\ D^\beta z_3 \end{bmatrix} = \begin{bmatrix} a(z_2 - z_1) \\ z_1 z_3 - z_1 \\ b - \frac{1}{2}(\bar{z}_1 z_2 + z_1 \bar{z}_2) - cz_3 \end{bmatrix} + u(y, z), \quad (31)$$

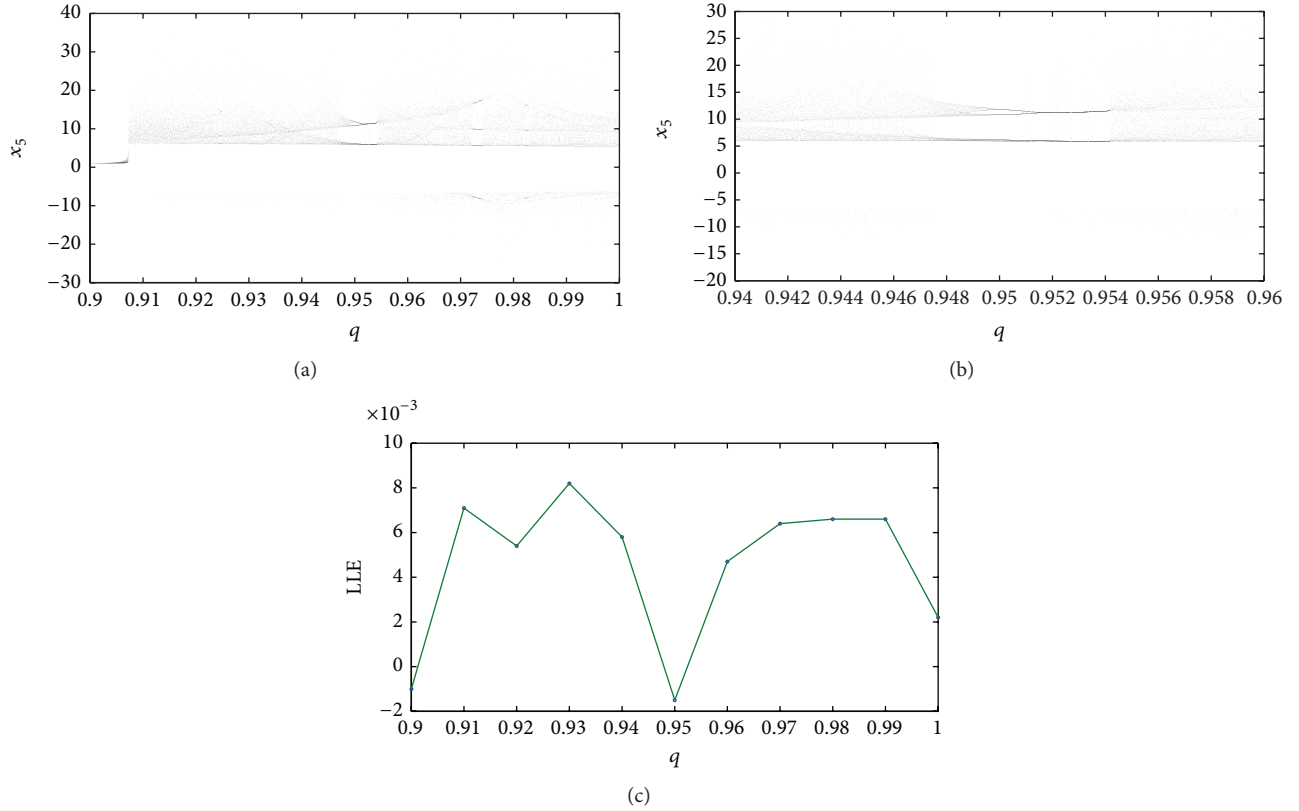


FIGURE 6: The bifurcation diagram of the system (14) with derivative order q . (a) $q \in [0.9, 1]$; (b) local zoom of (a); (c) the corresponding LLE diagram.

where $z_1 = x'_1 + ix'_2$ and $z_2 = x'_3 + ix'_4$ are complex variables and $z_3 = x'_5$ is real variable, and $i = \sqrt{-1}$. The system (31) can be expressed as

$$\begin{bmatrix} D^\beta x'_1 \\ D^\beta x'_2 \\ D^\beta x'_3 \\ D^\beta x'_4 \\ D^\beta x'_5 \end{bmatrix} = \begin{bmatrix} a(x'_3 - x'_1) \\ a(x'_4 - x'_2) \\ x'_1 x'_5 - x'_1 \\ x'_2 x'_5 - x'_2 \\ b - (x'_1 x'_3 + x'_2 x'_4) - cx'_5 \end{bmatrix} + u(y, z). \quad (32)$$

The synchronization error vector is taken as

$$e = z - y = e^{\text{real}} + ie^{\text{image}}, \quad (33)$$

where $e^{\text{real}} = (e_1, e_3, e_5)^T$ and $e^{\text{image}} = (e_2, e_4)^T$.

By computation, we can get $A(x_j, x'_j)$ as follows:

$$A(x_j, x'_j) = \begin{bmatrix} -a & 0 & a & 0 & 0 \\ 0 & -a & 0 & a & 0 \\ x'_5 & 0 & -1 & 0 & x_1 \\ 0 & x'_5 & 0 & -1 & x_2 \\ -x'_3 & -x'_4 & -x_1 & -x_2 & -c \end{bmatrix}. \quad (34)$$

By using the theorem, the matrix $B(x_j, x'_j)$ is designed as

$$B(x_j, x'_j) = \begin{bmatrix} 0 & 0 & 0 & 0 & x'_3 \\ 0 & 0 & 0 & 0 & x'_4 \\ -x'_5 - a & 0 & 0 & 0 & 0 \\ 0 & -x'_5 - a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (35)$$

Therefore, the error dynamical system is

$$\begin{bmatrix} D^\beta e_1 \\ D^\beta e_2 \\ D^\beta e_3 \\ D^\beta e_4 \\ D^\beta e_5 \end{bmatrix} = \begin{bmatrix} -a & 0 & a & 0 & x'_3 \\ 0 & -a & 0 & a & x'_4 \\ -a & 0 & -1 & 0 & x_1 \\ 0 & -a & 0 & -1 & x_2 \\ -x'_3 & -x'_4 & -x_1 & -x_2 & -c \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{bmatrix}. \quad (36)$$

A real symmetric positive definite matrix is chosen as $\mathbf{P} = \text{diag}(1, 1, 1, 1, 1)$; then,

$$\begin{aligned} & P[A(x_j, x'_j) + B(x_j, x'_j)] + [A(x_j, x'_j) + B(x_j, x'_j)]^H \mathbf{P} \\ &= \text{diag}(-2a, -2a, -2, -2, -2c). \end{aligned} \quad (37)$$

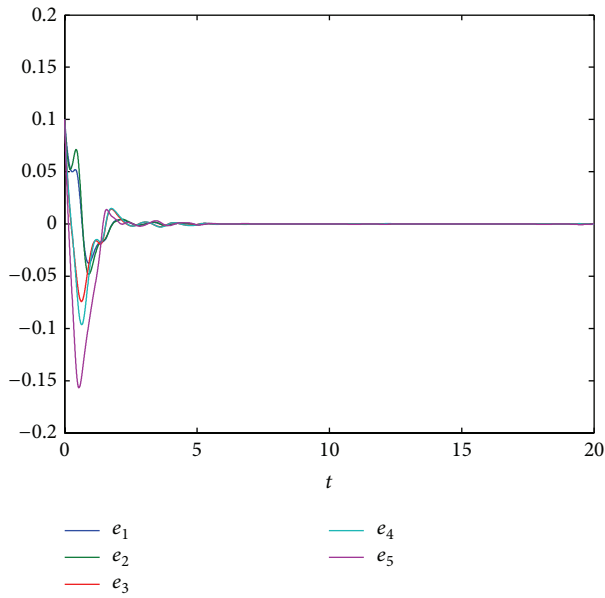


FIGURE 7: Curves of the synchronization errors of the systems (14) and (32).

Therefore, another real symmetric positive definite matrix is taken as $\mathbf{Q} = \text{diag}(2a, 2a, 2, 2, 2c)$, and expression (37) can be written as

$$\begin{aligned} & \mathbf{P} \left[\mathbf{A}(x_j, x'_j) + \mathbf{B}(x_j, x'_j) \right] \\ & + \left[\mathbf{A}(x_j, x'_j) + \mathbf{B}(x_j, x'_j) \right]^H \mathbf{P} = -\mathbf{Q}. \end{aligned} \quad (38)$$

According to the theorem, the synchronization of drive system (14) and response system (32) is achieved.

The values of the parameters of the drive and response systems are taken as $a = 5$, $b = 16$, $c = 1$. The orders of derivative for the two systems are $q = \beta = 0.99$. The initial conditions of the two systems are $(0.1, 0.1, 0.1, 0.1, 0.1)$ and $(1, 2, 3, 4, 5)$, respectively. The results of simulation are depicted in Figure 7, from which it is clear that the error variables e_1 , e_2 , e_3 , e_4 , and e_5 converge to zero as $t \rightarrow 20$ s, which implies the synchronization between the drive and response systems is achieved under the controllers.

5. Conclusions

In this paper, the dynamics and synchronization for a fractional-order system with complex variables have been investigated. The dynamics include symmetry, equilibrium points, chaos, and bifurcations with variation of both system parameters and derivative orders. Besides, based on the stability theory of fractional-order systems, the scheme of synchronization for the fractional-order complex system is presented, and numerical simulations are carried out to demonstrate the effectiveness and feasibility of the proposed scheme.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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