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Research Article

On Decompositions of Matrices over Distributive Lattices

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Let L be a distributive lattice and $M_{n,q}(L)(M_n(L)$, resp.) the semigroup (semiring, resp.) of $n\times q$ ($n\times n$, resp.) matrices over L. In this paper, we show that if there is a subdirect embedding from distributive lattice L to the direct product $\prod_{i=1}^{m} L_i$ of distributive lattices $L_1, L_2, \ldots, L_m$, then there will be a corresponding subdirect embedding from the matrix semigroup $M_{n,q}(L)$ (semiring $M_n(L)$, resp.) to semigroup $\prod_{i=1}^{m} M_{n,q}(L_i)$ (semiring $\prod_{i=1}^{m} M_n(L_i)$, resp.). Further, it is proved that a matrix over a distributive lattice can be decomposed into the sum of matrices over some of its special subchains. This generalizes and extends the decomposition theorems of matrices over finite distributive lattices, chain semirings, fuzzy semirings, and so forth. Finally, as some applications, we present a method to calculate the indices and periods of the matrices over a distributive lattice and characterize the structures of idempotent and nilpotent matrices over it. We translate the characterizations of idempotent and nilpotent matrices over a distributive lattice into the corresponding ones of the binary Boolean cases, which also generalize the corresponding structures of idempotent and nilpotent matrices over general Boolean algebras, chain semirings, fuzzy semirings, and so forth.

1. Introduction and Preliminaries

A semiring is an algebra $(R, +, \cdot)$ with two binary operations $+$ and $\cdot$ such that both $(R, +)$ and $(R, \cdot)$ are semigroups and such that the distributive laws

$$x(y+z) \approx xy + xz, \qquad (x+y)z \approx xz + yz \tag{1}$$

are satisfied. A partially ordered semiring means a semiring R equipped with a compatible ordering $\leq$; that is, $\leq$ is a partial order on R satisfying the following condition:

$$a \leq b, \quad c \leq d \Longrightarrow a + c \leq b + d, \quad ac \leq bd \tag{2}$$

for any $a, b, c, d \in R$.

A distributive lattice L is a lattice which satisfies either of the distributive laws and whose addition $+$ and the multiplication $\cdot$ on L are as follows:

$$a + b = a \vee b, \qquad ab = a \wedge b. \tag{3}$$

It is not hard to see that distributive lattice L is a partially ordered semiring.

In the following, we will introduce several kinds of distributive lattices which will often occur: general Boolean algebras (including binary Boolean algebras), chain semirings (including chains), and fuzzy semirings.

For a fixed positive integer k, let $\mathbb{B}_k$ be the general Boolean algebra of subsets of a k-element set $\mathbb{S}_k$ and $\sigma_1, \sigma_2, \ldots, \sigma_k$ denote the singleton subsets of $\mathbb{S}_k$. Union is denoted by $+$ and intersection by juxtaposition; 0 denotes the null set and 1 the set $\mathbb{S}_k$. Under these two operations, $\mathbb{B}_k$ is a finite distributive lattice. In particular, if $k = 1$, $\mathbb{B}_1$ is called the binary Boolean algebra.

Let $\mathbb{K}$ be any set of two or more elements. If $\mathbb{K}$ is totally ordered by $<$ (i.e., $x < y$ or $y < x$ for all distinct elements $x, y \in \mathbb{K}$), then define $x + y$ as $\max\{x, y\}$ and xy as $\min\{x, y\}$ for all $x, y \in \mathbb{K}$. If $\mathbb{K}$ has a universal lower bound and a universal upper bound, then $\mathbb{K}$ becomes a semiring and is called a chain semiring.

Let α, ω be real numbers with $\alpha < \omega$. Define $\mathbb{S} = \{\beta \in \mathbb{R} : \alpha \leq \beta \leq \omega\}$. Then, $\mathbb{S}$ is a chain semiring with $\alpha = 0$ and $\omega = 1$. Furthermore, if we choose the real numbers 0 and 1 as α and ω in the previous example, then $\mathbb{F} = \{\beta \in \mathbb{R} : 0 \leq \beta \leq 1\}$ is called fuzzy semiring.

For a distributive lattice L, denote $M_{n,p}(L)$ to be the set of all the $n \times p$ matrices over L. For any $A = (a_{ij})$, $B = (b_{ij}) \in M_{n,p}(L)$, define + in $M_{n,p}(L)$ by

$$A + B = \left(a_{ij} + b_{ij}\right). \tag{4}$$

Then, clearly, $(M_{n,p}(L), +)$ is a semigroup. On the other hand, $M_n(L)$ is denoted to be the set of all the $n \times n$ matrices over L. For any $A = (a_{ij})$, $B = (b_{ij}) \in M_n(L)$, the addition + in $M_n(L)$ is defined as above, and the multiplication $\cdot$ in $M_n(L)$ is defined by

$$AB = \left(c_{ij}\right), \tag{5}$$

where $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$. It is easy to verify that $(M_n(L), +, \cdot)$ is also a semiring. We will call it the $n\times n$ matrix semiring over L and simply write it as $M_n(L)$.

The theory of matrices over distributive lattices has important applications in optimization theory, models of discrete event networks, and graph theory. There are a series of papers in the literature considering matrices over distributive lattices and similar topics (e.g., see [1–31]).

It is well known that the decompositions of matrices over a distributive lattice play an important role in the studies of the lattice matrices. In this paper, we will firstly study the decompositions of matrices over a distributive lattice L in Section 2. We show that if there is a subdirect embedding from distributive lattice L to the direct product $\prod_{i=1}^{m} L_i$ of distributive lattices $L_1, L_2, \dots, L_m$, then there will be a corresponding subdirect embedding from the matrix semigroup $M_{n,q}(L)$ (semiring $M_n(L)$, resp.) to semigroup $\prod_{i=1}^{m} M_{n,q}(L_i)$ (semiring $\prod_{i=1}^{m} M_n(L_i)$, resp.). Further, it is proved that a matrix over a distributive lattice can be decomposed into the sum of matrices over some of its special subchains. This generalizes and extends the decomposition theorems of matrices over finite distributive lattices, chain semirings, fuzzy semirings, and so forth, including the corresponding results in [6, 8, 30]. In Section 3, as some applications, we present a method to calculate the indices and periods of the matrices over a distributive lattice and study the structures of idempotent and nilpotent matrices over L. We translate the characterizations of idempotent and nilpotent matrices over a distributive lattice into the corresponding ones of the binary Boolean cases, which generalize and extend the corresponding structures of idempotent and nilpotent matrices over Boolean algebras, chain semirings, fuzzy semirings, and so forth.

For notations and terminologies that occurred but are not mentioned in this paper, readers are referred to [32–34].

2. Decompositions of Matrices over a Distributive Lattice

Recall that an algebra A is said to be a subdirect product of an indexed family A_i $(i \in I)$ of algebras if it satisfies $A \le \prod_{i\in I} A_i$ and $A\pi_i = A_i$ for each $i \in I$, where π_i is the projective mapping from A to A_i. And in this case, we also say that A has a subdirect decomposition.

If a homomorphism φ from A to $\prod_{i\in I} A_i$ is injective, then it is called an embedding. An embedding $\varphi : A \to \prod_{i\in I} A_i$ is called subdirect if $A\varphi$ is a subdirect product of the A_i, and in this case we say that A is isomorphic to a subdirect product of A_i (see [32]).

An element a of a lattice L is called a join irreducible element of L if $a = x + y$ implies $x = a$ or $y = a$ for $x, y \in L$. In a finite lattice L, a nonzero element is join irreducible if and only if it has exactly one lower cover. Throughout this paper, the set of all join irreducible elements of L will be denoted by J.

Lemma 1 (see [6]). *Let L be a finite distributive lattice with $\ell \ge 2$ elements and $J = \{0, p_1, p_2, \dots, p_h\}$ the set of all join irreducible elements of L. Assume that C_i $(i = 1, 2, \dots, m)$ is a subchain $0 = p_{i0} < p_{i1} < \cdots < p_{is_i}$ of J such that $J = \bigcup_{i=1}^{m} C_i$ and $C_i \bigcap C_j = \{0\}$ for $i \ne j$; define the mapping ψ from L to $\prod_{i=1}^{m} C_i$ by*

$$(i)\ \ (a)\,\psi = \begin{cases} p_{is_i} & a \ge p_{is_i}, \\ p_{ik} & a \ge p_{ik} \text{ but } a \ngeq p_{i(k+1)}, \end{cases} \tag{6}$$
$$\left(\forall a \in L, 1 \le i \le m, 1 \le k < s_i\right).$$

Then, ψ is a subdirect embedding from lattice L to lattice $\prod_{i=1}^{m} C_i$ (i.e., lattice L is isomorphic to a subdirect product of the C_i).

Remark 2. Lemma 1 supplies us with a way to find subdirect decompositions for a finite distributive lattice L into some of its subchains.

By Lemma 1, the following results are easy to obtain.

Lemma 3 (see [6]). *Let L be a finite distributive lattice with $\ell \ge 2$ elements and $J = \{0, p_1, p_2, \dots, p_h\}$ the set of all join irreducible elements of L. Take $C_i = \{0, p_i\}$ $(i = 1, 2, \dots, h)$ and define the mapping ψ from L to $\prod_{i=1}^{h} C_i$ by*

$$(i)\ \ (a)\,\psi = \begin{cases} p_i & a \ge p_i, \\ 0 & \text{otherwise}. \end{cases} \quad (\forall a \in L, 1 \le i \le h) \tag{7}$$

Then, ψ is a subdirect embedding from lattice L to lattice $\prod_{i=1}^{h} C_i$ (i.e., lattice L is isomorphic to a subdirect product of the C_i).

Lemma 4 (see [6]). *Let $L = \{0, 1, \dots, \ell\}$ $(\ell > 1)$ be a chain with the usual ordering. Suppose that s is any (but fixed) positive integer with $1 < s < \ell$ and that $m = \lceil \ell/s \rceil$ is the least integer greater than or equal to ℓ/s. Take*

$$\begin{aligned} C_i &= \{0, s(i-1)+1, s(i-1)+2, \dots, si\} \\ &\qquad\qquad (i = 1, 2, \dots, m-1), \\ C_m &= \{0, s(m-1)+1, s(m-1)+2, \dots, \ell\}. \end{aligned} \tag{8}$$

Then, L is isomorphic to a subdirect product of chains C_i $(i = 1, 2, \dots, m)$.

In the following, we will study the decompositions of matrices over a distributive lattice L. Together with the subdirect decompositions of a finite distributive lattice obtained in the previous subsection, we will show that if there is a subdirect embedding from distributive lattice L to the direct product $\prod_{i=1}^{m} L_i$ of distributive lattices $L_1, L_2, \dots, L_m$, then there will be a corresponding subdirect embedding from the matrix semigroup $M_{n,q}(L)$ (semiring $M_n(L)$, resp.) to semigroup $\prod_{i=1}^{m} M_{n,q}(L_i)$ (semiring $\prod_{i=1}^{m} M_n(L_i)$, resp.). And then we will prove that a matrix over a distributive lattice can be decomposed into the sum of matrices over some of its special subchains.

Theorem 5. *Assume that distributive lattice L is a subdirect product of distributive lattices $L_1, L_2, \dots, L_m$. Define the mapping φ from $M_{n,q}(L)$ to $\prod_{i=1}^{m} M_{n,q}(L_i)$ by*

$$\left(\forall A = \left(a_{ij}\right) \in M_{n,q}(L)\right) \quad A\varphi = \left(A_1, A_2, \dots, A_m\right), \tag{9}$$

where $(\forall 1 \le k \le m)$ $A_k = (a_{ij}^{(k)})$ *and* $a_{ij}^{(k)} = (a_{ij})\pi_k$ *(π_k is the projective mapping from L to L_k). Then, φ is a subdirect embedding from semigroup $M_{n,q}(L)$ to semigroup $\prod_{i=1}^{m} M_{n,q}(L_i)$.*

Proof. Firstly, for any $A = (a_{ij})$, $B = (b_{ij}) \in M_{n,q}(L)$, we have

$$\begin{aligned} A\varphi = B\varphi &\Longleftrightarrow (\forall 1 \le k \le m) \quad A_k = B_k \\ &\Longleftrightarrow (\forall 1 \le k \le m, \forall 1 \le i, j \le n) \quad a_{ij}^{(k)} = b_{ij}^{(k)} \\ &\Longleftrightarrow (\forall 1 \le k \le m, \forall 1 \le i, j \le n) \\ &\qquad \left(a_{ij}\right)\pi_k = \left(b_{ij}\right)\pi_k \\ &\Longleftrightarrow (\forall 1 \le i, j \le n) \quad a_{ij} = b_{ij} \\ &\Longleftrightarrow A = B. \end{aligned} \tag{10}$$

This shows that φ is injective.

Secondly, let $C = (c_{ij}) = A + B$. Then, we have

$$c_{ij}^{(k)} = \left(a_{ij} + b_{ij}\right)\pi_k = \left(a_{ij}\right)\pi_k + \left(b_{ij}\right)\pi_k = a_{ij}^{(k)} + b_{ij}^{(k)} \quad \left(\forall 1 \le k \le m,\ 1 \le i, j \le n\right). \tag{11}$$

That is to say,

$$C_k = A_k + B_k \quad (\forall 1 \le k \le m). \tag{12}$$

This implies $(A + B)\varphi = A\varphi + B\varphi$.

Finally, for any $B = (b_{ij}) \in M_{n,q}(L_k)$ $(1 \le k \le m)$, there exists $a_{ij} \in L$ such that

$$\left(a_{ij}\right)\pi_k = b_{ij}, \tag{13}$$

since π_k is a surjection from L to L_k. That is to say, by taking $A = (a_{ij})$, we have $(A\varphi)\pi_k = B$. This shows that φ is a subdirect embedding from semigroup $M_{n,q}(L)$ to semigroup $\prod_{i=1}^{m} M_{n,q}(L_i)$. □

In particular, in Theorem 5, if we take $n = q$, then we also have the following.

Theorem 6. *Assume that distributive lattice L is a subdirect product of distributive lattices $L_1, L_2, \dots, L_m$. Define the mapping φ from $M_n(L)$ to $\prod_{i=1}^{m} M_n(L_i)$ by*

$$\left(\forall A = \left(a_{ij}\right) \in M_n(L)\right) \quad A\varphi = \left(A_1, A_2, \dots, A_m\right), \tag{14}$$

where $(\forall 1 \le k \le m)$ $A_k = (a_{ij}^{(k)})$ *and* $a_{ij}^{(k)} = (a_{ij})\pi_k$ *(π_k is the projective mapping from L to L_k). Then, φ is a subdirect embedding from semiring $M_n(L)$ to semiring $\prod_{i=1}^{m} M_n(L_i)$.*

Proof. We only need to show that, for any $A = (a_{ij})$, $B = (b_{ij}) \in M_n(L)$, $(AB)\varphi = (A\varphi)(B\varphi)$.

In fact, if we let $D = (d_{ij}) = AB$, then we have

$$d_{ij}^{(k)} = \left(\sum_{\ell=1}^{n} a_{i\ell} b_{\ell j}\right)\pi_k = \sum_{\ell=1}^{n} \left(a_{i\ell}\right)\pi_k \left(b_{\ell j}\right)\pi_k = \sum_{\ell=1}^{n} a_{i\ell}^{(k)} b_{\ell j}^{(k)} \quad \left(\forall 1 \le k \le m,\ 1 \le i, j \le n\right). \tag{15}$$

That is to say,

$$D_k = A_k B_k \quad (\forall 1 \le k \le m). \tag{16}$$

This implies $(AB)\varphi = (A\varphi)(B\varphi)$. □

Now, analogous with the discussions of the above two theorems, the following theorem is also not hard to prove.

Theorem 7. *Assume that ψ is a subdirect embedding from distributive lattice L to the direct product $\prod_{i=1}^{m} L_i$ of distributive lattices $L_1, L_2, \dots, L_m$. Define the mapping φ from $M_{n,q}(L)$ ($M_n(L)$, resp.) to $\prod_{i=1}^{m} M_{n,q}(L_i)$ ($\prod_{i=1}^{m} M_n(L_i)$, resp.) by*

$$\begin{aligned} &\left(\forall A = \left(a_{ij}\right) \in M_{n,q}(L)\right) \ \left(M_n(L), \text{resp.}\right) \\ &\qquad A\varphi = \left(A_1, A_2, \dots, A_m\right), \end{aligned} \tag{17}$$

where $(\forall 1 \le k \le m)$ $A_k = (a_{ij}^{(k)})$ *and* $a_{ij}^{(k)} = (k)(a_{ij})\psi$. *Then, φ is a subdirect embedding from semigroup $M_{n,q}(L)$ (semiring $M_n(L)$, resp.) to semigroup $\prod_{i=1}^{m} M_{n,q}(L_i)$ (semiring $\prod_{i=1}^{m} M_n(L_i)$, resp.).*

Thus, we have the following.

Theorem 8. *Let L be a finite distributive lattice with $\ell \ge 2$ elements and $J = \{0, p_1, p_2, \dots, p_h\}$ the set of all join irreducible elements of L. Assume that C_i $(i = 1, 2, \dots, m)$ and ψ are given as in Lemma 1. Define the mapping φ from $M_{n,q}(L)$ ($M_n(L)$, resp.) to $\prod_{i=1}^{m} M_{n,q}(C_i)$ ($\prod_{i=1}^{m} M_n(C_i)$, resp.) by*

$$\begin{aligned} &\left(\forall A = \left(a_{ij}\right) \in M_{n,q}(L)\right) \ \left(M_n(L), \text{resp.}\right) \\ &\qquad A\varphi = \left(A_1, A_2, \dots, A_m\right), \end{aligned} \tag{18}$$

where $(\forall 1 \le k \le m)$ $A_k = (a_{ij}^{(k)})$ *and* $a_{ij}^{(k)} = (k)(a_{ij})\psi$. *Then,* φ *is a subdirect embedding from semigroup* $M_{n,q}(L)$ *(semiring* $M_n(L)$*, resp.) to semigroup* $\prod_{i=1}^{m} M_{n,q}(C_i)$ *(semiring* $\prod_{i=1}^{m} M_n(C_i)$*, resp.) and* $A = \sum_{k=1}^{m} A_k$ *for any* $A \in M_{n,q}(L)$ $(M_n(L)$*, resp.).*

Proof. Let L be a finite distributive lattice with $\ell \ge 2$ elements and $J = \{0, p_1, p_2, \dots, p_h\}$ the set of all join irreducible elements of L. Assume that C_i $(i = 1, 2, \dots, m)$ and ψ are given as in Lemma 1; that is, C_i $(i = 1, 2, \dots, m)$ is a subchain $0 = p_{i0} < p_{i1} < \cdots < p_{is_i}$ of J such that $J = \bigcup_{i=1}^{m} C_i$ and $C_i \bigcap C_j = \{0\}$ for $i \ne j$; the mapping ψ from L to $\prod_{i=1}^{m} C_i$ is defined by

$$(i)\,(a)\,\psi = \begin{cases} p_{is_i} & a \ge p_{is_i}, \\ p_{ik} & a \ge p_{ik} \text{ but } a \ngeq p_{i(k+1)}, \end{cases} \qquad (19)$$
$$(\forall a \in L,\ 1 \le i \le m,\ 1 \le k < s_i).$$

We know by Lemma 1 that ψ is a subdirect embedding from lattice L to lattice $\prod_{i=1}^{m} C_i$. Thus, if we define the mapping φ from $M_{n,q}(L)$ $(M_n(L)$, resp.) to $\prod_{i=1}^{m} M_{n,q}(C_i)$ $(\prod_{i=1}^{m} M_n(C_i)$, resp.) by

$$\left(\forall A = (a_{ij}) \in M_{n,q}(L)\right) \ (M_n(L), \text{resp.}) \qquad (20)$$
$$A\varphi = (A_1, A_2, \dots, A_m),$$

where $(\forall 1 \le k \le m)$ $A_k = (a_{ij}^{(k)})$ and $a_{ij}^{(k)} = (k)(a_{ij})\psi$, then it follows from Theorem 7 that φ is a subdirect embedding from semigroup $M_{n,q}(L)$ (semiring $M_n(L)$, resp.) to semigroup $\prod_{i=1}^{m} M_{n,q}(C_i)$ (semiring $\prod_{i=1}^{m} M_n(C_i)$, resp.). Also, it is easy to verify that, for any $A = (a_{ij}) \in M_{n,q}(L)$ $(M_n(L)$, resp.), $a_{ij} = \sum_{k=1}^{m} a_{ij}^{(k)}$. That is to say, $A = \sum_{k=1}^{m} A_k$. □

Remark 9. Theorem 8 shows that a matrix over a finite distributive lattice L can be decomposed into the sum of matrices over some special subchains of L. That is to say, Theorem 7 generalizes Theorem 3.3 in [6]. Also, in Theorem 8, if we take $C_i = \{0, p_i\}$ $(1 \le i \le h)$ and define $A_{p_i} = (c_{kl})$, where

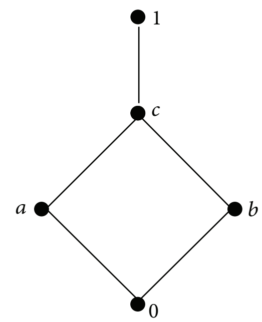
$$c_{kl} = \begin{cases} 1, & \text{if } a_{kl} \ge p_i, \\ 0, & \text{else}, \end{cases} \qquad (21)$$

for any $1 \le k \le n$, $1 \le l \le h$, then we can immediately obtain the following corollary.

Corollary 10. *Let* L *be a finite distributive lattice with* $\ell \ge 2$ *elements and* $J = \{0, p_1, p_2, \dots, p_h\}$ *the set of all join irreducible elements of* L*. Then, for any* $A = \in M_{n,q}(L)$*, the following holds:*

$$A = \sum_{i=1}^{h} p_i A_{p_i}. \qquad (22)$$

Example 11. Let $L = \{0, a, b, c, 1\}$ be a finite distributive lattice whose Hasse diagram is shown below



By Lemma 3, it is known that L is isomorphic to the subdirect product of $C_1 = \{0, a\}$, $C_2 = \{0, b\}$, and $C_3 = \{0, 1\}$. And then, by Theorem 8, $M_{n,p}(L)$ is isomorphic to the subdirect product of $M_{n,p}(C_1)$, $M_{n,p}(C_2)$, and $M_{n,p}(C_3)$.

Now, for given $A = \left(\begin{smallmatrix} a & b & 1 & c \\ b & b & a & a \\ 1 & a & b & c \end{smallmatrix}\right) \in M_{3,4}(L)$, we have $A = A_1 + A_2 + A_3$, where

$$A_1 = \begin{pmatrix} a & 0 & 0 & a \\ 0 & 0 & a & a \\ 0 & a & 0 & a \end{pmatrix} \in M_{3,4}(C_1),$$
$$A_2 = \begin{pmatrix} 0 & b & 0 & b \\ b & b & 0 & 0 \\ 0 & 0 & b & b \end{pmatrix} \in M_{3,4}(C_2), \qquad (23)$$
$$A_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \in M_{3,4}(C_3).$$

In particular, if we let finite distributive L to be general Boolean algebra $\mathbb{B}_k$ in Corollary 10, we can get the following.

Corollary 12. *Let* $\mathbb{B}_k$ *be general Boolean algebra and* $J = \{0, p_1, p_2, \dots, p_h\}$ *the set of all join irreducible elements of* $\mathbb{B}_k$*. Then, for any* $A \in M_{n,q}(\mathbb{B}_k)$*, the following holds:*

$$A = \sum_{i=1}^{h} p_i A_{p_i}. \qquad (24)$$

Also, consider finite distributive L to be a chain $\{0, 1, \dots, \ell\}$ $(\ell > 1)$ with the usual ordering. Given the subchains C_i $(i = 1, 2, \dots, m)$ of L as in Lemma 4, define A_k as in Theorem 8 and $\widetilde{\psi}_{s\ell}^{(k)} \overline{\varphi}_{\ell s}^{(k)}(A)$ as in Theorem 3 in [30]. Then, A_k is equal to $\widetilde{\psi}_{s\ell}^{(k)} \overline{\varphi}_{\ell s}^{(k)}(A)$. So we have the following.

Corollary 13. *Let* $L = \{0, 1, \dots, \ell\}$ $(\ell > 1)$ *be a chain with the usual ordering. Then, for any* $A \in M_n(L)$*, the following holds:*

$$A = \sum_{s=1}^{m} \widetilde{\psi}_{s\ell}^{(k)} \overline{\varphi}_{\ell s}^{(k)}(A). \qquad (25)$$

Summing up the above discussions, we have shown that a matrix over a finite distributive lattice L can be decomposed into the sum of matrices over some of its special subchains. Furthermore, we can also obtain some general decompositions of a matrix over a distributive lattice L.

Let L be a distributive lattice. For any $A = (a_{ij}) \in M_{n,p}(L)$, Φ_A is denoted to be the set of all the entries of A; that is,

$\Phi_A = \{a_{ij} \mid 1 \le i \le n,\ 1 \le j \le p\}$. Clearly, $\Phi_A = \{a_{ij} \mid 1 \le i \le n,\ 1 \le j \le p\}$ is a finite subset of L. Also, denote $\langle \Phi_A \rangle$ to be the sublattice of L generated by Φ_A.

Lemma 14 (see [8]). *Let L be a distributive lattice. If S is a finite subset of L, then $\langle S \rangle$ is a finite distributive sublattice of L, where $\langle S \rangle$ means a sublattice of L generated by S.*

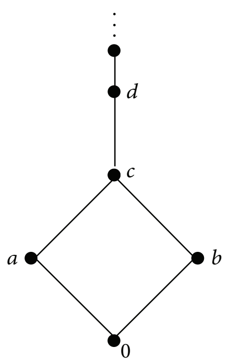
Theorem 15. *Let L be a distributive lattice. For any $A \in M_{n,q}(L)$ ($M_n(L)$, resp.), denote $\langle \Phi_A \rangle = \overline{L}$ and $J_{\langle \Phi_A \rangle} = \{0, p_1, p_2, \dots, p_h\}$. Assume that C_i ($i = 1, 2, \dots, m$) and ψ are given as in Lemma 1. Define the mapping φ from $M_{n,q}(\overline{L})$ ($M_n(\overline{L})$, resp.) to $\prod_{i=1}^{m} M_{n,q}(C_i)$ ($\prod_{i=1}^{m} M_n(C_i)$, resp.) by*

$$\begin{gathered}\left(\forall A = \left(a_{ij}\right) \in M_{n,q}\left(\overline{L}\right)\right)\ \left(M_n\left(\overline{L}\right), resp.\right) \\ A\varphi = \left(A_1, A_2, \dots, A_m\right),\end{gathered} \tag{26}$$

where $(\forall 1 \le k \le m)$ $A_k = (a_{ij}^{(k)})$ and $a_{ij}^{(k)} = (k)(a_{ij})\psi$. Then, φ is a subdirect embedding from semigroup $M_{n,q}(\overline{L})$ (semiring $M_n(\overline{L})$, resp.) to semigroup $\prod_{i=1}^{m} M_{n,q}(C_i)$ (semiring $\prod_{i=1}^{m} M_n(C_i)$, resp.) and $A = \sum_{k=1}^{m} A_k$ for any $A \in M_{n,q}(L)$ ($M_n(L)$, resp.).

Proof. By Lemma 14, it is not hard to see that $\langle \Phi_A \rangle$ is a finite distributive lattice. Now, denote $J_{\langle \Phi_A \rangle}$ to be the set of all the join irreducible elements of $\langle \Phi_A \rangle$. And then, analogous with the discussions of Theorem 8 or by Lemma 1 and Theorem 7, we can show the above theorem. □

Example 16. Let $L = \{0, a, b, c, d, \dots\}$ be an infinite distributive lattice whose Hasse diagram is shown below.



Given that $A = \begin{pmatrix} c & a & a & a \\ b & b & 0 & c \\ a & 0 & a & a \end{pmatrix} \in M_{3,4}(L)$, clearly, $\overline{L} = \langle \Phi_A \rangle = \{0, a, b, c\}$, $J_{\langle \Phi_A \rangle} = \{0, a, b\}$.

Note by Lemma 1 that finite distributive lattice $\overline{L} = \{0, a, b, c\}$ is isomorphic to a subdirect product of chains $\{0, a\}$ and $\{0, b\}$, and then by Theorem 15 we have

$$\begin{aligned} A &= \begin{pmatrix} c & a & a & a \\ b & b & 0 & c \\ a & 0 & a & a \end{pmatrix} \\ &= \begin{pmatrix} a & a & a & a \\ 0 & 0 & 0 & a \\ a & 0 & a & a \end{pmatrix} + \begin{pmatrix} b & 0 & 0 & 0 \\ b & b & 0 & b \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{27}$$

Let $L = \mathbb{K}$ be a chain semiring in Theorem 15. Then, for any $A \in M_n(\mathbb{K})$, notice that $\langle \Phi_A \rangle = \Phi_A$ for any $A \in M_n(\mathbb{K})$; we have the following corollary.

Corollary 17. *Let $\mathbb{K}$ be a chain semiring. For any $A \in M_{n,q}(\mathbb{K})$, denote $\Phi_A = \overline{L}$ and $J_{\Phi_A} = \{0, p_1, p_2, \dots, p_h\}$. Assume that C_i ($i = 1, 2, \dots, m$) and ψ are given as in Lemma 1 Define the mapping φ from $M_{n,q}$ to $\prod_{i=1}^{m} M_{n,q}(C_i)$ by*

$$\left(\forall A = \left(a_{ij}\right) \in M_{n,q}\left(\overline{L}\right)\right) \quad A\varphi = \left(A_1, A_2, \dots, A_m\right), \tag{28}$$

where $(\forall 1 \le k \le m)$ $A_k = (a_{ij}^{(k)})$ and $a_{ij}^{(k)} = (k)(a_{ij})\psi$. Then, φ is a subdirect embedding from semigroup $M_{n,q}(\overline{L})$ to semigroup $\prod_{i=1}^{m} M_{n,q}(C_i)$ and $A = \sum_{k=1}^{m} A_k$ for any $A \in M_{n,q}(\mathbb{K})$.

Similarly, let $L = \mathbb{F}$ be a fuzzy semiring in Theorem 15. Then, we will also have the following.

Corollary 18. *Let $\mathbb{F}$ be a fuzzy semiring. For any $A \in M_{n,q}(\mathbb{F})$, denote $\Phi_A = \overline{L}$ and $J_{\Phi_A} = \{0, p_1, p_2, \dots, p_h\}$. Assume that C_i ($i = 1, 2, \dots, m$) and ψ are given as in Lemma 1 Define the mapping φ from $M_{n,q}$ to $\prod_{i=1}^{m} M_{n,q}(C_i)$ by*

$$\left(\forall A = \left(a_{ij}\right) \in M_{n,q}\left(\overline{L}\right)\right) \quad A\varphi = \left(A_1, A_2, \dots, A_m\right), \tag{29}$$

where $(\forall 1 \le k \le m)$ $A_k = (a_{ij}^{(k)})$ and $a_{ij}^{(k)} = (k)(a_{ij})\psi$. Then, φ is a subdirect embedding from semigroup $M_{n,q}(\overline{L})$ to semigroup $\prod_{i=1}^{m} M_{n,q}(C_i)$ and $A = \sum_{k=1}^{m} A_k$ for any $A \in M_{n,q}(\mathbb{F})$.

Remark 19. Theorem 15 shows that a matrix over a distributive lattice can be decomposed into the sum of matrices over some of its special subchains. This generalizes and extends the decomposition theorems of matrices over finite distributive lattices, chain semirings, fuzzy semirings, and so forth, including the corresponding results in [6, 8, 30].

3. Some Applications

As a direct application, we will firstly use the decompositions of matrices obtained in Section 2 to give a way to calculate indices and periods of the square matrices over a distributive lattice.

Let L be a distributive lattice and let $M_n(L)$ be a semiring of matrices over L. Recall that if there exist positive integers d and p satisfying $A^{d+p} = A^d$ for any (but fixed) $A \in M_n(L)$, then the least such positive integers k and p are called the index and the period of A, respectively, and denoted by $k(A)$ and $p(A)$, respectively.

Now, by Theorem 7, it is not hard to obtain the following proposition.

Theorem 20. *Assume that ψ is a subdirect embedding from distributive lattice L to the direct product $\prod_{i=1}^{m} L_i$ of distributive lattices $L_1, L_2, \dots, L_m$. Define the mapping φ from $M_n(L)$ to $\prod_{i=1}^{m} M_n(L_i)$ by*

$$\left(\forall A = \left(a_{ij}\right) \in M_n\left(L\right)\right) \quad A\varphi = \left(A_1, A_2, \dots, A_m\right), \tag{30}$$

where $(\forall 1 \le k \le m)$ $A_k = (a_{ij}^{(k)})$ *and* $a_{ij}^{(k)} = (k)(a_{ij})\psi$. *Then, for any* $A \in M_n(L)$ *and any positive integers k and p, one has*

$$A^{d+p} = A^d \Longleftrightarrow (\forall 1 \le k \le m) \quad A_k^{d+p} = A_k^d. \tag{31}$$

Proof. For any $A \in M_n(L)$, assume that φ is a subdirect embedding from $M_n(L)$ to $\prod_{i=1}^m M_n(L_i)$ with $A\varphi = (A_1, A_2, \ldots, A_m)$. If $A^{d+p} = A^d$, then we have

$$A^{d+p}\varphi = A^d\varphi; \tag{32}$$

that is,

$$(A_1, A_2, \ldots, A_m)^{d+p} = (A_1, A_2, \ldots, A_m)^d, \tag{33}$$

and then

$$\left(A_1^{d+p}, A_2^{d+p}, \ldots, A_m^{d+p}\right) = \left(A_1^d, A_2^d, \ldots, A_m^d\right). \tag{34}$$

Thus, $A_i^{d+p} = A_i^d$ for $i = 1, 2, \ldots, m$.

Conversely, assume that $A\varphi = (A_1, A_2, \ldots, A_m)$, and, for $i = 1, 2, \ldots, m$, $A_i^{d+p} = A_i^d$. Then, we have

$$\left(A_1^{d+p}, A_2^{d+p}, \ldots, A_m^{d+p}\right) = \left(A_1, A_2, \ldots, A_m\right); \tag{35}$$

that is,

$$(A_1, A_2, \ldots, A_m)^{d+p} = (A_1, A_2, \ldots, A_h)^d, \tag{36}$$

and then

$$A^{d+p}\varphi = A^d\varphi. \tag{37}$$

Notice that φ is the subdirect embedding from $M_n(L)$ to $\prod_{i=1}^m M_n(L_i)$; we immediately get $A^{d+p} = A^d$. □

It is not hard to see that the indices and periods of the square matrices over a distributive lattice must exist. By Lemma 14, we can also get the following.

Theorem 21. *Let L be a distributive lattice. For any* $A \in M_n(L)$, *denote* $\langle \Phi_A \rangle = \overline{L}$, $J_{\langle \Phi_A \rangle} = \{0, p_1, p_2, \ldots, p_h\}$. *Assume that* C_i $(i = 1, 2, \ldots, m)$ *and* ψ *are given as in Lemma 1. Define the mapping* φ *from* $M_n(\overline{L})$ *to* $\prod_{i=1}^m M_n(C_i)$ *by*

$$\left(\forall A = \left(a_{ij}\right) \in M_n\left(\overline{L}\right)\right) \quad A\varphi = \left(A_1, A_2, \ldots, A_m\right), \tag{38}$$

where $(\forall 1 \le k \le m)$ $A_k = (a_{ij}^{(k)})$ *and* $a_{ij}^{(k)} = (k)(a_{ij})\psi$. *Then, the index* $k(A)$ *of A is equal to the maximum in the set of all indices of* A_k $(1 \le k \le m)$ *and the period* $p(A)$ *of A is equal to the least common multiple of all periods of* A_k $(1 \le k \le m)$.

In Theorem 21, take $C_i = \{0, p_i\}$; then, we have the following.

Corollary 22. *Let L be a distributive lattice. For any* $A \in M_n(L)$, *denote* $\langle \Phi_A \rangle = \overline{L}$, $J_{\langle \Phi_A \rangle} = \{0, p_1, p_2, \ldots, p_h\}$. *Then, the index* $k(A)$ *of A is equal to the maximum in the set of all indices of* A_{p_i} $(1 \le i \le h)$ *and the period* $p(A)$ *of A is equal to the least common multiple of all periods of* A_{p_i} $(1 \le i \le h)$.

Remark 23. Theorem 21 and Corollary 22 show that the indices and periods of a square matrix over a distributive lattice can be calculated by studying some ways of subdirect decompositions of the corresponding matrix semirings.

In the following, consider L to be a finite distributive lattice in Theorem 21. By Corollary 22, we immediately have the following.

Theorem 24. *Let L be a finite distributive lattice. For any* $A \in M_n(L)$, *denote* $\langle \Phi_A \rangle = \overline{L}$, $J_{\langle \Phi_A \rangle} = \{0, p_1, p_2, \ldots, p_h\}$. *Assume that* C_i $(i = 1, 2, \ldots, m)$ *and* ψ *are given as in Lemma 1. Define the mapping* φ *from* $M_n(\overline{L})$ *to* $\prod_{i=1}^m M_n(C_i)$ *by*

$$\left(\forall A = \left(a_{ij}\right) \in M_n\left(\overline{L}\right)\right) \quad A\varphi = \left(A_1, A_2, \ldots, A_m\right), \tag{39}$$

where $(\forall 1 \le k \le m)$ $A_k = (a_{ij}^{(k)})$ *and* $a_{ij}^{(k)} = (k)(a_{ij})\psi$. *Then, the index* $k(A)$ *of A is equal to the maximum in the set of all indices of* A_k $(1 \le k \le m)$ *and the period* $p(A)$ *of A is equal to the least common multiple of all periods of* A_k $(1 \le k \le m)$.

Corollary 25. *Let L be a finite distributive lattice. For any* $A \in M_n(L)$, *denote* $\langle \Phi_A \rangle = \overline{L}$, $J_{\langle \Phi_A \rangle} = \{0, p_1, p_2, \ldots, p_h\}$. *Then, the index* $k(A)$ *of A is equal to the maximum in the set of all indices of* A_{p_i} $(1 \le i \le h)$ *and the period* $p(A)$ *of A is equal to the least common multiple of all periods of* A_{p_i} $(1 \le i \le h)$.

Example 26. Let $L = \{0, a, b, c, d, \ldots\}$ be an infinite distributive lattice whose Hasse diagram is shown as in Example 16.

Now, for given $A = \left(\begin{smallmatrix} c & a & a \\ b & b & 0 \\ 0 & c & a \end{smallmatrix}\right) \in M_3(L)$, then, $\langle \Phi_A \rangle = \{0, a, b, c\}$, $J_{\langle \Phi_A \rangle} = \{0, a, b\}$.

Note by Lemma 1 that finite distributive lattice $\{0, a, b, c\}$ is isomorphic to a subdirect product of chains $\{0, a\}$ and $\{0, b\}$, and then by Theorem 15 we have

$$A = \begin{pmatrix} c & a & a \\ b & b & 0 \\ 0 & c & a \end{pmatrix} = \begin{pmatrix} a & a & a \\ 0 & 0 & 0 \\ 0 & a & a \end{pmatrix} + \begin{pmatrix} b & 0 & 0 \\ b & b & 0 \\ 0 & b & 0 \end{pmatrix} = A_1 + A_2. \tag{40}$$

It is easy to check that $k(A_1) = 1$, $k(A_2) = 2$, and $p(A_1) = p(A_2) = 1$. Hence, $k(A) = 2$, $p(A) = 1$.

On the other hand, by Theorem 20, we also have the following.

Theorem 27. *Let L be a finite distributive lattice with* $\ell \ge 2$ *elements and* $J = \{0, p_1, p_2, \ldots, p_h\}$ *the set of all join irreducible elements of L. Assume that* C_i $(i = 1, 2, \ldots, m)$ *and* ψ *are given as in Lemma 1. Define the mapping* φ *from* $M_n(L)$ *to* $\prod_{i=1}^m M_n(C_i)$ *by*

$$\left(\forall A = \left(a_{ij}\right) \in M_n\left(L\right)\right) \quad A\varphi = \left(A_1, A_2, \ldots, A_m\right), \tag{41}$$

where $(\forall 1 \le k \le m)$ $A_k = (a_{ij}^{(k)})$ *and* $a_{ij}^{(k)} = (k)(a_{ij})\psi$. *Then, the index* $k(A)$ *of A is equal to the maximum in the set of all indices of* A_k $(1 \le k \le m)$ *and the period* $p(A)$ *of A is equal to the least common multiple of all periods of* A_k $(1 \le k \le m)$.

Similarly, if we take $C_i = \{0, p_i\}$ in Theorem 27, then we will get the following theorem.

Theorem 28. *Let L be a finite distributive lattice with $\ell \geq 2$ elements and $A \in M_n(L)$ and $J = \{0, p_1, p_2, \ldots, p_h\}$ the set of all join irreducible elements of L. Then, the index $k(A)$ of A is equal to the maximum in the set of all indices of A_{p_i} $(1 \leq i \leq h)$ and the period $p(A)$ of A is equal to the least common multiple of all periods of A_{p_i} $(1 \leq i \leq h)$.*

By applying Theorem 27 to a finite chain $L = \{0, 1, \ldots, \ell\}$, together with the discussions of Corollary 13, we can also immediately obtain the following theorem which had been proved by [30].

Theorem 29. *Let $L = \{0, 1, \ldots, \ell\}$ $(\ell > 1)$ be a finite chain with the usual ordering. For any $A \in M_n(L)$, the index $k(A)$ of A is equal to the maximum in the set of all indices of $\overline{\varphi}_{\ell s}^{(k)}(A)$ $(1 \leq s \leq m)$ and the period $p(A)$ of A is equal to the least common multiple of all periods of $\overline{\varphi}_{\ell s}^{(k)}(A)$ $(1 \leq s \leq m)$.*

Next, using the decompositions of matrices obtained in Section 2, we will give another application to calculate the idempotent matrices over a distributive lattice. We will translate the characterizations of idempotent matrices over a distributive lattice into the corresponding ones of the binary Boolean cases, which generalize and extend the corresponding structures of idempotent matrices over general Boolean algebras, chain semirings, fuzzy semirings, and so forth.

Let L be a distributive lattice and $M_n(L)$ a semiring of matrices over L. Recall that a matrix A in $M_n(L)$ is called idempotent if $A^2 = A$.

Theorem 30. *Let L and L_i $(1 \leq i \leq h)$ be distributive lattices. Assume that φ is a subdirect embedding from semiring $M_n(L)$ to semiring $\prod_{i=1}^{h} M_n(L_i)$, and, for any $A \in M_n(L)$, $A\varphi = (A_1, A_2, \ldots, A_h)$. Then, $A^2 = A$ if and only if $A_i^2 = A_i$ $(i = 1, 2, \ldots, h)$.*

Proof. For any $A \in M_n(L)$, assume that φ is a subdirect embedding from $M_n(L)$ to $\prod_{i=1}^{h} M_n(L_i)$ with $A\varphi = (A_1, A_2, \ldots, A_h)$. If $A^2 = A$, then we have

$$A^2\varphi = A\varphi; \tag{42}$$

that is,

$$(A_1, A_2, \ldots, A_h)^2 = (A_1, A_2, \ldots, A_h), \tag{43}$$

and then

$$\left(A_1^2, A_2^2, \ldots, A_h^2\right) = (A_1, A_2, \ldots, A_h). \tag{44}$$

Thus, $A_i^2 = A_i$ for $i = 1, 2, \ldots, h$.

Conversely, assume that $A\varphi = (A_1, A_2, \ldots, A_h)$, and for $i = 1, 2, \ldots, h$, $A_i^2 = A_i$. Then, we have

$$\left(A_1^2, A_2^2, \ldots, A_h^2\right) = (A_1, A_2, \ldots, A_h); \tag{45}$$

that is,

$$(A_1, A_2, \ldots, A_h)^2 = (A_1, A_2, \ldots, A_h), \tag{46}$$

and then

$$A^2\varphi = A\varphi. \tag{47}$$

Notice that φ is the subdirect embedding from $M_n(L)$ to $\prod_{i=1}^{h} M_n(L_i)$; we immediately get $A^2 = A$. □

For any $A \in M_n(L)$, notice that $\langle\Phi_A\rangle$ is a finite distributive lattice; by Theorems 8 and 30, we immediately get the following.

Theorem 31. *Let L be a distributive lattice. For any $A \in M_n(L)$, denote $\langle\Phi_A\rangle = \overline{L}$. Assume that $\overline{C}_i$ $(1 \leq i \leq h)$ are the subdistributive lattices of $\overline{L}$ given as in Lemma 1 and φ is a subdirect embedding from semiring $M_n(\overline{L})$ to semiring $\prod_{i=1}^{h} M_n(\overline{C}_i)$ with $A\varphi = (A_1, A_2, \ldots, A_h)$. Then, $A^2 = A$ if and only if $A_i^2 = A_i$ $(i = 1, 2, \ldots, h)$.*

By Theorem 31, together with Theorem 8 or Corollary 10, we have the following.

Corollary 32. *Let L be a distributive lattice. For any $A \in M_n(L)$, $A^2 = A$ if and only if $A_{p_i}^2 = A_{p_i}$ $(i = 1, 2, \ldots, h)$, where $0 \neq p_i \in J_{\langle\Phi_A\rangle}$ and $A = \sum_{i=1}^{h} p_i A_{p_i}$.*

Example 33. Let $L = \{0, a, b, c, d, \ldots\}$ be an infinite distributive lattice whose Hasse diagram is shown as in Example 16.

Now, given $A = \begin{pmatrix} c & a & a \\ b & b & 0 \\ 0 & 0 & a \end{pmatrix} \in M_3(L)$, clearly, $\langle\Phi_A\rangle = \{0, a, b, c\}$, $J_{\langle\Phi_A\rangle} = \{0, a, b\}$.

Note by Lemma 1 that finite distributive lattice $\{0, a, b, c\}$ is isomorphic to a subdirect product of chains $\{0, a\}$ and $\{0, b\}$, and then, by Theorem 15, we have

$$A = \begin{pmatrix} c & a & a \\ b & b & 0 \\ 0 & 0 & a \end{pmatrix} = a \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + b \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{48}$$

By direct calculation (or by Theorem 2.11 in [14]), it is easy to check that

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ are idempotent matrices.} \tag{49}$$

Thus, by Corollary 32,

$$A = \begin{pmatrix} c & a & a \\ b & b & 0 \\ 0 & 0 & a \end{pmatrix} \text{ is an idempotent matrix.} \tag{50}$$

Let $L = \mathbb{K}$ be a chain semiring in Corollary 32. Then, for any $A \in M_n(\mathbb{K})$, notice that $\langle\Phi_A\rangle = \Phi_A$ for any $A \in M_n(\mathbb{K})$; we have the following corollary.

Corollary 34. *Let A be a matrix in $M_n(\mathbb{K})$. Then, A is idempotent if and only if all A_{p_i} are idempotent, where all p_i are the nonzero join irreducible elements of Φ_A and $A = \sum_{i=1}^{h} p_i A_{p_i}$.*

Similarly, let $L = \mathbb{F}$ be a fuzzy semiring in Corollary 32. Then, we will also have the following.

Corollary 35. *Let A be a matrix in $M_n(\mathbb{F})$. Then, A is idempotent if and only if all A_{p_i} are idempotent, where all p_i are the nonzero join irreducible elements of Φ_A and $A = \sum_{i=1}^{h} p_i A_{p_i}$.*

Remark 36. Theorem 31 and Corollary 32 show that the distinctions of idempotent matrices over a general distributive lattice can be also translated into the distinctions of idempotent matrices over the binary Boolean cases, which generalize and extend the corresponding ones of idempotent matrices over general Boolean algebras and chain semirings (including the fuzzy semirings).

In Theorem 30, consider L to be a finite distributive lattice. By Lemma 3 and Corollary 10, we immediately have the following.

Corollary 37. *Let L be a finite distributive lattice and p_i ($i = 1, 2, \ldots, h$) be its all nonzero join irreducible elements. Then, for any $A \in M_n(L)$, $A^2 = A$ if and only if $A_{p_i}^2 = A_{p_i}$ ($i = 1, 2, \ldots, h$), where $A = \sum_{i=1}^{h} p_i A_{p_i}$.*

In particular, if we let $L = \mathbb{B}_k$ be the general Boolean algebra in Corollary 37, then we will also obtain the following corollary which is just Theorem 4.1 in [24].

Corollary 38. *Let A be a matrix in $M_n(\mathbb{B}_k)$ with $k \geq 1$. Then, A is idempotent if and only if all pth constituents of A are idempotent in $M_n(\mathbb{B}_1)$, where the pth constituent of A is the matrix in $M_n(\mathbb{B}_1)$ whose (i, j)th entry is 1 if and only if $a_{ij} \geq \sigma_p$.*

On the other hand, by Theorem 31, we will also have the following results.

Corollary 39. *Let L be a finite distributive lattice. For any $A \in M_n(L)$, denote $\langle \Phi_A \rangle = \overline{L}$. Assume that $\overline{L}_i$ ($1 \leq i \leq h$) are the subdistributive lattices of $\overline{L}$ and φ is a subdirect embedding from semiring $M_n(\overline{L})$ to semiring $\prod_{i=1}^{h} M_n(\overline{L}_i)$ with $A\varphi = (A_1, A_2, \ldots, A_h)$. Then, $A^2 = A$ if and only if $A_i^2 = A_i$ ($i = 1, 2, \ldots, h$).*

Corollary 40. *Let L be a finite distributive lattice. For any $A \in M_n(L)$, $A^2 = A$ if and only if $A_{p_i}^2 = A_{p_i}$ ($i = 1, 2, \ldots, h$), where $0 \neq p_i \in J_{\langle \Phi_A \rangle}$ and $A = \sum_{i=1}^{h} p_i A_{p_i}$.*

Finally, we will use the decompositions of matrices obtained in Section 2 to calculate the nilpotent matrices over a distributive lattice. We will also translate the characterizations of nilpotent matrices over a distributive lattice into the corresponding ones of the binary Boolean cases, which generalize and extend the corresponding structures of nilpotent matrices over general Boolean algebras, chain semirings, fuzzy semirings, and so forth.

Let L be a distributive lattice and let $M_n(L)$ be a semiring of matrices over L. Recall that a matrix A in $M_n(L)$ is called nilpotent if $A^k = O$ for some $k \geq 1$, where O is the zero matrix in $M_n(L)$.

Theorem 41. *Let L and L_i ($1 \leq i \leq h$) be distributive lattices. Assume that φ is a subdirect embedding from semiring $M_n(L)$ to semiring $\prod_{i=1}^{h} M_n(L_i)$, and, for any $A \in M_n(L)$, $A\varphi = (A_1, A_2, \ldots, A_h)$. Then, $A^k = O$ if and only if $A_i^k = O$ ($i = 1, 2, \ldots, h$).*

Proof. Assume that φ is a subdirect embedding from $M_n(L)$ to $\prod_{i=1}^{h} M_n(L_i)$ and, for any $A \in M_n(L)$, $A\varphi = (A_1, A_2, \ldots, A_h)$. If $A^n = O$, then we have $A^n\varphi = O$; that is,

$$(A_1, A_2, \ldots, A_h)^n = (O, O, \ldots, O), \tag{51}$$

and then

$$(A_1^n, A_2^n, \ldots, A_h^n) = (O, O, \ldots, O). \tag{52}$$

Thus, we have $A_i^n = O$ for $i = 1, 2, \ldots, h$.

Conversely, for $i = 1, 2, \ldots, h$, if $A_i^n = O$, then we have $A^n\varphi = (O, O, \ldots, O)$. Notice that φ is the subdirect embedding from $M_n(L)$ to $\prod_{i=1}^{h} M_n(L_i)$; then, we immediately have $A^n = O$. □

By Theorems 8 and 41, we can get the following.

Theorem 42. *Let L be a distributive lattice. For any $A \in M_n(L)$, denote $\langle \Phi_A \rangle = \overline{L}$. Assume that $\overline{C}_i$ ($1 \leq i \leq h$) are the subdistributive lattices of $\overline{L}$ given as in Lemma 1 and φ is a subdirect embedding from semiring $M_n(\overline{L})$ to semiring $\prod_{i=1}^{h} M_n(\overline{C}_i)$ with $A\varphi = (A_1, A_2, \ldots, A_h)$. Then, $A^k = O$ if and only if $A_i^k = O$ ($i = 1, 2, \ldots, h$).*

By Theorem 42, together with Theorem 8 or Corollary 10, we have the following.

Corollary 43. *Let L be a distributive lattice. For any $A \in M_n(L)$, $A^k = O$ if and only if $A_{p_i}^k = O$ ($i = 1, 2, \ldots, h$), where $0 \neq p_i \in J_{\langle \Phi_A \rangle}$ and $A = \sum_{i=1}^{h} p_i A_{p_i}$.*

Example 44. Let $L = \{0, a, b, c, d, \ldots\}$ be an infinite distributive lattice whose Hasse diagram is shown as in Example 16.

For given $A = \begin{pmatrix} c & a & a \\ b & b & 0 \\ 0 & 0 & a \end{pmatrix} \in M_3(L)$, clearly, $\langle \Phi_A \rangle = \{0, a, b, c\}$, $J_{\langle \Phi_A \rangle} = \{0, a, b\}$.

Note by Lemma 1 that finite distributive lattice $\{0, a, b, c\}$ is isomorphic to a subdirect product of chains $\{0, a\}$ and $\{0, b\}$, and then, by Theorem 15, we have

$$A = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}. \tag{53}$$

By direct calculation, it is easy to check that

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ are nilpotent matrices.} \tag{54}$$

And then by Corollary 43,

$$A = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \text{ is a nilpotent matrix.} \quad (55)$$

Let $L = \mathbb{K}$ be a chain semiring in Corollary 43. Then, for any $A \in M_n(\mathbb{K})$, notice that $\langle \Phi_A \rangle = \Phi_A$ for any $A \in M_n(\mathbb{K})$; we have the following corollary.

Corollary 45. *Let A be a matrix in $M_n(\mathbb{K})$. Then, A is nilpotent if and only if all A_{p_i} are nilpotent, where all p_i are the nonzero join irreducible elements of Φ_A and $A = \sum_{i=1}^{h} p_i A_{p_i}$.*

Similarly, let $L = \mathbb{F}$ be a fuzzy semiring in Corollary 43. Then, we will also have the following.

Corollary 46. *Let A be a matrix in $M_n(\mathbb{F})$. Then, A is nilpotent if and only if all A_{p_i} are nilpotent, where all p_i are the nonzero join irreducible elements of Φ_A and $A = \sum_{i=1}^{h} p_i A_{p_i}$.*

Remark 47. Theorem 42 or Corollary 43 shows that the distinction of nilpotent matrices over a distributive lattice can be translated into the distinction of the binary Boolean cases, which generalize and extend the corresponding ones of nilpotent matrices over Boolean algebras and chain semirings (including the fuzzy semirings).

In Theorem 41, consider L to be a finite distributive lattice. By Lemma 3 and Corollary 10, we immediately have the following.

Corollary 48. *Let L be a finite distributive lattice and let p_i $(i = 1, 2, \ldots, h)$ be its all nonzero join irreducible elements. Then, for any $A \in M_n(L)$, $A^n = O$ if and only if $A_{p_i}^n = O$ $(i = 1, 2, \ldots, h)$, where $A = \sum_{i=1}^{h} p_i A_{p_i}$.*

In particular, if we let $L = \mathbb{B}_k$ be the general Boolean algebra in Corollary 48, then we will obtain the following corollary.

Corollary 49. *Let A be a matrix in $M_n(\mathbb{B}_k)$ with $k \geq 1$. Then, A is nilpotent if and only if all pth constituents of A are nilpotent in $M_n(\mathbb{B}_1)$, where the pth constituent of A is the matrix in $M_n(\mathbb{B}_1)$ whose (i, j)th entry is 1 if and only if $a_{ij} \supseteq \sigma_p$.*

On the other hand, by Theorem 42, we will also obtain the following corollaries.

Corollary 50. *Let L be a finite distributive lattice. For any $A \in M_n(L)$, denote $\langle \Phi_A \rangle = \overline{L}$. Assume that $\overline{L}_i$ $(1 \leq i \leq h)$ are the subdistributive lattices of $\overline{L}$ and φ is a subdirect embedding from semiring $M_n(\overline{L})$ to semiring $\prod_{i=1}^{h} M_n(\overline{L}_i)$ with $A\varphi = (A_1, A_2, \ldots, A_h)$. $A^k = O$ if and only if $A_i^k = O$ $(i = 1, 2, \ldots, h)$.*

Corollary 51. *Let L be a finite distributive lattice. For any $A \in M_n(L)$, $A^k = O$ if and only if $A_{p_i}^k = O$ $(i = 1, 2, \ldots, h)$, where $0 \neq p_i \in J_{\langle \Phi_A \rangle}$ and $A = \sum_{i=1}^{h} p_i A_{p_i}$.*

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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