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*Research Article*

# The Global Existence of Solutions in Time for a Chemotaxis Model with Two Chemicals

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This paper concerns the uniform boundedness and global existence of solutions in time for the chemotaxis model with two chemicals. We prove the system has global existence of solutions in time for any dimension $n$.

## 1. Introduction and Statement of Main Result

Chemotaxis is the influence of chemical substances in the environment on the movement of mobile species. Keller and Segel [1] proposed the general chemotaxis system

$$\begin{aligned} u_t &= \nabla \cdot \left(D_1(u,v)\nabla u - D_2(u,v)\nabla v\right), \\ v_t &= D_v \Delta v - k(v)v + f(u,v), \end{aligned} \tag{1}$$

where $u$ is the density function of cells (e.g., *Dictyostelium discoideum*) that are attracted by a chemical substance (e.g., cAMP) produced by themselves and the movement towards a higher concentration of the chemical substance, whose concentration function is $v$. $D_1$ and $D_v$ are the random diffusion rates of cells and the chemical, respectively; $D_2(u,v)\nabla v$ represents the chemotactic flux of cells and $D_2(u,v)$ is positive for positive $u$ and $v$ and is called the sensitivity function; and $f(u,v)$ is the creation rate of the chemical, while $k(v)v$ is the degradation rate of the chemical.

The simplest case of (1) is that $D_1$, $D_v$, and $k$ are all positive constants, $f(u,v) = v$, and $D_2(u,v) = \chi u$ with $\chi$ being a positive constant. This was called by Childress and Perkus the "minimal model." When dimension $n = 1$, solutions exist globally; see [2]. For $n = 2$, global existence depends on a threshold: when the initial mass lies below the threshold solutions exist globally, while above the threshold solutions blow up in finite time; these results were derived by various authors; see the review article [3, 4]. Many authors have analyzed system (1) for several variants, such as global existence, blow up solutions, and many other results; see [3–5].

Painter et al. [6] proposed a chemotaxis model with two chemicals. They considered a Turing system [7] as a mechanism for providing spatially heterogeneous chemical distributions to which a cell population chemotactically responds. That was the following model:

$$\begin{aligned} &\frac{\partial w}{\partial t} = \nabla\left[D_w \nabla w - w\chi_1(u,v)\nabla u - w\chi_2(u,v)\nabla v\right], \\ &\qquad\qquad x \in \Omega,\ t > 0, \\ &\frac{\partial u}{\partial t} = \nabla\left(D_u \cdot \nabla u\right) + f(u,v), \quad x \in \Omega,\ t > 0, \\ &\frac{\partial v}{\partial t} = \nabla\left(D_v \cdot \nabla v\right) + g(u,v), \quad x \in \Omega,\ t > 0, \\ &\frac{\partial w}{\partial \nu} = \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial\Omega, \\ &w(x,0) = w_0(x), \qquad u(x,0) = u_0(x), \\ &v(x,0) = v_0(x), \end{aligned} \tag{2}$$

where $w$ is a cell population, $u$ and $v$ are chemicals, and the cell population responds chemotactically to both chemical species. $\chi_1$ and $\chi_2$ are the chemotactic sensitivity functions and $f$ and $g$ define the chemical kinetics. $D_w$, $D_u$, and $D_v$ are taken as constants.

A special form of system (2) is as follows:

$$\begin{aligned}
&\frac{\partial w}{\partial t} = D\nabla\left(\nabla w - \frac{w}{(k_1+u)^2}\nabla u - \frac{w}{(k_2+v)^2}\nabla v\right), \\
&\qquad\qquad x \in \Omega,\ t > 0, \\
&\frac{\partial u}{\partial t} = \Delta u + \delta - Ku - uv^2, \quad x \in \Omega,\ t > 0, \\
&\frac{\partial v}{\partial t} = \Delta v + Ku + uv^2 - v, \quad x \in \Omega,\ t > 0, \\
&\frac{\partial w}{\partial \nu} = \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t > 0, \\
&w(x,0) = w_0(x) \ge 0, \qquad u(x,0) = u_0(x) \ge 0, \\
&v(x,0) = v_0(x) \ge 0,
\end{aligned} \tag{3}$$

where $\Omega \in \mathbb{R}^n$ is a bounded domain. The kinetics of (3) are described by a simplified model for the glycolysis reaction [8, 9]. As we know, the global existence of solutions in time of two species such as (1) is investigated by many authors; however, the global existence of solutions in time of three species is studied little. In this paper, we study the global existence of solutions in time of (3); by applying analysis semigroup and energy method we will prove that system (3) has global solutions in time in any dimension $n$.

We state the main result of this paper as follows.

**Theorem 1.** *For any $w_0, u_0, v_0 \in W^{1,p}(\Omega)$, $p > n$ satisfying $w_0 \ge 0$, $u_0 \ge 0$, $v_0 \ge 0$ on $\Omega$, (3) has a unique positive global solution $(w, u, v)$ such that*

$$\begin{aligned}
(w,u,v) \in C\big(&[0,+\infty), \\
&W^{1,p}(\Omega) \times W^{1,p}(\Omega) \times W^{1,p}(\Omega)\big) \\
&\cap C^{2+2\epsilon,1+\epsilon}_{\mathrm{loc}}(\Omega \times (0,+\infty)).
\end{aligned} \tag{4}$$

## 2. The Proof of Theorem 1

**Theorem 2.** *For any $w_0, u_0, v_0 \in W^{1,p}(\Omega)$, $p > n$ satisfying $w_0 \ge 0$, $u_0 \ge 0$, $v_0 \ge 0$ on $\Omega$, one has the following conclusions.*

(i) *(3) has a unique solution $(w(x,t),\ u(x,t),\ v(x,t))$ on $\Omega \times [0, T_{(u_0,v_0)})$ with $0 < T_{(u_0,v_0)} \le \infty$ satisfying*

$$\begin{aligned}
(w(\cdot,t), u(\cdot,t), v(\cdot,t)) \in C\Big(&\big[0, T_{(w_0,u_0,v_0)}\big), \\
&W^{1,p}(\Omega) \times W^{1,p}(\Omega) \\
&\times W^{1,p}(\Omega)\Big) \\
&\cap C^{2+2\epsilon,1+\epsilon}_{\mathrm{loc}}\Big(\Omega \times \big(0, T_{(w_0,u_0,v_0)}\big)\Big), \\
&\textit{for any } 0 < \epsilon < \frac{1}{4}.
\end{aligned} \tag{5}$$

(ii) *Moreover, if, for small $\delta > 0$, $t \in [\delta, T_{(w_0,u_0,v_0)})$, $\|(w,u,v)(\cdot,t)\|_{L^\infty(\Omega)}$ is bounded; then $T_{(w_0,u_0,v_0)} = \infty$; that is, $(w,u,v)$ has global existence, and for any $0 \le \rho \le \sigma \le 1$, $(w,u,v) \in C^\rho([\delta,\infty), C^{2(1-\sigma)}(\Omega) \times C^{2(1-\sigma)}(\Omega) \times C^{2(1-\sigma)}(\Omega))$.*

*Proof.* Equation (3) can be written as

$$\begin{aligned}
&\begin{pmatrix} w \\ u \\ v \end{pmatrix}_t = \nabla\left[A(u,v)\nabla\begin{pmatrix} w \\ u \\ v \end{pmatrix}\right] + \begin{pmatrix} 0 \\ \delta - Ku - uv^2 \\ Ku + uv^2 - v \end{pmatrix}, \\
&\qquad\qquad x \in \Omega,\ t > 0, \\
&A(u,v)\nabla\begin{pmatrix} w \\ u \\ v \end{pmatrix} = 0, \quad x \in \partial\Omega,\ t > 0,
\end{aligned} \tag{6}$$

where

$$A(u,v) = \begin{pmatrix} D & -\dfrac{w}{(k_1+u)^2} & -\dfrac{w}{(k_2+v)^2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{7}$$

Since the eigenvalues of $A$ are positive, (6) is normally parabolic; then (i) follows from [10]. Note that (6) is also a "triangular system," so in virtue of [11], we complete the proof of Theorem 2. □

In the following, we always assume $w_0$, $u_0$, $v_0 \in W^{1,p}(\Omega)$, $p > n$ satisfying $w_0 \ge 0$, $u_0 \ge 0$, $v_0 \ge 0$ on $\Omega$.

**Lemma 3.** *For any dimension $n$, any solution $u$ of (3) has the following estimate:*

$$\|u\|_{L^2(\Omega)} \le \mu_1, \quad 0 \le t < T, \tag{8}$$

*where $T = T_{(w_0,u_0,v_0)}$ and $\mu_1$ depends only on $\|u_0\|_{L^1(\Omega)}$, $\|u_0\|_{L^2(\Omega)}$, $\delta$, $k$ and $|\Omega|$.*

*Proof.* Integrating the second equation of (3) over $\Omega$, we have

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u\,dx &= \delta|\Omega| - K\int_\Omega u\,dx - \int_\Omega uv^2 dx \\
&\le -K\int_\Omega u\,dx + \delta|\Omega|.
\end{aligned} \tag{9}$$

Integrating (9) with respect to $t$, we get

$$\begin{aligned}
&\int_\Omega u\,dx \le e^{-Kt}\|u_0\|_{L^1(\Omega)} + \frac{\delta|\Omega|}{K}, \\
&\text{that is } \|u\|_{L^1(\Omega)} \le C, \quad C \sim \|u_0\|_{L^1(\Omega)}, \quad \delta, K, |\Omega|.
\end{aligned} \tag{10}$$

Multiplying the second equation of (3) by $u$ and integrating with respect to $x$ over $\Omega$, we get

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u^2 dx &= -2\int_\Omega |\nabla u|^2 dx + 2\delta\int_\Omega u\,dx \\
&\quad - 2K\int_\Omega u^2 dx - 2\int_\Omega u^2v^2 dx \\
&\le -2K\int_\Omega u^2 dx + 2\delta\|u\|_{L^1(\Omega)}.
\end{aligned} \tag{11}$$

Integrating (11) with respect to $t$ and together with (10), we obtain

$$\|u\|_{L^2(\Omega)} \le \mu_1, \quad \mu_1 \sim \|u_0\|_{L^1(\Omega)}, \quad \|u_0\|_{L^2(\Omega)}, \quad \delta, K, |\Omega| . \tag{12}$$

□

**Lemma 4.** *For any dimension $n$ and small constants $\tau_0 > 0$, any solution $u$ of (3) has the following estimate:*

$$\|u\|_{L^\infty(\Omega)} \le \mu_2, \qquad \tau_0 \le t < T, \tag{13}$$

*where $T = T_{(w_0,u_0,v_0)}$ and $\mu_2$ is a constant depending on $\|u_0\|_{L^1(\Omega)}$, $\|u_0\|_{L^2(\Omega)}$, $\delta$, $k$, $|\Omega|$ and $\|u(\cdot,\tau_0)\|_{L^\infty(\Omega)}$.*

*Proof.* Multiplying the second equation of (3) by $u^{p-1} (p \ge 2)$ and integrating with respect to $x$ over $\Omega$, we get

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_\Omega u^p dx = &-\frac{4(p-1)}{p^2}\int_\Omega \left|\nabla(u^{p/2})\right|^2 dx \\ &+ \delta\int_\Omega u^{p-1}dx - K\int_\Omega u^p dx - \int_\Omega u^p v^2 dx.\end{aligned} \tag{14}$$

Here (14) implies

$$\begin{aligned}\frac{d}{dt}\int_\Omega u^p dx + \frac{4(p-1)}{p}\int_\Omega \left|\nabla(u^{p/2})\right|^2 dx \\ \le \delta p\int_\Omega u^{p-1}dx - Kp\int_\Omega u^p dx.\end{aligned} \tag{15}$$

Define $z = u^{p/2}$, by Hölder's inequality; (15) can be written as

$$\begin{aligned}\frac{d}{dt}\int_\Omega z^2 dx + \frac{4(p-1)}{p}\int_\Omega |\nabla z|^2 dx \\ \le \delta p\int_\Omega z^{2(p-1)/p}dx - Kp\int_\Omega z^2 dx \\ \le \delta p|\Omega|^{1/p}\|z\|_{L^2}^{2(p-1)/p}.\end{aligned} \tag{16}$$

Gagliardo-Nirenberg inequality implies

$$\begin{aligned}\|z\|_{L^2(\Omega)} \le C_1\|\nabla z\|_{L^2(\Omega)}^\theta \|z\|_{L^1(\Omega)}^{1-\theta} \\ + C_2\|z\|_{L^1(\Omega)}, \quad \text{with } \theta = \frac{n}{n+2}.\end{aligned} \tag{17}$$

In virtue of (17) and Young's inequality, we have

$$\begin{aligned}\|z\|_{L^2}^{2(p-1)/p} \le &\left(2C_1^2\|\nabla z\|_{L^2(\Omega)}^{2\theta}\|z\|_{L^1(\Omega)}^{2(1-\theta)} + 2C_2^2\|z\|_{L^1(\Omega)}^2\right)^{(p-1)/p} \\ \le &\, 2^{(p-1)/p}C_1^{2(p-1)/p}\|\nabla z\|_{L^2(\Omega)}^{2(p-1)\theta/p} \times \|z\|_{L^1(\Omega)}^{2(p-1)(1-\theta)/p} \\ &+ 2^{(p-1)/p}C_2^{(2(p-1)/p)}\|z\|_{L^1(\Omega)}^{2(p-1)/p} \\ \le &\,\frac{(p-1)\theta}{p}\varepsilon^{p/(p-1)\theta}\|\nabla z\|_{L^2(\Omega)}^2 \\ &+ \frac{p-(p-1)\theta}{p}\varepsilon^{-p/(p-(p-1)\theta)}2^{(p-1)/(p-(p-1)\theta)} \\ &C_1^{2(p-1)/(p-(p-1)\theta)}\|z\|_{L^1(\Omega)}^{2(p-1)(1-\theta)/(p-(p-1)\theta)} \\ &+ 2^{(p-1)/p}C_2^{2(p-1)/p}\|z\|_{L^1(\Omega)}^{2(p-1)/p} \\ \le &\,\frac{(p-1)\theta}{p}\varepsilon^{p/(p-1)\theta}\|\nabla z\|_{L^2(\Omega)}^2 \\ &+ C_3\varepsilon^{-p/(p-(p-1)\theta)}\|z\|_{L^1(\Omega)}^{2(p-1)(1-\theta)/(p-(p-1)\theta)} \\ &+ C_4\|z\|_{L^1(\Omega)}^{2(p-1)/p},\end{aligned} \tag{18}$$

with $C_3 = 2^{(n+2)/2}C_1^{n+2}$ and $C_4 = 2C_2^2$.

Taking suitable $\varepsilon$ such that $\delta p|\Omega|^{1/p}((p-1)\theta/p)\varepsilon^{p/(p-1)\theta} = 3(p-1)/p$, we obtain

$$\begin{aligned}\varepsilon^{-p/(p-(p-1)\theta)} &= \left[\left(\frac{3}{\delta p|\Omega|^{1/p}\theta}\right)^{(p-1)\theta/p}\right]^{-p/(p-(p-1)\theta)} \\ &< C_5 p^{n/2},\end{aligned} \tag{19}$$

with $C_5 = (\delta|\Omega|^{1/p}\theta/3)^{n/2}$.

In view of (16)–(19), we get

$$\begin{aligned}\frac{d}{dt}\int_\Omega z^2 dx + \frac{(p-1)}{p}\int_\Omega |\nabla z|^2 dx \\ \le C_3C_5p^{(n+2)/2}\delta|\Omega|^{1/p}\|z\|_{L^1(\Omega)}^{2(p-1)(1-\theta)/(p-(p-1)\theta)} \\ + C_4\delta p|\Omega|^{(1/p)}\|z\|_{L^1(\Omega)}^{2(p-1)/p} \\ \le C_6\delta p^{(n+2)/2}\|z\|_{L^1(\Omega)}^{2(p-1)(1-\theta)/(p-(p-1)\theta)} \\ + C_7\delta p\|z\|_{L^1(\Omega)}^{2(p-1)/p},\end{aligned} \tag{20}$$

with $C_6 = C_3C_5|\Omega|^{1/p}$ and $C_7 = C_4|\Omega|^{1/p}$.

By Poincaré inequality, there exists a constant $\sigma > 0$ depending on $n$, $p$, $\Omega$ such that

$$\|z\|_{L^2(\Omega)}^2 \le \frac{1}{\sigma}\left(\|\nabla z\|_{L^2(\Omega)}^2 + \|z\|_{L^1(\Omega)}^2\right); \tag{21}$$

we have

$$\frac{p-1}{p}\int_\Omega |\nabla z|^2 dx \ge \frac{p-1}{p}\sigma\int_\Omega z^2 dx - \frac{p-1}{p}\|z\|_{L^1(\Omega)}^2. \tag{22}$$

It follows from (20) and (22) that

$$\begin{aligned}\frac{d}{dt}\int_{\Omega} z^2 dx \leq & -\frac{p-1}{p}\sigma\int_{\Omega} z^2 dx \\ & + C_6\delta p^{(n+2)/2}\|z\|_{L^1(\Omega)}^{2(p-1)(1-\theta)/(p-(p-1)\theta)} \\ & + C_7\delta p\|z\|_{L^1(\Omega)}^{(2(p-1)/p)} + \frac{p-1}{p}\|z\|_{L^1(\Omega)}^2.\end{aligned} \tag{23}$$

After simple calculation, we obtain

$$\begin{aligned}\|z\|_{L^2(\Omega)}^2 \leq \max\Big\{ & C_8 p^{(n+2)/2}\sup_{t\geq\tau_0}\|z\|_{L^1(\Omega)}^{2(p-1)(1-\theta)/(p-(p-1)\theta)} \\ & + C_9 p \sup_{t\geq\tau_0}\|z\|_{L^1(\Omega)}^{2(p-1)/p} \\ & + C_{10}\sup_{t\geq\tau_0}\|z\|_{L^1(\Omega)}^2, \|z(\tau_0)\|_{L^2(\Omega)}^2\Big\},\end{aligned} \tag{24}$$

with $C_8 = C_6\delta/\sigma$, $C_9 = C_7\delta/\sigma$, and $C_{10} = 1/\sigma$.

Since $2(p-1)(1-\theta)/(p-(p-1)\theta) < 2$ and $2(p-1)/p < 2$, we have

$$\begin{aligned}\sup_{t\geq\tau_0}\|z\|_{L^2(\Omega)}^2 \leq \max\Big\{ & C_{11}p^{(n+2)/2} \\ & \times \max\Big\{\sup_{t\geq\tau_0}\|z\|_{L^1(\Omega)}^2, 1\Big\}, \\ & \|z(\tau_0)\|_{L^2(\Omega)}^2\Big\},\end{aligned} \tag{25}$$

with $C_{11} = \max\{C_8, C_9, C_{10}\}$.

Substituting $z = u^{p/2}$, $p = 2^k$, $k = 1, 2, 3, \ldots$ into (25) yields

$$\begin{aligned}&\sup_{t\geq\tau_0}\int_{\Omega} u^{2^k} dx \\ &\quad \leq \max\Big\{C_{11}2^{((n+2)/2)k}\max\Big\{\Big(\sup_{t\geq\tau_0}\int_{\Omega} u^{2^{k-1}}dx\Big)^2, 1\Big\}, \\ &\qquad \|u(\tau_0)\|_{L^\infty(\Omega)}^{2^k}\Big\}.\end{aligned} \tag{26}$$

Without loss of generality, we can assume

$$\sup_{t\geq\tau_0}\int_{\Omega} u^{2^k} dx \leq C_{11}2^{((n+2)/2)k}\Big(\sup_{t\geq\tau_0}\int_{\Omega} u^{2^{k-1}}dx\Big)^2 \tag{27}$$

$$\begin{aligned}\sup_{t\geq\tau_0}\int_{\Omega} u^{2^k} dx \leq & \Big[C_{11}2^{((n+2)/2)k}\Big]^{2^0}\Big[C_{11}2^{((n+2)/2)(k-1)}\Big]^{2^1}\cdots \\ & \Big[C_{11}2^{((n+2)/2)(k-(k-3))}\Big]^{2^{k-3}}, \\ & \Big[C_{11}2^{((n+2)/2)(k-(k-2))}\Big]^{2^{k-2}}\Big(\sup_{t\geq\tau_0}\int_{\Omega} u^{2^{k-(k-1)}}dx\Big)^{2^{k-1}} \\ \leq & C_{11}^{2^{(k-1)}-1} \\ & \times 2^{((n+2)/2)(k+(k-1)2^1+\cdots+4\cdot 2^{k-4}+3\cdot 2^{k-3}+2\cdot 2^{k-2}+2^{k-1})} \\ & \times \Big(\sup_{t\geq\tau_0}\int_{\Omega} u^2 dx\Big)^{2^{k-1}}.\end{aligned} \tag{28}$$

Taking $1/2^k$ and letting $k \to \infty$ of (28), together with Lemma 3, we have

$$\sup_{t\geq\tau_0}\|u\|_{L^\infty(\Omega)} \leq C_{11}^{1/2}2^{(n+2)/2}\Big(\sup_{t\geq\tau_0}\int_{\Omega} u^2 dx\Big)^{1/2} \leq \mu_2. \tag{29}$$

□

With the notation

$$\overline{w} = u + v, \tag{30}$$

we have the following Lemma 5.

**Lemma 5.** *For any dimension $n$, $\overline{w}$ has the following estimate:*

$$\|\overline{w}\|_{C^1(\Omega)} \leq \mu_3, \qquad \tau_0 \leq t < T, \tag{31}$$

*where $T = T_{(w_0,u_0,v_0)}$ and $\mu_3$ is a constant depending on $\|u_0\|_{L^1(\Omega)}$, $\|u_0\|_{L^2(\Omega)}$, $\delta$, $k$, $|\Omega|$, $\|u(\cdot,\tau_0)\|_{L^\infty(\Omega)}$, and $\|\overline{w}(\cdot,\tau_0)\|_{W^{2,p}(\Omega)}$.*

*Proof.* In view of (3), $\overline{w}$ satisfies the following equation:

$$\frac{d\overline{w}}{dt} = \Delta\overline{w} + \delta + u - \overline{w}; \tag{32}$$

let $X = L^p(\Omega)$, $A = \Delta - I$ with domain $D = W^{2,p}(\Omega)$; then $A$ generates a linear analysis semigroup on $X$ satisfying $\|e^{At}\| \leq Ce^{-at}$ and $\|e^{At}\|_{X^\alpha} \leq C_\alpha t^{-\alpha}e^{-at}$ for $0 < a < 1$. Taking $p > n$, the fractional space $X^\alpha \hookrightarrow C^\gamma$ with $0 \leq \gamma < 2\alpha - (n/p)$; taking $\alpha = (1/2) + (n/2p)$ we get $\gamma = 1$.

In virtue of (32), we obtain

$$\overline{w}(\cdot,t) = e^{A(t-\tau_0)}\overline{w}(\cdot,\tau_0) + \int_{\tau_0}^{t} e^{A(t-s)}(\delta + u(\cdot,s))\,ds,$$

$$\begin{aligned}\|\overline{w}(\cdot,t)\|_{C^1(\Omega)} & \leq C_1\|\overline{w}(\cdot,t)\|_{X^\alpha} \\ & \leq C_1\Big\|e^{A(t-\tau_0)}\overline{w}(\cdot,\tau_0)\Big\|_{X^\alpha} \\ & \quad + C_1\int_{\tau_0}^{t}\Big\|e^{A(t-s)}(\delta + u(\cdot,s))\Big\|_{X^\alpha}ds\end{aligned}$$

$$\begin{aligned}
&\leq C_2 e^{-a(t-\tau_0)} \|\overline{w}(\cdot,\tau_0)\|_{X^\alpha} \\
&\quad + \int_{\tau_0}^{t} C_\alpha (t-s)^{-\alpha} e^{-a(t-s)} \|\delta + u(\cdot,s)\|_{L^p(\Omega)} ds \\
&\leq C_3 + C_4 \sup_{\tau_0 \leq s \leq t} \|u(\cdot,t)\|_{L^p(\Omega)} \\
&\leq \mu_3.
\end{aligned} \tag{33}$$

□

By Lemma 4 and (30), one has

$$\|v\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\Omega)} + \|\overline{w}\|_{L^\infty(\Omega)} \leq \mu_4, \tag{34}$$

where $\mu_4$ is a constant depending on $\|u_0\|_{L^1(\Omega)}$, $\|u_0\|_{L^2(\Omega)}$, $\delta$, $k$, $|\Omega|$, $\|u(\cdot,\tau_0)\|_{L^\infty(\Omega)}$, and $\|\overline{w}(\cdot,\tau_0)\|_{W^{2,p}}(\Omega)$.

Similar as the proof of Lemma 5, we can prove

$$\|v\|_{C^1(\Omega)} \leq \mu_5, \tag{35}$$

where $\mu_5$ is a constant depending on $\|u_0\|_{L^1(\Omega)}$, $\|u_0\|_{L^2(\Omega)}$, $\delta$, $k$, $|\Omega|$, $\|u(\cdot,\tau_0)\|_{L^\infty(\Omega)}$, $\|\overline{w}(\cdot,\tau_0)\|_{W^{2,p}(\Omega)}$, and $\|v(\cdot,\tau_0)\|_{W^{2,p}(\Omega)}$.

Lemma 5 and (35) yield

$$\|u\|_{C^1(\Omega)} \leq \mu_6, \tag{36}$$

where $\mu_6$ is a constant depending on $\|u_0\|_{L^1(\Omega)}$, $\|u_0\|_{L^2(\Omega)}$, $\delta, k, |\Omega|$, $\|u(\cdot,\tau_0)\|_{L^\infty(\Omega)}$, $\|\overline{w}(\cdot,\tau_0)\|_{W^{2,p}(\Omega)}$, and $\|v(\cdot,\tau_0)\|_{W^{2,p}(\Omega)}$.

**Lemma 6.** *For any dimension $n$, any solution $\omega$ of* (3) *has the following estimate:*

$$\|w\|_{L^\infty(\Omega)} \leq \mu_7, \qquad \tau_0 \leq t < T, \tag{37}$$

*where $T = T_{(w_0,u_0,v_0)}$ and $\mu_7$ is a constant depending on $\|u_0\|_{L^1(\Omega)}$, $\|u_0\|_{L^2(\Omega)}$, $\delta$, $k$, $|\Omega|$, $\|u(\cdot,\tau_0)\|_{L^\infty(\Omega)}$, $\|\overline{w}(\cdot,\tau_0)\|_{W^{2,p}(\Omega)}$, $\|v(\cdot,\tau_0)\|_{W^{2,p}(\Omega)}$, $\|w_0(x)\|_{L^1(\Omega)}$, and $\|w(\cdot,\tau_0)\|_{L^\infty(\Omega)}$.*

*Proof.* Integrating the first equation of (3) with respect to $x$ over $\Omega$ and together with the boundary condition, we get

$$\int_\Omega w(x,t)\,dx = \int_\Omega w_0(x)\,dx \leq C_1. \tag{38}$$

In the following, we will use the inequality as follows:

$$\|u\|^2_{L^2(\Omega)} \leq \varepsilon \|\nabla u\|^2_{L^2(\Omega)} + C\left(1+\varepsilon^{-n/2}\right)\|u\|^2_{L^1(\Omega)}, \tag{39}$$

with $C$ depending only on $n$ and $\Omega$.

Multiplying the first equation of (3) by $w^{s-1}$ $(s \geq 2)$ and integrating with respect to $x$ over $\Omega$ imply

$$\begin{aligned}
\frac{1}{s}\frac{d}{dt}\int_\Omega w^s dx &= -D(s-1)\int_\Omega w^{s-2}|\nabla w|^2 dx \\
&\quad + (s-1)\int_\Omega \frac{w^{s-1}}{(k_1+u)^2}\nabla u \cdot \nabla w\,dx \\
&\quad + (s-1)\int_\Omega \frac{w^{s-1}}{(k_2+v)^2}\nabla v \cdot \nabla w\,dx \\
&\leq \frac{-4D(s-1)}{s^2}\int_\Omega \left|\nabla (w^{s/2})\right|^2 dx \\
&\quad + C_2(s-1)\int_\Omega w^{s-1}\nabla w\,dx \\
&\quad + C_3(s-1)\int_\Omega w^{s-1}\nabla w\,dx \\
&\leq \frac{-4D(s-1)}{s^2}\int_\Omega \left|\nabla (w^{s/2})\right|^2 dx \\
&\quad + C_4\int_\Omega w^{s/2}\nabla\left(w^{s/2}\right)dx.
\end{aligned} \tag{40}$$

(39) and Hölder's inequality yield

$$\begin{aligned}
\frac{d}{dt}\int_\Omega w^s dx &\leq \frac{-4D(s-1)}{s}\int_\Omega \left|\nabla (w^{s/2})\right|^2 dx \\
&\quad + C_4 s\left(\int_\Omega w^s dx\right)^{1/2}\left(\int_\Omega \left|\nabla (w^{s/2})\right|^2 dx\right)^{1/2} \\
&\leq -2D\int_\Omega \left|\nabla (w^{s/2})\right|^2 dx \\
&\quad + D\int_\Omega \left|\nabla (w^{s/2})\right|^2 dx + \frac{C_4^2 s^2}{4D}\int_\Omega w^s dx \\
&\leq -D\int_\Omega \left|\nabla (w^{s/2})\right|^2 dx + C_5 s^2\int_\Omega w^s dx \\
&\leq \frac{D\left(C\varepsilon^{-n/2}+1\right)}{\varepsilon}\left(\int_\Omega w^{s/2}dx\right)^2 \\
&\quad - \frac{D}{\varepsilon}\int_\Omega w^s dx + C_5 s^2\int_\Omega w^s dx \\
&\leq -C_5 s^2\int_\Omega w^s dx + C_6 s^{n+2}\left(\int_\Omega w^{s/2}dx\right)^2.
\end{aligned} \tag{41}$$

For $\tau_0 \leq t \leq T$, by (41), we have

$$\frac{d}{dt}\left(e^{C_5 s^2 t}\int_\Omega w^s dx\right) \leq C_6 s^{n+2} e^{C_5 s^2 t}\left(\int_\Omega w^{s/2}dx\right)^2. \tag{42}$$

Integrating (42) with respect to $t$ over $[\tau_0, t]$, we obtain

$$\begin{aligned}
\int_\Omega w^s(x,t)\,dx &\leq e^{C_5 s^2(\tau_0-t)}\int_\Omega w^s(x,\tau_0)\,dx \\
&\quad + C_7 s^n\left(1-e^{C_5 s^2(\tau_0-t)}\right) \\
&\quad \times \sup_{\tau_0\leq t\leq T}\left(\int_\Omega w^{s/2}dx\right)^2 \\
&\leq \|w(x,\tau_0)\|^s_{L^\infty(\Omega)} \\
&\quad + C_8 s^n \sup_{\tau_0\leq t\leq T}\left(\int_\Omega w^{s/2}dx\right)^2.
\end{aligned} \tag{43}$$

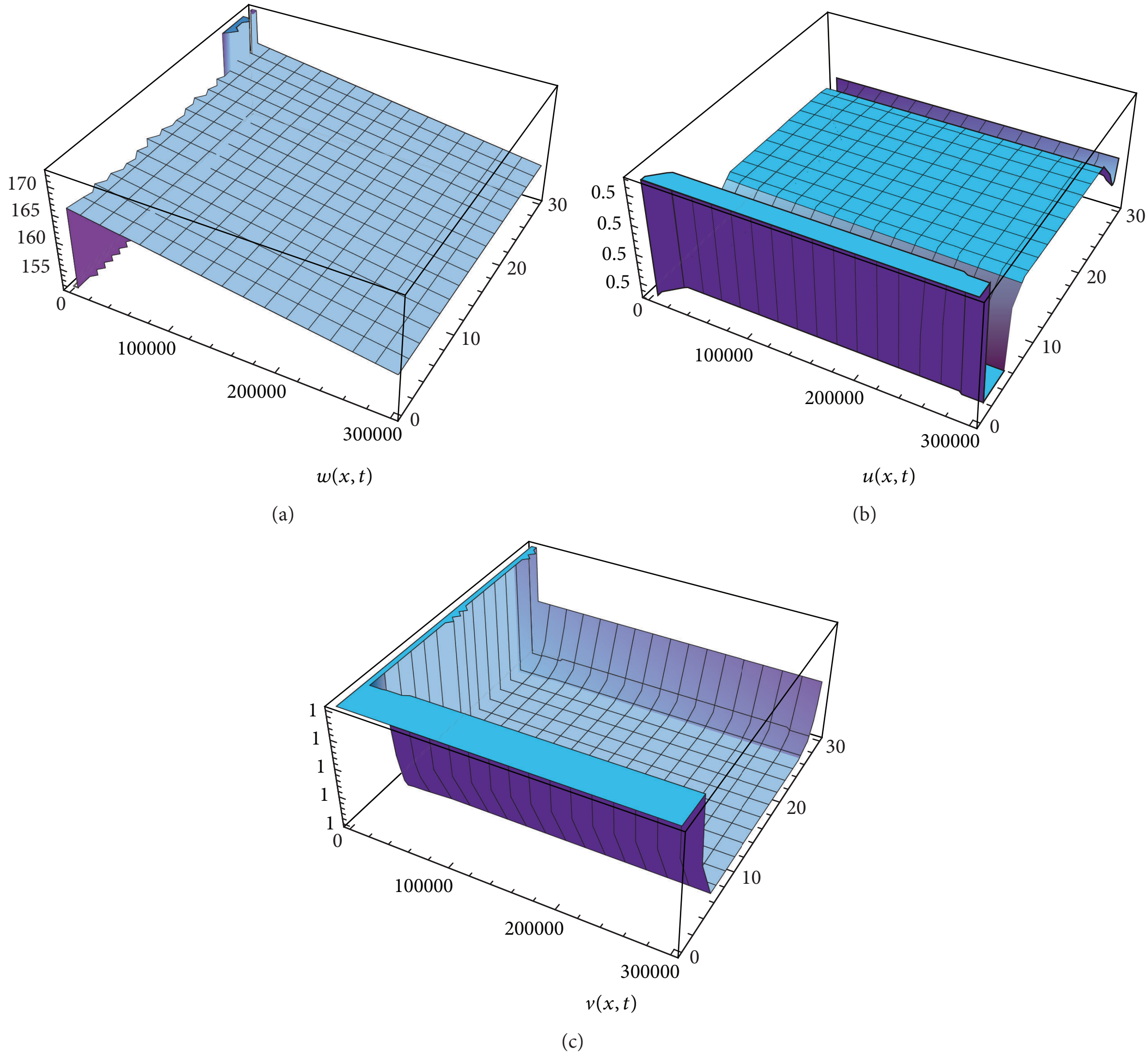


Figure 1

Let

$$M(s) = \max \left\{ \|w(x,\tau_0)\|_{L^\infty(\Omega)}, \sup_{\tau_0 \le t \le T} \left( \int_\Omega w^s dx \right)^{1/s} \right\}; \tag{44}$$

then we get

$$M(s) \le \left(C_9 s^n\right)^{(1/s)} M\left(\frac{s}{2}\right), \quad s \ge 2. \tag{45}$$

Taking $s = 2^k$, $k = 1, 2, \ldots$, we get

$$\begin{aligned} M\left(2^k\right) &\le C_9^{1/2^k} 2^{kn/2^k} M\left(2^{k-1}\right) \\ &\le C_9^{(1/2^k)+\cdots+(1/2)} 2^{(kn/2^k)+\cdots+(n/2)} M(1) \\ &\le CM(1). \end{aligned} \tag{46}$$

For $\tau_0 < +\infty$ and (38) one gets

$$\|w(\cdot,t)\|_{L^\infty(\Omega)} \le \sup_{\tau_0 \le t \le T} \int_\Omega w\, dx \le \mu_7, \tag{47}$$

where $\mu_7$ is a constant depending on $\|u_0\|_{L^1(\Omega)}$, $\|u_0\|_{L^2(\Omega)}$, $\delta$, $k$, $|\Omega|$, $\|u(\cdot,\tau_0)\|_{L^\infty(\Omega)}$, $\|\overline{w}(\cdot,\tau_0)\|_{W^{2,p}(\Omega)}$, $\|v(\cdot,\tau_0)\|_{W^{2,p}(\Omega)}$, $\|w_0(x)\|_{L^1(\Omega)}$, and $\|w(\cdot,\tau_0)\|_{L^\infty(\Omega)}$. Now by Theorem 2(ii) and Lemma 3–Lemma 6, we have proved the Theorem 1. □

For $D = k_1 = k_2 = \delta = K = 1$, $T = 300000$, $\Omega = [0, 30]$, $w_0(x) = (1/10000)x^3(30 - x)^2$, $u_0(x) = x^3(30 - x)^2$, and $v_0(x) = x^2(30 - x)^2$, we have the numerical simulation solutions of (3) as shown in Figure 1.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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