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Research Article

Boundary Layer Flow due to the Vibration of a Sphere

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Boundary layer flow of the Newtonian fluid that is caused by the vibration of inner sphere while the outer sphere is at rest is calculated. Vishik-Lyusternik (Nayfeh refers to this method as the method of composite expansions) method is employed to construct an asymptotic expansion of the solution of the Navier-Stokes equations in the limit of high-frequency vibrations for Reynolds number of $O(1)$. The effect of the Stokes drift of fluid particles is also considered.

1. Introduction

In the present paper we consider boundary layer flow due to the vibrating sphere in a viscous fluid. Boundary layer flow for Newtonian and non-Newtonian fluid's is studied by many researchers [1, 2]. The interesting effect of viscosity is the flow produced by high-frequency oscillations of a boundary where the flow oscillations do not average to zero, such steady flow is referred to as "steady streaming" or "acoustic streaming" or "nonlinear streaming" [3, 4].

Mathematical modeling of the problem involves two parameters: the inverse of Reynolds number ν and the inverse Strouhal number α, given by

$$\alpha = \frac{V_0^*}{\omega a}, \qquad \nu = \frac{1}{\mathrm{Re}} = \frac{\nu^*}{a V_0^*}, \tag{1}$$

where V_0^* is the amplitude of the velocity of the oscillating sphere, a is the radius of the sphere, ω is the angular frequency of the oscillations, and ν^* is the kinematic viscosity of the fluid. Parameter α measures the ratio of the amplitude of the displacement of the oscillating sphere to its radius and is assumed to be small, that is, $\alpha \ll 1$. An asymptotic solution of Navier-Stokes equations in the limit $\alpha \to 0$ is obtained using the Vishik-Lyusternik method.

Steady secondary flow was investigated numerically in [5] for $0.2 \le \sqrt{\nu} \le 1$. The study was carried out to investigate the steady streaming near the rotating cylinder. In [6] steady streaming between two cylinders was studied experimentally for $1/(\alpha\nu)^{1/2}$ at different amplitude of oscillations. The study was carried out to investigate the behavior of outer boundary on the inner circulations. The unsteady flow around rotating and oscillating cylinder in a viscous incompressible micropolar fluid is studied numerically by [7] for $0 < \alpha < 2.8$ and obtained a good agreement with the experiments. Stream lines pattern was observed between two oscillating walls by Thomas et al. [8]. The study is carried out to show that less force is required by the flat plate to move the fluid rather than wavy wall. Steady flow between two cylinders was studied by [9] where the inner cylinder performs transverse oscillation and outer cylinder performs rotatory oscillations for $1/(\alpha\nu)$.

Wang [10] discussed the steady streaming due to oscillations of a sphere for $\mathrm{Re}_s = \alpha/\nu = O(1)$ using the method of inner and outer expansion. However his solution is incomplete, which had been understood and explained by Riley [11]. Riley discussed the steady streaming due to a sphere fixed in an oscillating fluid for $\mathrm{Re}_s \ll 1$ using method of matched asymptotic expansions. He discussed two cases: (i) for $\mathrm{Re} = 1/\nu \ll 1$ and (ii) for $|1/\alpha\nu| \gg 1$, where in the second case choice of Re is arbitrary such that $\mathrm{Re}_s \ll 1$. Dohara [12] studied steady streaming for $1/\sqrt{\alpha\nu} \approx O(1)$ and obtained good results with the experimental results. Recently the experimental work of Kotas et al. [13–15] produced the visualization of steady streaming due to the oscillating spheroids for moderate Reynolds number and small amplitude of oscillations. In recent paper [13], numerical results for steady streaming are presented for $\mathrm{Re}_s \gg 1$ and compared with the experimental results of Kotas et al.

In the present paper, steady streaming between two spheres is studied. The results are obtained for $\mathrm{Re} = O(1)$. Section 2 describes the mathematical formulation of the problem. Section 3 briefly explains the asymptotic procedure. To construct an asymptotic expansion of the solution of the Navier-Stokes equations, we apply the Vishik-Lyusternik method [16, 17]. This method has an essential advantage that it does not require the procedure of matching inner and outer expansion as in the case of matched asymptotic expansions. The Vishik-Lyusternik method had been used to study viscous boundary layer at a fixed impermeable boundary by Chudov [17]. Recently it has been applied to viscous boundary layers in high Reynolds number flows through a fixed domain with an inlet and an outlet [18] and to viscous flows in a half-plane produced by tangential vibrations on its boundary [19]. Vishik-Lyusternik method has successfully applied to the steady streaming due to a vibrating cylinder and by vibrating wavy wall [20–22]. In Section 4 asymptotic equations are solved. In Section 5 the steady Eulerian velocity is corrected through Stokes drift. This produces the steady Lagrangian velocity which is important because: (i) it is the Lagrangian velocity that is observed in experiments; (ii) it is the Lagrangian velocity that is invariant under the change of the frame of reference from the one fixed in the oscillating sphere to the one fixed in space. Section 6 contains the discussion of the results.

2. Formulation of the Problem

We consider a three-dimensional flow of a viscous incompressible fluid between two spheres with radii a and R^* ($R^* > a$) produced by small translational vibrations of an inner sphere about the axis of the outer sphere which is fixed in space. Let $\mathbf{x}^* = (x^*, y^*, z^*)$ be Cartesian coordinates in space, and let $\mathbf{x}_0^* = (0, 0, z_0^*(t^*))$ be the position of the center of the inner sphere at time t^*. We assume that $z_0^*(t^*)$ is oscillating in t^* with angular frequency ω and period $T = 2\pi/\omega$ and has zero mean value, that is,

$$\overline{z^*} \equiv \frac{1}{T}\int_0^T z^*(t^*)\,dt^* = 0. \quad (2)$$

The motion of the fluid is governed by the two-dimensional Navier-Stokes equations:

$$\mathbf{v}^*_{t^*} + (\mathbf{v}^* \cdot \nabla^*)\mathbf{v}^* = -\nabla^* p^* + \nu^* \nabla^{*2}\mathbf{v}^*, \quad \nabla^* \cdot \mathbf{v}^* = 0. \quad (3)$$

The velocity of the fluid satisfies the standard no-slip condition on the surfaces of the spheres which is:

$$\mathbf{v}^*(z^*, t^*)\,|_{\text{outer sphere}} = 0,$$
$$\mathbf{v}^*(z^*, t^*)\,|_{\text{inner sphere}} = \frac{d\mathbf{z}_0^*}{dt^*}. \quad (4)$$

Let $1/\omega$, a, V_0^*, and $a\omega V_0^*$ be the characteristic scales for time, length, velocity, and pressure, respectively. Using the dimensionless variables

$$\tau = \omega t^*, \qquad \mathbf{x} = \frac{1}{a}\mathbf{x}^*, \qquad \mathbf{x}_0 = \frac{1}{a}\mathbf{x}_0^*,$$
$$\mathbf{v} = \frac{1}{V_0^*}\mathbf{v}^*, \qquad p = \frac{p^*}{a\omega V_0^*}, \quad (5)$$

we rewrite (3) in the form

$$\mathbf{v}_\tau + \alpha\left[(\mathbf{v}\cdot\nabla)\mathbf{v} - \nu\nabla^2\mathbf{v}\right] = -\nabla p, \quad \nabla\cdot\mathbf{v} = 0, \quad (6)$$

where

$$\alpha = \frac{V_0^*}{a\omega}, \qquad \nu = \frac{\nu^*}{V_0^* a^*}, \quad (7)$$

are the inverse Strouhal number and the dimensionless viscosity (the inverse Reynolds number). Equations (4) become

$$\mathbf{v}(z, \tau)|_{\text{outer sphere}} = 0,$$
$$\mathbf{v}(z, \tau)|_{\text{inner sphere}} = \frac{1}{\alpha}\dot{\mathbf{z}}_0(\tau) = \frac{1}{\alpha}\dot{z}_0(\tau)\,\mathbf{e}_z. \quad (8)$$

Here dots denote differentiation with respect to τ and $z_0(\tau)$ is a given function which describes the motion of the inner sphere.

We are interested in the asymptotic behavior of periodic solution of (6) in the high-frequency limit $\alpha \to 0$.

It is convenient to introduce parameter ϵ such that $\alpha = \epsilon^2$. Then (6) becomes

$$\mathbf{v}_\tau = -\nabla p + \epsilon^2\left(-(\mathbf{v}\cdot\nabla)\mathbf{v} + \nu\nabla^2\mathbf{v}\right), \quad \nabla\cdot\mathbf{v} = 0. \quad (9)$$

Boundary condition (8) takes the form

$$\mathbf{v}(z, \tau)|_{\text{outer sphere}} = 0, \qquad \mathbf{v}(z, \tau)|_{\text{inner sphere}} = \dot{f}(\tau)\,\mathbf{e}_z. \quad (10)$$

We assume that the flow is axisymmetric (in spherical coordinates (r, θ, φ), axisymmetric flow means $\mathbf{v} = v_r(r,\theta)\mathbf{e}_r + v_\theta(r,\theta)\mathbf{e}_\theta$, where φ is the azimuthal angle and $v_\varphi = 0$, $\partial/\partial\varphi = 0$) in which the flow quantities do not depend upon the azimuthal angle φ. Thus the time-dependent boundary of the sphere can be described in the parametric form by the equations

$$x = \sin\tilde{\theta}\cos\varphi, \qquad y = \sin\tilde{\theta}\sin\varphi,$$
$$z = \cos\tilde{\theta} + \epsilon^2 f(\tau), \quad (11)$$

where $\tilde{\theta} \in [0, \pi)$ is the parameter on the sphere boundary. Now the boundary condition on the sphere can be written as follows:

$$\mathbf{v}\big|_{\substack{z=\cos\tilde{\theta}+\epsilon^2 f(\tau)\\ \sqrt{x^2+y^2}=\sin\tilde{\theta}}} = f'(\tau)\,\mathbf{e}_z. \quad (12)$$

Using the assumption that ϵ is small, we expand u and v in Taylor's series at $\epsilon = 0$. This yields

$$\begin{aligned}
&\mathbf{v}\Big|_{\substack{z=\cos\tilde{\theta}\\ \sqrt{x^2+y^2}=\sin\tilde{\theta}}} + \epsilon^2 f(\tau)\,\partial_z \mathbf{v}\Big|_{\substack{z=\cos\tilde{\theta}\\ \sqrt{x^2+y^2}=\sin\tilde{\theta}}} \\
&\quad + \frac{\epsilon^4 f^2(\tau)}{2}\partial_z^2 \mathbf{v}\Big|_{\substack{z=\cos\tilde{\theta}\\ \sqrt{x^2+y^2}=\sin\tilde{\theta}}} + \cdots = f'(\tau)\,\mathbf{e}_z.
\end{aligned} \tag{13}$$

Note that each term on the left side of (13) is evaluated at the averaged position of the sphere.

In spherical polar coordinates (r, θ, φ), with origin at the axis of the outer sphere, the axisymmetric form of (9) takes the form

$$\begin{aligned}
u_\tau &= -p_r + \epsilon^2\left[-uu_r - \frac{v}{r}u_\theta + \frac{v^2}{r}\right. \\
&\qquad \left. + \nu\left(\nabla^2 u - \frac{2u}{r^2} - \frac{2(\sin\theta v)_\theta}{r^2\sin^2\theta}\right)\right], \\
v_\tau &= -\frac{1}{r}p_\theta \\
&\quad + \epsilon^2\left[-uv_r - \frac{v}{r}v_\theta - \frac{uv}{r} + \nu\left(\nabla^2 v - \frac{v}{r^2\sin^2\theta} + \frac{2u_\theta}{r^2}\right)\right], \\
&u_r + \frac{2u}{r} + \frac{(\sin\theta v)_\theta}{r\sin\theta} = 0,
\end{aligned} \tag{14}$$

where u and v are the velocity components along r and θ directions and subscripts "τ," "r," and "θ" denote partial derivatives, and

$$\nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2\sin^2\theta}\frac{\partial(\sin\theta f_\theta)}{\partial\theta}. \tag{15}$$

The boundary condition on the outer sphere becomes

$$u|_{r=R} = 0, \qquad v|_{r=R} = 0, \tag{16}$$

where $R = R^*/a$. The boundary condition (13) on the inner sphere takes the form

$$\begin{aligned}
&u|_{r=1} + \epsilon^2 f\left[Lu + \frac{\sin\theta}{r}v\right]\Big|_{r=1} \\
&\quad + \frac{\epsilon^4 f^2}{2}\left[L^2 u - \frac{\sin^2\theta}{r}u + \frac{2\sin\theta}{r}Lv - \frac{2\sin\theta\cos\theta}{r^2}v\right]\Big|_{r=1} \\
&\quad + \cdots = f'(\tau)\cos\theta,
\end{aligned} \tag{17}$$

$$\begin{aligned}
&v|_{r=1} + \epsilon^2 f\left[Lv - \frac{\sin\theta}{r}u\right]\Big|_{r=1} \\
&\quad + \frac{\epsilon^4 f^2}{2}\left[L^2 v - \frac{\sin^2\theta}{r}v - \frac{2\sin\theta}{r}Lu + \frac{2\sin\theta\cos\theta}{r^2}u\right]\Big|_{r=1} \\
&\quad + \cdots = -f'(\tau)\sin\theta.
\end{aligned} \tag{18}$$

Here

$$L = \cos\theta\partial_r - \frac{\sin\theta}{r}\partial_\theta. \tag{19}$$

Equation (14) is to be solved according to the boundary conditions (16)–(18).

3. Asymptotic Expansion

We seek a solution of (14)–(18) in the form

$$u = u^i(r,\theta,\tau,\epsilon) + \epsilon u^a(\xi,\theta,\tau,\epsilon) + \epsilon u^b(\eta,\theta,\tau,\epsilon), \tag{20}$$

$$v = v^i(r,\theta,\tau,\epsilon) + v^a(\xi,\theta,\tau,\epsilon) + v^b(\eta,\theta,\tau,\epsilon), \tag{21}$$

$$p = p^i(r,\theta,\tau,\epsilon) + p^a(\xi,\theta,\tau,\epsilon) + p^b(\eta,\theta,\tau,\epsilon). \tag{22}$$

Here $\xi = (r-1)/\epsilon$ and $\eta = (R-r)/\epsilon$ are the boundary layer variables. Functions u^i, v^i, and p^i represent a regular expansion of the solution in power series in ϵ (an outer solution), and u^a, v^a, and p^a and u^b, v^b, and p^b correspond to boundary layer corrections to this regular expansion. Superscripts "a" and "b" correspond to the boundary layers at the inner and outer spheres, respectively. We assume that the boundary layer part of the expansion rapidly decays outside the thin boundary layers, that is,

$$\begin{aligned}
&u^a, v^a, p^a \longrightarrow 0 \quad \text{as } \xi \longrightarrow \infty, \\
&u^b, v^b, p^b \longrightarrow 0 \quad \text{as } \eta \longrightarrow \infty.
\end{aligned} \tag{23}$$

3.1. Regular Part of the Expansion. Substituting (20)–(22) in (14), we get the following equations:

$$u^i_{k_\tau} = -p^i_{k_r}, \tag{24}$$

$$v^i_{k_\tau} = -\frac{1}{r}p^i_{k_\theta}, \tag{25}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u^i_k\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(v^i_k\sin\theta\right) = 0, \tag{26}$$

where $k = 0, 1$ and

$$u^i_{k_\tau} = -p^i_{k_r} + \nu\left(\nabla^2 u^i_{k-2} - \frac{2u^i_{k-2}}{r^2} - \frac{\left(2v^i_{k-2}\sin\theta\right)_\theta}{\sin\theta r^2}\right) + M_k, \tag{27}$$

$$v^i_{k_\tau} = -\frac{1}{r}p^i_{k_\theta} + \nu\left(\nabla^2 v^i_{k-2} - \frac{v^i_{k-2}}{r^2\sin^2\theta} + \frac{2u^i_{k-2_\theta}}{r^2}\right) + N_k, \tag{28}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u^i_k\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(v^i_k\sin\theta\right) = 0, \tag{29}$$

for $k = 2, 3, 4, 5$. Explicit expression of M_k and N_k for $k = 1, 2, 3$ is given in Appendix A.

3.2. Boundary Layer Expansion

Boundary Layer at the Inner Sphere. Let us assume that

$$u^a = u_0^a + \epsilon u_1^a + \epsilon^2 u_2^a + \cdots,$$
$$v^a = v_0^a + \epsilon v_1^a + \epsilon^2 v_2^a + \cdots, \quad (30)$$
$$p^a = p_0^a + \epsilon p_1^a + \epsilon^2 p_2^a + \cdots.$$

Then we substitute (20)–(22) and (30) into (14) and take into account that u_k^i, v_k^i, and p_k^i $(k = 0, 1, \ldots)$ satisfy (24)–(29). Then we make the change of variables $r = 1+\epsilon\xi$, expand every function of $\epsilon\xi$ in Taylor's series at $\epsilon = 0$, and collect terms of equal powers in ϵ. This produces the following sequence of equations:

$$v_{0_\tau}^a + p_{0_\theta}^a - \nu v_{0_{\xi\xi}}^a = 0, \quad (31)$$

$$p_{0_\xi}^a = 0, \quad (32)$$

$$u_{0_\xi}^a + \frac{(\sin\theta v_0^a)_\theta}{\sin\theta} = 0, \quad (33)$$

$$v_{k_\tau}^a + p_{k_\theta}^a - \nu v_{k_{\xi\xi}}^a = F_k^a, \quad (34)$$

$$p_{k_\xi}^a = G_k^a, \quad (35)$$

$$u_{k_\xi}^a + \frac{(\sin\theta v_k^a)_\theta}{\sin\theta} = H_k^a, \quad (36)$$

for $k \geq 1$. Functions F_k^a, G_k^a, and H_k^a are defined in terms of $\mathbf{v}_0^i, \ldots, \mathbf{v}_{k-1}^i, v_0^a, \ldots, v_{k-1}^a$, and $u_0^a, \ldots, u_{k-1}^a$. Explicit expression of these functions for $k = 1, 2, 3$ is given in Appendix A.

Boundary Layer at the Outer Sphere. Let us assume that

$$u^b = u_0^b + \epsilon u_1^b + \epsilon^2 u_2^b + \cdots,$$
$$v^b = v_0^b + \epsilon v_1^b + \epsilon^2 v_2^b + \cdots, \quad (37)$$
$$p^b = p_0^b + \epsilon p_1^b + \epsilon^2 p_2^b + \cdots.$$

Then we substitute (20)–(22) and (37) into (14) and take into account that u_k^i, v_k^i, and p_k^i $(k = 0, 1, \ldots)$ satisfy (24)–(29). Then we make the change of variables $r = R-\epsilon\eta$, expand every function of ϵs in Taylor's series at $\epsilon = 0$ and collect terms of equal powers in ϵ. This produces the following sequence of equations:

$$v_{0_\tau}^a + \frac{1}{R}\, p_{0_\theta}^a - \nu v_{0_{\eta\eta}}^a = 0, \quad (38)$$

$$p_{0_\eta}^a = 0, \quad (39)$$

$$-u_{0_\eta}^a + \frac{1}{R}\, \frac{(\sin\theta v_0^a)_\theta}{\sin\theta} = 0, \quad (40)$$

$$v_{k_\tau}^b + \frac{1}{R}\, p_{k_\theta}^b - \nu v_{k_{\eta\eta}}^b = F_k^b, \quad (41)$$

$$p_{k_\eta}^b = G_k^b, \quad (42)$$

$$-u_{k_\eta}^b + \frac{1}{R}\, \frac{(\sin\theta v_k^b)_\theta}{\sin\theta} = H_k^b, \quad (43)$$

for $k \geq 1$. Functions F_k^b, G_k^b, and H_k^b are defined in terms of $\mathbf{v}_0^i, \ldots, \mathbf{v}_{k-1}^i, v_0^b, \ldots, v_{k-1}^b$, and $u_0^b, \ldots, u_{k-1}^b$. Explicit expression of these functions for $k = 1, 2, 3$ is given in Appendix A. We require that in all orders the boundary layer corrections to the outer solution rapidly decay outside boundary layers, that is,

$$u_k^a \longrightarrow 0, \quad v_k^a \longrightarrow 0, \quad p_k^a \longrightarrow 0 \quad \text{as } \xi \longrightarrow \infty, \quad (44)$$

$$u_k^b \longrightarrow 0, \quad v_k^b \longrightarrow 0, \quad p_k^b \longrightarrow 0 \quad \text{as } \eta \longrightarrow \infty, \quad (45)$$

for $k = 0, 1, \ldots$.

Before we try to solve these equations, it is convenient to discuss boundary conditions at $r = 1$ and $r = R$.

3.3. Boundary Conditions. Substituting (20)–(22), (30), and (37) in (16)–(18) and collecting terms of equal powers in ϵ, we obtain the following boundary conditions:

$$u_0^i\big|_{r=1} = f'(\tau)\cos\theta, \quad (46)$$

$$v_0^i\big|_{r=1} + v_0^a\big|_{\xi=0} = -f'(\tau)\sin\theta, \quad (47)$$

$$u_0^i\big|_{r=R} = 0, \quad (48)$$

$$v_0^i\big|_{r=R} + v_0^b\big|_{\eta=0} = 0, \quad (49)$$

$$u_k^i\big|_{r=1} + u_{k-1}^a\big|_{\xi=0} = Q_k^a, \quad (50)$$

$$v_k^i\big|_{r=1} + v_k^a\big|_{\xi=0} = S_k^a, \quad (51)$$

$$u_k^i\big|_{r=R} + u_{k-1}^b\big|_{\eta=0} = 0, \quad (52)$$

$$v_k^i\big|_{r=R} + v_k^b\big|_{\eta=0} = 0, \quad (53)$$

for $k \geq 1$. Functions Q_k and S_k are defined in terms of $\mathbf{v}_0^i, \ldots, \mathbf{v}_{k-1}^i, v_0^a, \ldots, v_{k-1}^a$, and $u_0^a, \ldots, u_{k-1}^a$. Explicit expression of these functions for $k = 1, 2, 3$ is given in Appendix A.

4. Analysis of the Asymptotic Equations

4.1. Leading Order Equations. Taking the derivative of (26) (for $k = 0$) with respect to τ and substituting (24)-(25) we get

$$p_{0rr}^i + \frac{2}{r} p_{0r}^i + \frac{1}{r^2\sin\theta}\left(p_{0\theta}^i \sin\theta\right)_\theta = 0. \quad (54)$$

The boundary conditions for the above equation can be obtained from (46) and (48) and with the relation $u_{0_\tau}^i = -p_{0_r}^i$. The boundary conditions are

$$p_{0_r}^i\big|_{r=1} = -\operatorname{Re}\left(Ce^{i\tau}\right)\cos\theta, \quad p_{0_r}^i\big|_{r=R} = 0. \quad (55)$$

The solution of (54) together with (55) and (26) gives u_0^i and v_0^i, the (leading order) oscillating outer flow which are

$$u_0^i = \frac{f'(\tau)}{1-R^3}\left(1-\frac{R^3}{r^3}\right)\cos\theta,$$
$$v_0^i = -\frac{f'(\tau)}{2(1-R^3)}\left(2+\frac{R^3}{r^3}\right)\sin\theta. \quad (56)$$

Inner Sphere. Consider now the leading order equations (31)–(33). The condition of decay at infinity (in the boundary layer variable ξ) for p_0^a and (32) has a consequence that $p_0^a \equiv 0$. Hence, we have the standard heat equation

$$v_{0\tau}^a = \nu v_{0\xi\xi}^a. \quad (57)$$

Boundary condition for v_0^a at $\xi = 0$ follows from (47):

$$v_0^a(r,\theta,\tau)\,|_{\xi=0} = -v_0^i(r,\theta,\tau)\,|_{r=1} - f'(\tau)\sin\theta. \quad (58)$$

The solution of (57) must also satisfy the decay condition (44).

Substitution of (56) in the boundary condition (58) yields

$$v_0^a(\xi,\theta,\tau)\big|_{\xi=0} = \frac{3R^3}{2(1-R^3)}f'(\tau)\sin\theta$$
$$= \frac{3R^2}{2(1-R^3)}\operatorname{Re}\left(iCe^{i\tau}\right)\sin\theta. \quad (59)$$

Equation (57) subject to the boundary conditions (59) and (44) can be solved by standard methods. The solution is given by

$$v_0^a = \frac{3R^3}{2(1-R^3)}\operatorname{Re}\left(iCe^{-\gamma\xi+i\tau}\right)\sin\theta, \quad (60)$$

where

$$\gamma = \frac{1+i}{\sqrt{2\nu}}. \quad (61)$$

On averaging (57), we obtain

$$\overline{v}_{0\xi\xi}^a = 0. \quad (62)$$

The only solution of this equation that satisfies the decay condition at infinity and the boundary condition $\overline{v}_0^a|_{\xi=0} = 0$ (which follows from averaging the boundary condition (59)) is zero solution. Thus, *in the leading order the boundary layer at the inner sphere is purely oscillatory*. This fact implies that the boundary condition for $\overline{v}_0^i$ at $r = 1$ (i.e., obtained by averaging the condition (47)) is $\overline{v}_0^i|_{r=1} = 0$. Similarly, averaging the condition (46) yields $\overline{u}_0^i|_{r=1} = 0$. Thus, we have

$$\overline{\mathbf{v}}_0^i|_{r=1} = 0. \quad (63)$$

The normal velocity u_0^a is determined from (33):

$$u_0^a(\xi,\theta,\tau) = \int_\xi^\infty \frac{(\sin\theta v_0^a)_\theta}{\sin\theta}\left(\xi',\theta,\tau\right)d\xi'$$
$$= \frac{3R^3}{2(1-R^3)}\operatorname{Re}\left(\frac{i}{\gamma}Ce^{-\gamma\xi+i\tau}\right)\cos\theta. \quad (64)$$

Here the constant of integration is chosen so as to guarantee that $u_0^a(\xi,\theta,\tau)$ decays as $\xi \to \infty$. Evidently, u_0^a, given by (64), does not satisfy the boundary condition $u_0^a|_{\xi=0} = 0$. Now we recall that the correction to u_0^i is ϵu_0^a, and, therefore, $u_0^a|_{\xi=0}$ gives us the boundary condition for the next approximation of the outer solution. Indeed, according to (50) (For $k = 1$), we must have

$$u_1^i(r,\theta,\tau)|_{r=0} = -u_0^a(\xi,\theta,\tau)\big|_{\xi=0}$$
$$= -\frac{3R^3}{2(1-R^3)}\operatorname{Re}\left(\frac{i}{\gamma}Ce^{i\tau}\right)\cos\theta. \quad (65)$$

Outer Sphere. Consider now (38)–(40). It follows from (39) and the condition of decay at infinity (in the boundary layer variable η) for p_0^b that $p_0^b \equiv 0$. Hence, we have

$$v_{0\tau}^b = \nu v_{0\eta\eta}^b. \quad (66)$$

Boundary condition for v_0^b at $\eta = 0$ follows from (49):

$$v_0^b(\eta,\theta,\tau)|_{\eta=0} = -v_0^i(r,\theta,\tau)|_{r=R} = \frac{3f'}{2(1-R^3)}\sin\theta$$
$$= \frac{3}{2(1-R^3)}\operatorname{Re}\left(iCe^{i\tau}\right)\sin\theta. \quad (67)$$

The solution of (66) that satisfies (67) and the decay condition (45) are given by

$$v_0^b = \frac{3}{2(1-R^3)}\operatorname{Re}\left(iCe^{-\gamma\eta+i\tau}\right)\sin\theta. \quad (68)$$

As mentioned before, the radial velocity u_0^b is determined from the incompressibility condition (40):

$$u_0^b(\eta,\theta,\tau) = -\frac{1}{R}\int_\eta^\infty \frac{(\sin\theta v_0^b)_\theta}{\sin\theta}\left(\eta',\theta,\tau\right)d\eta'$$
$$= -\frac{3}{2R(1-R^3)}\operatorname{Re}\left(\frac{i}{\gamma}Ce^{-\gamma\xi+i\tau}\right)\cos\theta. \quad (69)$$

Here again the constant of integration is chosen so as to guarantee that u_0^b decays as $\eta \to \infty$. $u_0^b|_{\eta=0}$ gives us the boundary condition for the next approximation of the outer solution:

$$u_1^i(r,\theta,\tau)\big|_{r=R} = -u_0^b(\eta,\theta,\tau)\big|_{\eta=0}$$
$$= \frac{3}{2R(1-R^3)}\operatorname{Re}\left(\frac{i}{\gamma}Ce^{i\tau}\right)\cos\theta. \quad (70)$$

The same arguments as in the case of the inner sphere lead to a conclusion that *in the leading order the boundary layer at the outer sphere is purely oscillatory*. This fact implies that the boundary condition for $\overline{v}_0^i$ at $r = R$ (i.e., obtained by averaging the condition (49)) is $\overline{v}_0^i|_{r=R} = 0$. Similarly, averaging the condition (48) yields $\overline{u}_0^i|_{r=R} = 0$. Hence,

$$\overline{\mathbf{v}}_0^i|_{r=R} = 0. \tag{71}$$

Averaged Outer Flow. Averaging (27) and (28) (for $k = 2$) together with boundary conditions (63) and (71) implies that $\overline{\mathbf{v}}_0^i \equiv 0$, that is, *there is no steady streaming in the leading order of the expansion*. First order solution.

4.2. Higher Order Solutions. Using the same procedure as for the leading order equation we get the steady streaming at higher orders of ϵ. Equations (20)-(21) can be written as follows:

$$\begin{aligned} u &= u_0^i + \epsilon\left(u_1^i + u_0^a + u_0^b\right) + \epsilon^2\left(u_2^i + u_1^a + u_1^b\right) + \cdots, \\ v &= v_0^i + v_0^a + v_0^b + \epsilon\left(v_1^i + v_1^a + v_1^b\right) \\ &\quad + \epsilon^2\left(v_2^i + v_2^a + v_2^b\right) + \cdots. \end{aligned} \tag{72}$$

To rewrite our asymptotic expansion in terms of the stream function, we get

$$\begin{aligned} \psi &= \psi_0^i + \epsilon\left[\psi_1^i + \psi_0^a + \psi_0^b\right] + \epsilon^2\left[\psi_2^i + \psi_1^a + \psi_1^b\right] \\ &\quad + \epsilon^3\left[\psi_3^i + \psi_2^a + +\psi_2^b\right] + O\left(\epsilon^4\right), \end{aligned} \tag{73}$$

where ψ_k^i is such that $u_k^i = (1/r^2 \sin\theta)\psi_{k\theta}^i$ and $v_k^i = -(1/r\sin\theta)\psi_{kr}^i$ for $k = 0, 1, \ldots$ and where ψ_k^a is defined as

$$\begin{aligned} \psi_k^a &= \sin\theta \int_\xi^\infty v_k^a\left(\xi', \theta, \tau\right) d\xi', \\ \psi_k^b &= -\sin\theta \int_\eta^\infty v_k^b\left(\eta', \theta, \tau\right) d\eta'. \end{aligned} \tag{74}$$

Taking the average of (73), we get

$$\overline{\psi} = \epsilon^2\left[\overline{\psi}_2^i + \overline{\psi}_1^a\right] + \epsilon^3\left[\overline{\psi}_3^i + \overline{\psi}_2^a + +\overline{\psi}_2^b\right] + O\left(\epsilon^4\right), \tag{75}$$

where

$$\begin{aligned} \overline{\psi}_1^a &= -\frac{3R^3}{4\left(R^3-1\right)} e^{-s/\sqrt{2\nu}} \sin\left(\frac{s}{\sqrt{2\nu}}\right)\sin\theta\sin 2\theta, \\ \overline{\psi}_2^a &= \frac{3R^2\sqrt{2\nu}\, e^{-s/\sqrt{2\nu}}}{64\left(R^3-1\right)^2} \\ &\quad \times\left[3R^4 e^{-s/\sqrt{2\nu}} + \left(40R^4 - 16R\right)\frac{s}{\sqrt{2\nu}}\sin\frac{s}{\sqrt{2\nu}}\right. \\ &\quad \left. + \left(24R^4 - 12\right)\sin\frac{s}{\sqrt{2\nu}} + \left(48R^4 - 12\right)\cos\frac{s}{\sqrt{2\nu}}\right] \\ &\quad \times \sin 2\theta\sin\theta, \\ \overline{\psi}_2^b &= -\frac{9\sqrt{2\nu}\, e^{-\eta/\sqrt{2\nu}}}{64R^2\left(R^3-1\right)^2} \\ &\quad \times\left[e^{-\eta/\sqrt{2\nu}} + 8\frac{\eta}{\sqrt{2\nu}}\sin\frac{\eta}{\sqrt{2\nu}}\right. \\ &\quad \left. +12\sin\frac{\eta}{\sqrt{2\nu}} + 20\cos\frac{\eta}{\sqrt{2\nu}}\right]\sin 2\theta\sin\theta. \end{aligned} \tag{76}$$

Similarly we get

$$\overline{\psi}_2^i = \left(C_1 - \frac{C_2}{r^2} + C_3 r^2 + C_4 r^5\right)\sin\theta\sin 2\theta, \tag{77}$$

where

$$\begin{aligned} C_1 &= \frac{\left(2R^5 + 4R^4 + 6R^3 + 8R^2 + 10R + 5\right)R^3 A}{H} \\ &\quad + \frac{BR\left(5R^5 + 10R^4 + 8R^3 + 6R^2 + 4R + 2\right)}{H}, \\ C_2 &= \frac{\left(2R^3 + 4R^2 + 6R + 3\right)AR^5}{H} \\ &\quad + \frac{BR^3\left(3R^3 + 6R^2 + 4R + 2\right)}{H}, \\ C_3 &= -\frac{\left(5R^5 + 10R^4 + 8R^3 + 6R^2 + 4R + 2\right)A}{H} \\ &\quad - \frac{R\left(2R^5 + 4R^4 + 6R^3 + 8R^2 + 10R + 5\right)B}{H}, \\ C_4 &= \frac{\left(3R^3 + 6R^2 + 4R + 2\right)A}{H} \\ &\quad + \frac{R\left(2R^3 + 4R^2 + 6R + 3\right)B}{H}, \\ H &= 4R^8 + 8R^7 + 12R^6 - 9R^5 - 30R^4 - 9R^3 + 12R^2 \\ &\quad + 8R + 4, \\ A &= \frac{45}{32}\frac{R^6}{\left(R^3-1\right)^2}, \qquad B = \frac{45}{32}\frac{1}{R\left(R^3-1\right)^2}, \\ \overline{\psi}_3^i &= \left(D_1 - \frac{D_2}{r^2} + D_3 r^3 + D_4 r^5\right)\sin 2\theta\sin\theta, \end{aligned} \tag{78}$$

where

$$\begin{aligned} D_1 &= \frac{\left(4R^6 + 4R^5 + 4R^4 + 4R^3 + 4R^2 + 25R + 25\right)R^3 A}{G} \\ &\quad - \frac{\left(25R^6 + 25R^5 + 4R^4 + 4R^3 + 4R^2 + 4R + 4\right)B}{G} \end{aligned}$$

$$+\frac{\left(2R^5+4R^4+6R^3+8R^2+10R+5\right)R^3}{G}C$$
$$+\frac{R\left(5R^5+10R^4+8R^3+6R^2+4R+2\right)}{G}D,$$
$$D_2=\frac{15(R+1)R^5}{G}A-\frac{15(R+1)R^5}{G}B$$
$$+\frac{\left(2R^3+4R^2+6R+3\right)R^5}{G}C$$
$$+\frac{R^3\left(3R^3+6R^2+4R+2\right)}{G}D,$$
$$D_3=-\frac{10\left(R^6+R^5+R^4+R^3+R^2+R+1\right)}{G}A$$
$$+\frac{10\left(R^6+R^5+R^4+R^3+R^2+R+1\right)}{G}B$$
$$-\frac{5R^5+10R^4+8R^3+6R^2+4R+2}{G}C$$
$$-\frac{R\left(2R^5+4R^4+6R^3+8R^2+10R+5\right)}{G}D,$$
$$D_4=\frac{6\left(R^4+R^3+R^2+R+1\right)}{G}A$$
$$-\frac{6\left(R^4+R^3+R^2+R+1\right)}{G}B$$
$$+\frac{\left(3R^3+6R^2+4R+2\right)}{G}C$$
$$+\frac{R\left(2R^3+4R^2+6R+3\right)}{G}D,$$
$$G=4R^9+4R^8+4R^7-21R^6-21R^5$$
$$+21R^4+21R^3-4R^2-4R-4,$$
$$A=-\frac{\sqrt{2\nu}\,9R^2\left(17R^4-4\right)}{64(R^3-1)^2},\qquad B=\frac{189\sqrt{2\nu}}{64R^2(R^3-1)^2},$$
$$C=-\sqrt{2\nu}\,\frac{9R^2\left(46R^7-61R^4-11R^3-4\right)}{32(R^3-1)^3},$$
$$D=\sqrt{2\nu}\,\frac{9\left(6R^8+10R^7-7R^4-19R^3+16\right)}{32R^2(R^3-1)^3}. \tag{79}$$

5. Stokes Drift

It is well-known that in oscillatory flows the observed averaged Lagrangian velocity differs from the Eulerian velocity by the term known as Stokes drift. The velocity observed in the experiments is the velocity of fluid particles, that is, the Lagrangian velocity. Our asymptotic expansion for the averaged Eulerian velocity has the form

$$\overline{u}^E=\epsilon^2\left[\overline{u}_2^i+\overline{u}_1^a\right]+O\left(\epsilon^3\right), \tag{80}$$
$$\overline{v}^E=\epsilon\overline{v}_1^i+\epsilon^2\left[\overline{v}_2^i+\overline{v}_2^a+\overline{v}_2^b\right]+O\left(\epsilon^3\right). \tag{81}$$

It is shown in Appendix B that the Lagrangian velocity of fluid particles is given by

$$\overline{u}^L=\epsilon^2\overline{u}_2^i+O\left(\epsilon^3\right), \tag{82}$$
$$\overline{v}^L=\epsilon^2\left[\overline{v}_2^i+\overline{v}_2^a+\overline{v}_2^b+\overline{v}_2^s\right]+O\left(\epsilon^3\right), \tag{83}$$

where $\overline{v}_2^s$ is the Stokes drift velocity of the fluid particles. Comparing (80) with (82) and (81) with (83), we observe that the Stokes drift eliminates (i) $\overline{u}_a^1$ from (80) and (ii) the $O(\epsilon)$ term in (83). It also results in the additional $O(\epsilon^2)$ term $\overline{\mathbf{v}}_s$ in the expansion of azimuthal velocity. Thus, the $O(\epsilon)$ steady boundary layer at the inner cylinder disappears when we take account of the Stokes drift. This is a consequence of the fact that the steady Lagrangian velocity rather than the steady Eulerian velocity is invariant to the change of reference frame. In the reference frame fixed in the inner cylinder, we would have no $O(\epsilon)$ steady boundary layer at the inner cylinder. Further calculations with the help of the known formula for $\overline{\mathbf{v}}^a$, $\overline{\mathbf{v}}^b$, and $\overline{\mathbf{v}}^s$ show that $\overline{\mathbf{v}}^L$ can be written as follows:

$$\overline{u}^L=\epsilon^2\overline{u}_2^i+O\left(\epsilon^3\right),$$
$$\overline{v}^L=\epsilon^2\left[\overline{v}_2^i+\left(\overline{v}_2^a\right)^L+\left(\overline{v}_2^b\right)^L\right]+O\left(\epsilon^3\right), \tag{84}$$

where

$$\left(\overline{v}_2^a\right)^L=\frac{9R^6e^{-\xi/\sqrt{2\nu}}}{32(R^3-1)^2}\left(12\sin\frac{\xi}{\sqrt{2\nu}}+5e^{-\xi/\sqrt{2\nu}}\right)\sin(2\theta),$$
$$\left(\overline{v}_2^b\right)^L=\frac{9e^{-\eta/\sqrt{2\nu}}}{32R(R^3-1)^2}\times\left(5e^{-\eta/\sqrt{2\nu}}+8\sin\frac{\eta}{\sqrt{2\nu}}-4\frac{\eta}{\sqrt{2\nu}}\left(\cos\frac{\eta}{\sqrt{2\nu}}-\sin\frac{\eta}{\sqrt{2\nu}}\right)\right)\sin(2\theta). \tag{85}$$

Asymptotic Expansion for Stream Function. To rewrite our asymptotic expansion of terms of the averaged stream function, we get

$$\overline{\psi}^E=\epsilon^2\left[\overline{\psi}_2^i+\overline{\psi}_1^a\right]+\epsilon^3\left[\overline{\psi}_3^i+\overline{\psi}_2^a+\overline{\psi}_2^b\right]+O\left(\epsilon^4\right),$$
$$\overline{\psi}^L=\epsilon^2\overline{\psi}_2^i+\epsilon^3\left[\overline{\psi}_3^{i^L}+\overline{\psi}_2^{a^L}+\overline{\psi}_2^{b^L}\right]+O\left(\epsilon^4\right). \tag{86}$$

In the last formula, $\overline{\psi}_2^{a^L}$ and $\overline{\psi}_2^{b^L}$ are obtained from (85):

$$\overline{\psi}_2^{a^L} = \frac{9R^6\sqrt{2\nu}\,e^{-s/\sqrt{2\nu}}}{64(R^3-1)^2}\left[5e^{-s/\sqrt{2\nu}} + 12\sin\frac{s}{\sqrt{2\nu}} + 12\cos\frac{s}{\sqrt{2\nu}}\right]\sin 2\theta\sin\theta,$$

$$\overline{\psi}_2^{b^L} = -\frac{9\sqrt{2\nu}\,e^{-\eta/\sqrt{2\nu}}}{64R^2(R^3-1)^2} \times\left[5e^{-\eta/\sqrt{2\nu}} + 8\frac{\eta}{\sqrt{2\nu}}\sin\frac{\eta}{\sqrt{2\nu}} + 12\sin\frac{\eta}{\sqrt{2\nu}} + 12\cos\frac{\eta}{\sqrt{2\nu}}\right]\sin 2\theta\sin\theta. \tag{87}$$

$\overline{\psi}_3^{i^L}$ is the stream function for the third-order Lagrangian velocity:

$$\overline{\psi}_3^{i^L} = \left(\widetilde{D}_1 - \frac{\widetilde{D}_2}{r^2} + \widetilde{D}_3 r^3 + \widetilde{D}_4 r^5\right)\sin 2\theta\sin\theta, \tag{88}$$

where

$$\widetilde{D}_1 = \frac{\left(4R^6+4R^5+4R^4+4R^3+4R^2+25R+25\right)R^3\widetilde{A}}{G} - \frac{\left(25R^6+25R^5+4R^4+4R^3+4R^2+4R+4\right)\widetilde{B}}{G} + \frac{\left(2R^5+4R^4+6R^3+8R^2+10R+5\right)R^3}{G}\widetilde{C} + \frac{R\left(5R^5+10R^4+8R^3+6R^2+4R+2\right)}{G}\widetilde{D},$$

$$\widetilde{D}_2 = \frac{15(R+1)R^5}{G}\widetilde{A} - \frac{15(R+1)R^5}{G}\widetilde{B} + \frac{\left(2R^3+4R^2+6R+3\right)R^5}{G}\widetilde{C} + \frac{R^3\left(3R^3+6R^2+4R+2\right)}{G}\widetilde{D},$$

$$\widetilde{D}_3 = -\frac{10\left(R^6+R^5+R^4+R^3+R^2+R+1\right)}{G}\widetilde{A} + \frac{10\left(R^6+R^5+R^4+R^3+R^2+R+1\right)}{G}\widetilde{B} - \frac{5R^5+10R^4+8R^3+6R^2+4R+2}{G}\widetilde{C} - \frac{R\left(2R^5+4R^4+6R^3+8R^2+10R+5\right)}{G}\widetilde{D},$$

$$\widetilde{D}_4 = \frac{6\left(R^4+R^3+R^2+R+1\right)}{G}\widetilde{A} - \frac{6\left(R^4+R^3+R^2+R+1\right)}{G}\widetilde{B} + \frac{\left(3R^3+6R^2+4R+2\right)}{G}\widetilde{C} + \frac{R\left(2R^3+4R^2+6R+3\right)}{G}\widetilde{D},$$

$$\widetilde{G} = 4R^9+4R^8+4R^7-21R^6-21R^5+21R^4 + 21R^3-4R^2-4R-4,$$

$$\widetilde{A} = -\frac{\sqrt{2\nu}\,153R^6}{64(R^3-1)^2}, \qquad \widetilde{B} = \frac{153\sqrt{2\nu}}{64R^2(R^3-1)^2},$$

$$\widetilde{C} = -\sqrt{2\nu}\left(\left(3\left(8+4R-12R^2-10R^3-2R^4-33R^5 - 181R^6-2R^7+138R^9\right)\right) \times\left(32\left(R^3-1\right)^3\right)^{-1}\right),$$

$$\widetilde{D} = \sqrt{2\nu}\,\frac{9\left(-4+16R+4R^3-19R^4-7R^5+10R^8+6R^9\right)}{32R^3(R^3-1)^3}. \tag{89}$$

6. Results Discussion

Equation (77) together with (87) and (88) represent nonzero terms in the asymptotic expansion of the stream function for the Lagrangian velocity. Let us first discuss the domain of applicability of formula (86). Steady streaming due to the vibration of the sphere is calculated. Our asymptotic expansion is formally valid for $\nu = O(1)$ and for $\epsilon \ll 1$. Also we may expect that it will be valid for all ϵ and ν such that the contribution of the $O(\epsilon^3)$ term to the right side of (86) is smaller than the contribution of the $O(\epsilon^2)$ term. It is convenient to rewrite (86) in the form

$$\overline{\psi}^L = \epsilon^2\sqrt{\nu}\,\left[\Phi_0(r) + \epsilon\Phi_1(\epsilon,\nu,r)\right]\sin\theta\sin 2\theta + O\left(\epsilon^4\right), \tag{90}$$

where

$$\Phi_0(r) = \frac{1}{\sin\theta\sin 2\theta}\overline{\psi}_2^i(r,\theta),$$

$$\Phi_1(r,\nu) = \frac{1}{\sqrt{\nu}\sin\theta\sin 2\theta} \times\left[\overline{\psi}_3^i(r,\theta) + \overline{\psi}_2^a\big|_{\xi=(r-1)/\epsilon} + \overline{\psi}_2^b\big|_{\eta=(R-r)/\epsilon}\right]. \tag{91}$$

Consider now the following quantity:

$$\chi(\nu,\epsilon) = \epsilon\sqrt{\nu}\frac{\max_{r\in[1,\infty]}\left|\Phi_1(\nu,r,\epsilon)\right|}{\max_{r\in[1,\infty]}\left|\Phi_0(r)\right|}, \tag{92}$$

which measures the magnitude of the second nonzero term relative to the first term. We expect that our theory will work

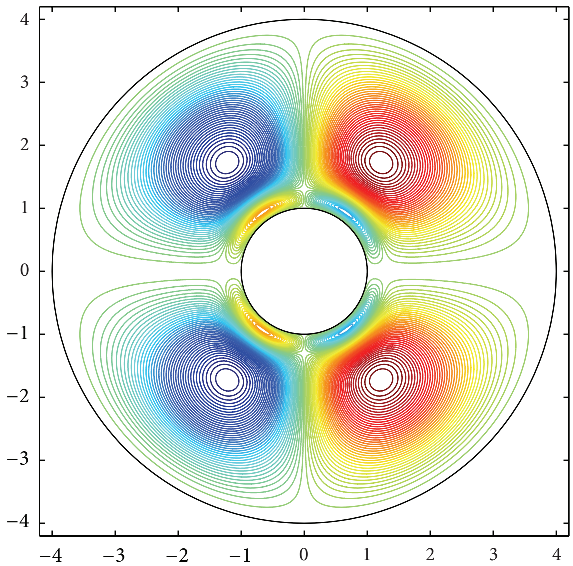

Figure 1: Steady streaming between two spheres.

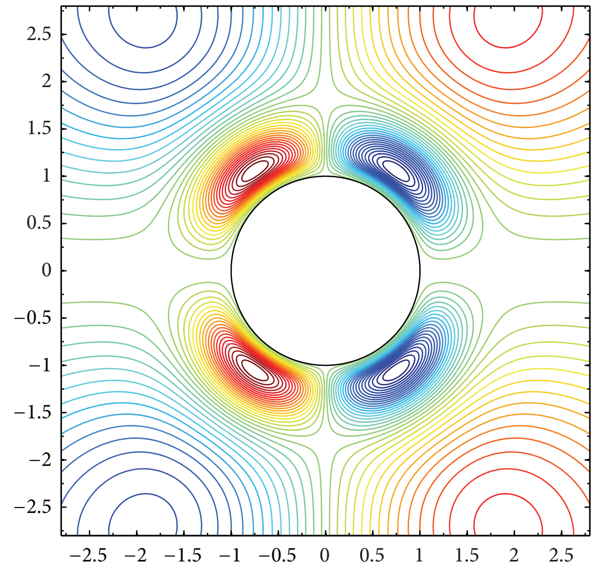

Figure 2: Steady streaming near inner sphere.

for all ϵ, ν such that $\chi(\nu, \epsilon) < 1$, and the smaller the χ is the better the theory should work.

In (86) if we take $R \to \infty$ we get the results of steady streaming due to single sphere in infinite viscous incompressible fluid. The results of single vibrating sphere in an infinite fluid are

$$\overline{\psi}_2^i = \frac{45}{64}\left(1 - \frac{1}{r^2}\right)\sin\theta\sin 2\theta,$$

$$\overline{\psi}_2^a = \frac{9\sqrt{2\nu}\, e^{-\xi/\sqrt{2\nu}}}{64}\left[5e^{-\xi/\sqrt{2\nu}} + 12\sin\frac{\xi}{\sqrt{2\nu}} + 12\cos\frac{\xi}{\sqrt{2\nu}}\right]\sin\theta\sin 2\theta, \quad (93)$$

$$\overline{\psi}_3^i = \frac{1}{2}\sqrt{2\nu}\left(-\frac{567}{32} + \frac{207}{16r^2}\right)\sin\theta\sin 2\theta.$$

Wang [10] studied the steady streaming when the sphere is at rest and fluid is oscillating at infinity for $\mathrm{Re}_s = O(1)$. However, the averaged stream function was incomplete because of the absence of the terms $\overline{\psi}_2^a$ and $\overline{\psi}_3^i$ which appears at $O(\epsilon^3)$ in the expansion of stream function. Riley [11] studied the flow produced by a fixed sphere placed in an oscillating fluid using matched asymptotic expansion method and obtained a uniformly valid expansion of the stream function under the same assumption as in the present paper. He did not compute the $\overline{\psi}_3^i$, which appears in the expansion of stream function at $O(\epsilon^3)$. In order to obtain the invariant velocity field one should consider the Lagrangian velocity which is different from the Eulerian velocity by Stokes drift velocity of the fluid particles. In order to support the argument, here the results for steady streaming are also presented for the case when the sphere is at rest and fluid is oscillating at infinity using the Vishik-Lyusternik method. In Eulerian coordinates system the steady streaming is given as follows:

$$\overline{\psi}_2^i = \frac{45}{64}\left(1 - \frac{1}{r^2}\right)\sin\theta\sin 2\theta,$$

$$\overline{\psi}_2^a = \frac{9\sqrt{2\nu}e^{-\xi/\sqrt{2\nu}}}{64}\left[e^{-\xi/\sqrt{2\nu}} + 8\frac{\xi}{\sqrt{2\nu}}\sin\frac{\xi}{\sqrt{2\nu}} + 12\sin\frac{\xi}{\sqrt{2\nu}} + 20\cos\frac{\xi}{\sqrt{2\nu}}\right]\sin\theta\sin 2\theta,$$

$$\overline{\psi}_3^i = \frac{1}{2}\sqrt{2\nu}\left(-\frac{567}{32} + \frac{189}{16r^2}\right)\sin\theta\sin 2\theta. \quad (94)$$

After computing the Stokes drift the results of (93) are achieved.

Typical stream line pattern due to vibrating sphere is shown in Figure 1 for $\epsilon = 0.06$, $\nu = 1$, and $R = 1$. Graph of streamlines produced near the inner sphere produced by the formula (86) are shown in Figure 2 for $\nu = 0.1$ and $\epsilon = 0.316$ with outer sphere at $R = 7$. In Figure 3 two sets of circulations with opposite direction can be seen in each quadrant for $\epsilon = 0.316$, $R = 7$, and $\nu = 0.1, 0.12, 0.14, 0.16$, respectively. Figure 3 shows that with the increase in ν the thickness of the inner circulation also increases. Similarly if we fix ν and change ϵ, it is observed that with the increase of ϵ the thickness of inner circulation also increases and outer circulations become thinner. Figure 3 shows the streamlines for $\nu = 0.1$ and $\epsilon = 0.3, 0.34, 0.37, 0.4$, respectively. The center of the circulation lies on the line at angle $\theta \cong 54^\circ$. It is observed that as ν and ϵ increase the stagnation point moves away from the surface of inner sphere. Figure 2 is in good comparison with the experimental results of Kotas et al. [13–15]. A recent paper [23] claims that distance of stagnation point to the surface of the sphere along the axis of oscillation

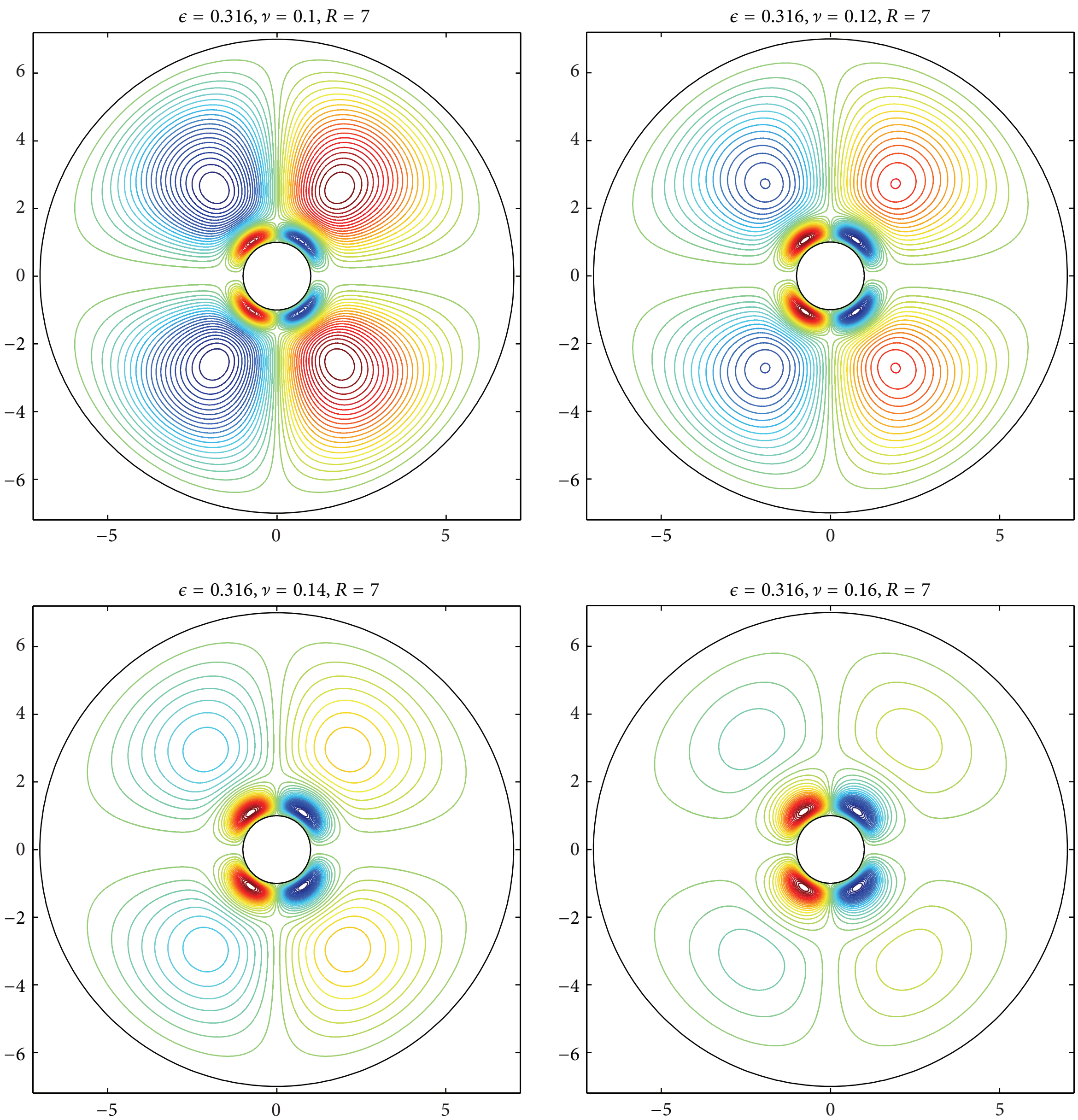

FIGURE 3: Steady streaming between two spheres.

is different from the distance of stagnation point to the surface of sphere along perpendicular axis. According to our results this does not happen. In Figure 4 the distance of stagnation points UPS (upper stagnation point) and LSP (left stagnation point) from the surface of the sphere is the same. Typical profiles of the radial velocity are shown in Figure 5 for $\epsilon = 0.316$ and $\nu - 0.1, 0.12, 0.14, 0.16$, respectively.

7. Conclusion

(I) Vishik-Lyusternik method is successfully applied to study the steady streaming between two spheres. In comparison with the method of matched asymptotic expansions, it has two advantages: (i) it does not require the procedure of matching the inner and outer expansions and (ii) the boundary layer part of the expansion satisfies the condition of decay at infinity (in boundary layers) in all the orders of the expansion, which is not the case in the method of matched asymptotic expansion.

(II) Stokes drift is calculated not for the outer flow but for the entire flow domain including the boundary layers.

(III) We also calculated the averaged lagrangian velocity which remains invariant with the change of reference frame and is observed in experiment.

(IV) The distance of stagnation points from the surface of sphere in each quadrant remains the same.

(V) Qualitatively the graphs of the streamlines are very much similar to the experimental results which are done for $\epsilon > 0.3$; our theory is valid for the small ϵ.

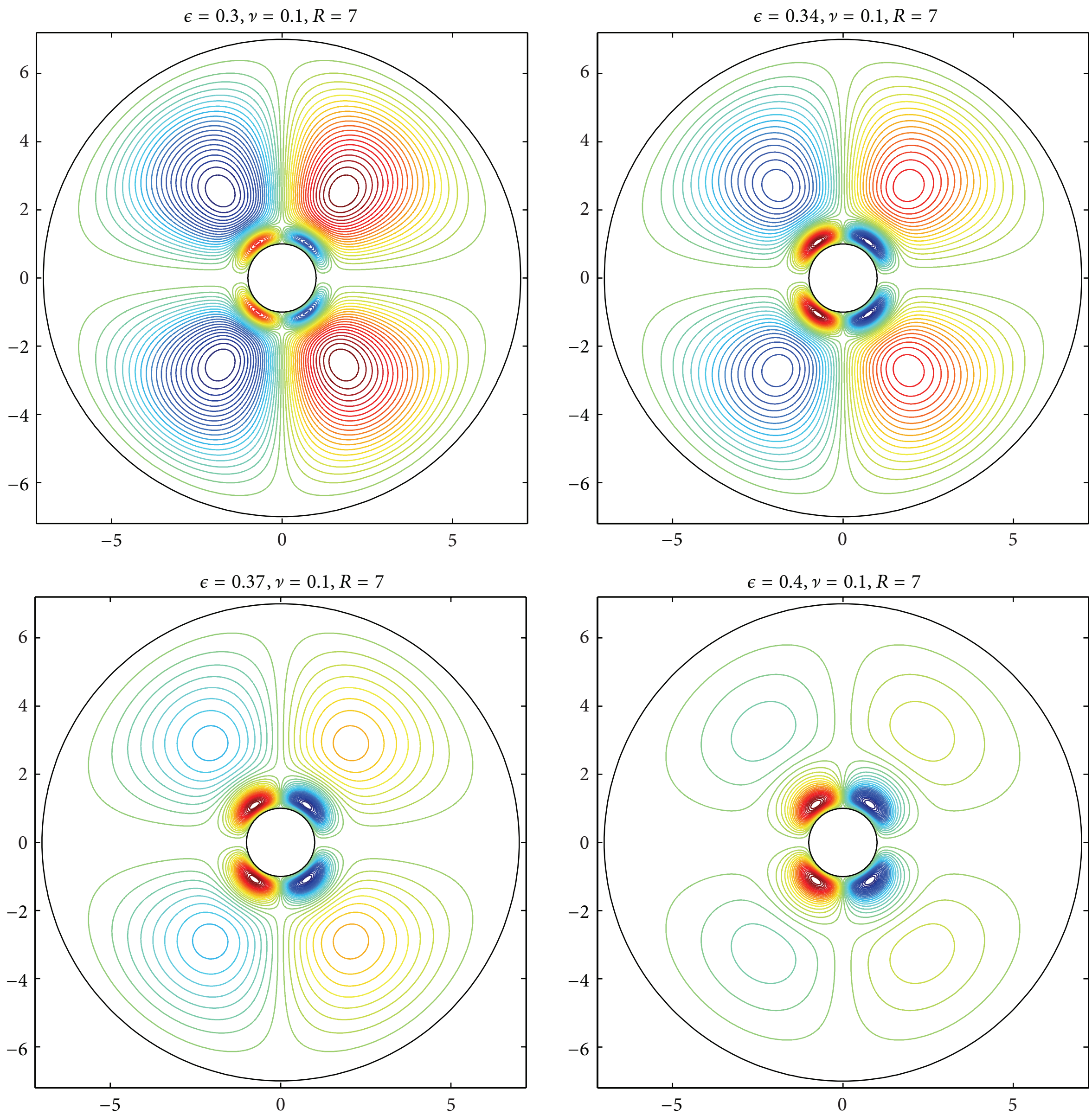

FIGURE 4: Steady streaming between two spheres.

Appendix

A. Explicit Expression for the Right Hand Side of (27)–(29), (34)–(36), (41)–(43), and (50)-(51)

Functions M_k, N_k, and (29) for $k = 2, 3, 4, 5$ in (27)-(28) are given by

$$M_2 = -u_0^i u^i_{0_r} - \frac{v_0^i u^i_{0_\theta}}{r} + \frac{\left(v_0^i\right)^2}{r},$$

$$N_2 = -u_0^i v^i_{0_r} - \frac{v_0^i v^i_{0_\theta}}{r} - \frac{u_0^i v_0^i}{r},$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_2^i\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(v_2^i \sin\theta\right) = 0,$$

$$M_3 = -u_0^i u^i_{1_r} - u_1^i u^i_{0_r} - \frac{v_0^i u^i_{1_\theta}}{r} - \frac{v_1^i u^i_{0_\theta}}{r} + 2\frac{v_0^i v_1^i}{r},$$

$$N_3 = -u_0^i v^i_{1_r} - u_1^i v^i_{0_r} - \frac{v_0^i v^i_{1_\theta}}{r} - \frac{v_1^i v^i_{0_\theta}}{r} - \frac{u_0^i v_1^i}{r} - \frac{u_1^i v_0^i}{r},$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_3^i\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(v_3^i \sin\theta\right) = 0,$$

$$M_4 = -u_0^i u^i_{2_r} - u_1^i u^i_{1_r} - u_2^i u^i_{0_r} - \frac{v_0^i u^i_{2_\theta}}{r} - \frac{v_1^i u^i_{1_\theta}}{r} - \frac{v_2^i u^i_{0_\theta}}{r} + \frac{v_0^i v_2^i}{r} + \frac{v_1^i v_1^i}{r} + \frac{v_2^i v_0^i}{r},$$

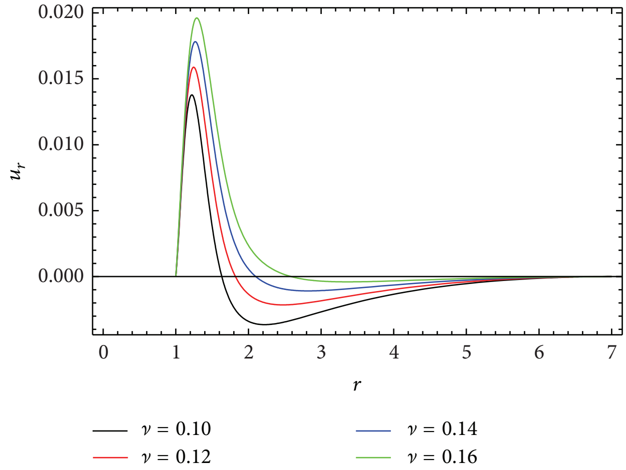

FIGURE 5: Radial component of the velocity for $\epsilon = 0.316$ and $R = 7$.

$$
\begin{aligned}
N_4 &= -u_0^i v_{2_r}^i - u_1^i v_{1_r}^i - u_2^i v_{0_r}^i - \frac{v_0^i v_{2_\theta}^i}{r} - \frac{v_1^i v_{1_\theta}^i}{r} \\
&\quad - \frac{v_2^i v_{0_\theta}^i}{r} - \frac{u_0^i v_1^i}{r} - \frac{u_1^i v_1^i}{r} - \frac{u_2^i v_1^i}{r}, \\
&\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_4^i\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(v_4^i \sin\theta\right) = 0, \\
M_5 &= -u_0^i u_{3_r}^i - u_1^i u_{2_r}^i - u_2^i u_{1_r}^i - u_3^i u_{0_r}^i - \frac{v_0^i u_{3_\theta}^i}{r} - \frac{v_1^i u_{2_\theta}^i}{r} \\
&\quad - \frac{v_2^i u_{1_\theta}^i}{r} - \frac{v_3^i u_{0_\theta}^i}{r} + \frac{v_0^i v_3^i}{r} + \frac{v_1^i v_2^i}{r} + \frac{v_2^i v_1^i}{r} + \frac{v_3^i v_0^i}{r}, \\
N_5 &= -u_0^i v_{3_r}^i - u_1^i v_{2_r}^i - u_2^i v_{1_r}^i - u_3^i v_{0_r}^i - \frac{v_0^i v_{3_\theta}^i}{r} - \frac{v_1^i v_{2_\theta}^i}{r} \\
&\quad - \frac{v_2^i v_{1_\theta}^i}{r} - \frac{v_3^i v_{0_\theta}^i}{r} - \frac{u_0^i v_3^i}{r} - \frac{u_1^i v_2^i}{r} - \frac{u_2^i v_1^i}{r} - \frac{u_3^i v_0^i}{r}, \\
&\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_5^i\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(v_5^i \sin\theta\right) = 0.
\end{aligned}
\tag{A.1}
$$

Functions F_k^a, G_k^a, and H_k^a for $k = 1, 2, 3$ in (34)–(36) are given by

$$
\begin{aligned}
F_1^a &= -u_0^i\Big|_{r=1} v_{0_\xi}^a + 2\nu v_{0_\xi}^a + \xi p_{0_\theta}^a, \\
G_1^a &= 0, \\
H_1^a &= \frac{\xi(\sin\theta v_0^a)_\theta}{\sin\theta} - 2u_0^a, \\
F_2^a &= -\xi^2 p_{0_\theta}^a + \xi p_{1_\theta}^a - u_1^i\Big|_{r=1} v_{0_\xi}^a - u_0^a v_{0_\xi}^a - u_0^i\Big|_{r=1} v_0^a \\
&\quad - v_0^i\Big|_{r=1} v_{0_\theta}^a - v_0^a v_{0_\theta}^i\Big|_{r=1} - v_0^a v_{0_\theta}^a - \xi u_{0_r}^i\Big|_{r=1} v_{0_\xi}^a \\
&\quad - u_0^i\Big|_{r=1} v_{1_\xi}^a + \nu\left(2v_{1_\xi}^a - 4\xi v_{0_\xi}^a + \frac{\left(\sin\theta v_{0_\theta}^a\right)_\theta}{\sin\theta} - \frac{2v_0^a}{\sin^2\theta}\right), \\
G_2^a &= -u_{0_\tau}^a + \nu\, u_{0_{\xi\xi}}^a, \\
H_2^a &= -2\left(u_1^a - \xi u_0^a\right) + \frac{\xi(\sin\theta v_1^a)_\theta}{\sin\theta} - \frac{\xi^2(\sin\theta v_0^a)_\theta}{\sin\theta}, \\
F_3^a &= -u_0^i\Big|_{r=1} v_{2_\xi}^a - \left(u_1^i + \xi u_{0_r}^i\right)\Big|_{r=1} v_{1_\xi}^a - u_0^a v_{0_r}^i\Big|_{r=1} \\
&\quad - \left[u_2^i + \xi u_{1_r}^i + \frac{\xi^2 u_{0_{rr}}^i}{2}\right]\Bigg|_{r=1} v_{0_\xi}^a - u_0^a v_{1_\xi}^a - u_1^a v_{0_\xi}^a \\
&\quad - \partial_\theta\left[v_0^i\Big|_{r=1} v_1^a + \left(v_1^i + \xi v_{0_r}^i\right)\Big|_{r=1} v_0^a + v_0^a v_1^a - \frac{\xi\left(v_0^a\right)^2}{2}\right. \\
&\quad \left. -\xi v_0^i\Big|_{r=1} v_0^a\right] - u_0^i\Big|_{r=1} v_1^a \\
&\quad - \left(u_1^i + \xi u_{0_r}^i\right)\Big|_{r=1} v_0^a - u_0^a v_0^i\Big|_{r=1} - u_0^a v_0^a + \xi u_0^i\Big|_{r=1} v_0^a \\
&\quad + s^3 p_{0_\theta}^a - \xi^2 p_{1_\theta}^a + \xi p_{2_\theta}^a \\
&\quad + \nu\left(2v_{2_\xi}^a - 4s v_{1_\xi}^a + 6\xi^2 v_{0\xi}^a + \frac{\left(\sin\theta v_{1_\theta}^a\right)_\theta}{\sin\theta}\right. \\
&\quad \left. - \frac{2\xi\left(\sin\theta v_{0_\theta}^a\right)_\theta}{\sin\theta} - \frac{2v_1^a}{\sin^2\theta} + \frac{4\xi v_0^a}{\sin^2\theta} + 2u_{0_\theta}^a\right), \\
G_3^a &= -u_{1_\tau}^a + \nu u_{1_{\xi\xi}}^a - u_0^i\Big|_{r=1} u_{0_\xi}^a - v_0^a u_{0_\theta}^i\Big|_{r=1} \\
&\quad + \left(v_0^a\right)^2 + 2v_0^a v_0^i\Big|_{r=1} + \nu\left[u_{0_\xi}^a - 2\,\frac{(\sin\theta v_0^a)_\theta}{\sin\theta}\right].
\end{aligned}
\tag{A.2}
$$

Functions F_k^b, G_k^b, and H_k^b for $k = 1, 2, 3$ in (41)–(43) are given by

$$
\begin{aligned}
F_1^b &= -\frac{2\nu}{R} v_{0\eta}^b - \frac{\eta}{R^2} p_{0\theta}^b, \\
G_1^b &= 0, \\
H_1^b &= -\frac{\eta}{R^2}\frac{\left(\sin\theta v_0^b\right)_\theta}{\sin\theta} - \frac{2}{R} u_0^b, \\
F_2^b &= u_0^i\Big|_{r=R} v_{1\eta}^b + \left(u_1^i\Big|_{r=R} - \eta u_{0r}^i\Big|_{r=R}\right) v_{0\eta}^b + u_0^b v_{0\eta}^b \\
&\quad - \frac{1}{R}\partial_\theta\left(\frac{\left(v_0^b\right)^2}{2} + v_0^i\Big|_{r=R} v_0^b\right) - \frac{1}{R} u_0^i|_{r=R} v_0^b \\
&\quad + \nu\left(-\frac{1}{R} v_{1\eta}^b - \frac{\eta}{R^2} v_{0\eta}^b + \frac{1}{R^2}\left(\frac{\left(\sin\theta v_{0_\theta}^b\right)_\theta}{\sin\theta} - \frac{2v_0^b}{\sin^2\theta}\right)\right) \\
&\quad - \frac{\eta}{R^2} p_{1\theta}^b - \frac{\eta^2}{R^3} p_{0\theta}^b, \\
G_2^b &= u_{0\tau}^b - \nu u_{0\eta\eta}^b,
\end{aligned}
$$

$$H_2^b = -\frac{\eta}{R^2}\frac{\left(\sin\theta v_1^b\right)_\theta}{\sin\theta} - \frac{\eta^2}{R^3}\frac{\left(\sin\theta v_0^b\right)_\theta}{\sin\theta} - \frac{1}{R}u_1^b - \frac{\eta}{R^2}u_0^b,$$

$$\begin{aligned}F_3^b = {} & u_0^i\big|_{r=R} v_{2\eta}^b + \left(u_1^i\big|_{r=R} - \eta u_{0r}^i\big|_{r=R}\right) v_{1\eta}^b \\ & + \left(u_2^i\big|_{r=R} - \eta u_{1r}^i\big|_{r=R} + \frac{\eta^2}{2} u_{0rr}^i\big|_{r=R}\right) v_{0\eta}^b \\ & + u_0^b v_{1\eta}^b + u_1^b v_{0\eta}^b \\ & - \frac{1}{R}\partial_\theta \Bigg[v_0^i\big|_{r=R} v_1^b + \left(v_1^i\big|_{r=R} - \eta v_{0r}^i\big|_{r=R}\right) v_0^b + v_0^b v_1^b \\ & \qquad + \frac{\eta}{R^2}\left(\frac{\left(v_0^b\right)^2}{2} + v_0^i\big|_{r=R} v_0^b\right)\Bigg] - v_{0r}^i\big|_{r=R} u_0^b \\ & - \frac{1}{R}\left[u_0^i\big|_{r=R} v_1^a + \left(u_1^i\big|_{r=R} - \eta u_{0r}^i\big|_{r=R}\right) v_0^b \right. \\ & \qquad \left. + v_0^i\big|_{r=R} u_0^b + u_0^a v_0^a\right] - \frac{\eta}{R^2} u_0^i\big|_{r=R} v_0^b \\ & + \nu\Bigg(-\frac{1}{R}v_{2\eta}^b - \frac{\eta}{R^2}v_{1\eta}^b - \frac{\eta^2}{R^3}v_{0\eta}^b \\ & \qquad + \frac{1}{R^2}\left(\frac{\left(\sin\theta v_{1_\theta}^b\right)_\theta}{\sin\theta} - \frac{2v_1^b}{\sin^2\theta}\right) \\ & \qquad + 2\frac{\eta}{R^3}\left(\frac{\left(\sin\theta v_{0_\theta}^b\right)_\theta}{\sin\theta} - \frac{v_0^b}{\sin^2\theta}\right)\Bigg) \\ & + \nu\frac{2}{R^2}u_{0\theta}^b - \frac{\eta}{R^2}p_{2\theta}^b - \frac{\eta^2}{R^3}p_{1\theta}^b - \frac{\eta^3}{R^4}p_{0\theta}^b,\end{aligned}$$

$$\begin{aligned}G_3^b = {} & u_{1\tau}^b - \nu u_{1\eta\eta}^b - u_0^i\big|_{r=R} u_{0\eta}^b + u_{0\theta}^i\big|_{r=R} v_0^b \\ & - \frac{1}{R}\left(\left(v_0^b\right)^2 + 2v_0^i\big|_{r=R} v_0^b\right) \\ & + \nu\left(\frac{1}{R}u_{0\eta}^b + \frac{2}{R^2}\frac{\left(\sin\theta v_0^b\right)_\theta}{\sin\theta}\right).\end{aligned} \tag{A.3}$$

Function Q_k^a and S_k^a in (50) and (51) are given by

$$Q_1^a = 0,$$

$$S_1^a = -\cos\theta f(\tau)\, v_{0\xi}^a\big|_{\xi=0},$$

$$Q_2^a = -\cos\theta f\left(u_{0r}^i\big|_{r=1} + u_{0\xi}^a\big|_{\xi=0}\right) + \sin\theta f u_{0\theta}^i\big|_{r=1},$$

$$\begin{aligned}\xi_2^a = {} & -\cos\theta f\left(v_{0r}^i\big|_{r=1} + v_{1\xi}^a\big|_{\xi=0}\right) \\ & + \sin\theta f\left(v_{0\theta}^i\big|_{r=1} + v_{0\theta}^a\big|_{\xi=0}\right) \\ & + \sin\theta f u_0^i\big|_{r=1} - \cos^2\theta\frac{f^2}{2} v_{0\xi\xi}^a\big|_{\xi=0}.\end{aligned}$$

$$\begin{aligned}Q_3^a = {} & -\cos\theta f\left(u_{1r}^i\big|_{r=1} + u_{1\xi}^a\big|_{\xi=0}\right) \\ & + \sin\theta f\left(u_{1\theta}^i\big|_{r=1} + u_{0\theta}^a\big|_{\xi=0}\right) \\ & - \sin\theta f\left(v_1^i\big|_{r=1} + v_1^a\big|_{\xi=0}\right) - \cos^2\theta\frac{f^2}{2} u_{0\xi\xi}^a\big|_{\xi=0} \\ & - \sin\theta\cos\theta\, f^2 v_{0\xi}^a\big|_{\xi=0},\end{aligned}$$

$$\begin{aligned}S_3^a = {} & -\cos\theta f\left(v_{1r}^i\big|_{r=1} + v_{2\xi}^a\big|_{\xi=0}\right) \\ & + \sin\theta f\left(v_{1\theta}^i\big|_{r=1} + v_{1\theta}^a\big|_{\xi=0}\right) \\ & + \sin\theta f\left(u_1^i\big|_{r=1} + u_0^a\big|_{\xi=0}\right) - \cos^2\theta\frac{f^2}{2} v_{1\xi\xi}^a\big|_{\xi=0} \\ & + \sin\theta\cos\theta f^2 v_{0\xi\theta}^a\big|_{\xi=0} - \sin^2\theta\frac{f^2}{2} v_{0\xi}^a\big|_{\xi=0} \\ & - \cos^3\theta\frac{f^3}{6} v_{0\xi\xi\xi}^a\big|_{\xi=0}.\end{aligned} \tag{A.4}$$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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