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Research Article

On the Generalized Hyers-Ulam Stability of an n-Dimensional Quadratic and Additive Type Functional Equation

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We investigate the generalized Hyers-Ulam stability of a functional equation $f\left(\sum_{j=1}^{n} x_j\right)+(n-2)\sum_{j=1}^{n} f(x_j)-\sum_{1\le i<j\le n} f(x_i+x_j)=0$.

1. Introduction

Throughout this paper, let X be a normed space and Y a Banach space. For a given mapping $f : X \to Y$, we define

$$\begin{gathered} Af(x,y) := f(x+y) - f(x) - f(y), \\ Qf(x,y) := f(x+y) + f(x-y) - 2f(x) - 2f(y), \end{gathered} \tag{1}$$

for all $x, y \in X$. A mapping $f : X \to Y$ is called an additive mapping (a quadratic mapping, resp.) if f satisfies the functional equation $Af = 0$ ($Qf = 0$, resp.). If a mapping is represented by sum of an additive mapping and a quadratic mapping, we call the mapping a quadratic-additive mapping. For a functional equation $Ef = 0$ if all of the solutions of $Ef = 0$ are quadratic-additive mappings and all of quadratic-additive mappings are the solutions of $Ef = 0$, then we call the functional equation $Ef = 0$ a quadratic-additive type functional equation.

In 1940, Ulam [1] raised a question concerning the stability of homomorphisms. Hyers [2], Aoki [3], Rassias [4], and Găvruța [5] made important role to study the stability of the functional equation. During the last decades, the stability problems of functional equations have been extensively investigated by a number of mathematicians (see also [6–9]).

In 2006, Jun and Kim [10] obtained the stability of the functional equation

$$\begin{aligned} f\left(\sum_{j=1}^{n} x_j\right) + (n-2)\sum_{j=1}^{n} f\left(x_j\right) \\ - \sum_{1\le i<j\le n} f\left(x_i + x_j\right) = 0, \end{aligned} \tag{2}$$

for all $x_1, x_2, \ldots, x_n \in X$ $(n > 2)$ (see also [11–15]). The functional equation (2) is a quadratic-additive type functional equation (see Theorem 2.6 in [16]). For the case $n = 3$, Jung [17] proved the stability of the functional equation (2) (see also [18–20]) and, for the case $n = 4$, Chang et al. [21] proved the stability of the functional equation (2) (see also [22–25]).

In this paper, we will generalize the previous results of the stability problem of the functional equation (2) on the punctured domain. In particular, we will show the superstability (if $p < 0$) of the functional equation (2) in the sense of Rassias.

2. Stability of the Functional Equation (2) (n Is Even)

Let (s,t) be a fixed element in $\{(1,1),(1,-1),(-1,-1)\}$ and let $\varphi : (X \setminus \{0\})^n \to [0,\infty)$ be a function satisfying the conditions:

$$\sum_{j=0}^{\infty} 4^{-sj}\varphi\left(2^{sj}x_1, 2^{sj}x_2, \ldots, 2^{sj}x_n\right) < \infty, \tag{3}$$

$$\sum_{j=0}^{\infty} 2^{-tj}\varphi\left(2^{tj}x_1, 2^{tj}x_2, \ldots, 2^{tj}x_n\right) < \infty, \tag{4}$$

for all $x_1, x_2, \ldots, x_n \in X \setminus \{0\}$, where n is a fixed even integer greater than 2 in this section. For convenience, we use the following abbreviations in this section for a given mapping $f : X \to Y$:

$$\begin{aligned} &Df\left(x_1, x_2, \ldots, x_n\right) \\ &\quad := f\left(\sum_{j=1}^{n} x_j\right) + (n-2)\sum_{j=1}^{n} f\left(x_j\right) - \sum_{1\le i<j\le n} f\left(x_i + x_j\right), \\ &J_m f(x) \\ &\quad = \frac{1}{2}\left(4^{-sm}\left(f\left(2^{sm}x\right) + f\left(-2^{sm}x\right) - \frac{2(n+2)}{3n} f(0)\right)\right. \\ &\qquad \left. + 2^{-tm}\left(f\left(2^{tm}x\right) - f\left(-2^{tm}x\right)\right)\right), \\ &\qquad \overline{x} := \left(\overbrace{x, \ldots, x}^{n/2+1}, \overbrace{-x, \ldots, -x}^{n/2-1}\right), \end{aligned} \tag{5}$$

for all $x, x_1, x_2, \ldots, x_n \in X$. From these, we get the equality

$$\begin{aligned} &J_m f(x) - J_{m+1} f(x) \\ &\quad = \frac{2\cdot 4^{\tau_{-s,m}}}{n(n-2)}\left(Df\left(\overline{2^{\tau_{s,m}}x}\right) + Df\left(\overline{-2^{\tau_{s,m}}x}\right)\right)s \\ &\qquad + \frac{2^{\tau_{-t,m}-1}}{n-2}\left(Df\left(\overline{2^{\tau_{t,m}}x}\right) - Df\left(\overline{-2^{\tau_{t,m}}x}\right)\right)t \end{aligned} \tag{6}$$

for all $x \in X \setminus \{0\}$ and all nonnegative integers m, where $\tau_{k,m}$ are the integers defined by

$$\tau_{k,m} = k\left(m + \frac{1}{2}\right) - \frac{1}{2} \tag{7}$$

for $k \in \{-1, 1\}$.

Lemma 1. *If $f : X \to Y$ is a mapping such that*

$$Df\left(x_1, x_2, \ldots, x_n\right) = 0, \tag{8}$$

for all $x_1, x_2, \ldots, x_n \in X \setminus \{0\}$, then

$$J_m f(x) = f(x) - \frac{n+2}{3n} f(0) \tag{9}$$

for all $x \in X \setminus \{0\}$ and all nonnegative integers m.

Proof. We can easily get

$$\begin{aligned} &f(x) - \frac{n+2}{3n} f(0) - J_m f(x) \\ &\quad = \sum_{j=0}^{m-1}\left(\frac{2\cdot 4^{\tau_{-s,j}}}{n(n-2)}\left(Df\left(\overline{2^{\tau_{s,j}}x}\right) + Df\left(\overline{-2^{\tau_{s,j}}x}\right)\right)s\right. \\ &\qquad \left. + \frac{2^{\tau_{-t,j}-1}}{n-2}\left(Df\left(\overline{2^{\tau_{t,j}}x}\right) - Df\left(\overline{-2^{\tau_{t,j}}x}\right)\right)t\right) \\ &\quad = 0 \end{aligned} \tag{10}$$

for all $x \in X \setminus \{0\}$ and all nonnegative integers m. □

Theorem 2. *Suppose that $f : X \to Y$ is a mapping such that*

$$\left\|Df\left(x_1, x_2, \ldots, x_n\right)\right\| \le \varphi\left(x_1, x_2, \ldots, x_n\right) \tag{11}$$

for all $x_1, x_2, \ldots, x_n \in X \setminus \{0\}$ with $\lim_{m\to\infty} J_m f(0) = 0$. Then, there exists a unique mapping $F : X \to Y$ satisfying (8) for all $x_1, x_2, \ldots, x_n \in X \setminus \{0\}$ and

$$\left\|f(x) - \frac{n+2}{3n} f(0) - F(x)\right\| \le \sum_{j=0}^{\infty} \Phi_j(x) \tag{12}$$

for all $x \in X \setminus \{0\}$ with $F(0) = 0$, where Φ_j are the mappings defined by

$$\begin{aligned} \Phi_j(x) :=\ & \frac{2\cdot 4^{\tau_{-s,j}}}{n(n-2)}\left(\varphi\left(\overline{2^{\tau_{s,j}}x}\right) + \varphi\left(\overline{-2^{\tau_{s,j}}x}\right)\right) \\ & + \frac{2^{\tau_{-t,j}-1}}{n-2}\left(\varphi\left(\overline{2^{\tau_{t,j}}x}\right) + \varphi\left(\overline{-2^{\tau_{t,j}}x}\right)\right) \end{aligned} \tag{13}$$

for all $x \in X \setminus \{0\}$.

Proof. It follows from (6) and (11) that

$$\begin{aligned} &\left\|J_m f(x) - J_{m+m'} f(x)\right\| \\ &\quad = \sum_{j=m}^{m+m'-1}\left\|\frac{2\cdot 4^{\tau_{-s,j}}}{n(n-2)}\left(Df\left(\overline{2^{\tau_{s,j}}x}\right) + Df\left(\overline{-2^{\tau_{s,j}}x}\right)\right)s\right. \\ &\qquad \left. + \frac{2^{\tau_{-t,j}-1}}{n-2}\left(Df\left(\overline{2^{\tau_{t,j}}x}\right) - Df\left(\overline{-2^{\tau_{t,j}}x}\right)\right)t\right\| \\ &\quad \le \sum_{j=m}^{m+m'-1} \Phi_j(x) \end{aligned} \tag{14}$$

for all $x_1, x_2, \ldots, x_n \in X \setminus \{0\}$ and all nonnegative integers m, m' with $m' > 0$. From (3), (4), and (14), it follows that the sequence $\{J_m f(x)\}$ is Cauchy for all $x \in X \setminus \{0\}$. Since Y is complete, the sequence $\{J_m f(x)\}$ converges. From this and $\lim_{m\to\infty} J_m f(0) = 0$, we can define the mapping $F : X \to Y$ by

$$F(x) := \lim_{m\to\infty} J_m f(x) \tag{15}$$

for all $x \in X$. Moreover, letting $m = 0$ and taking the limit as $m' \to \infty$ in (14), we get the inequality (12). Notice that

$\lim_{m\to\infty} J_m f(0) = 0$, $\lim_{m\to\infty} 4^{-sm}\varphi(2^{sm}x_1,\ldots,2^{sm}x_n) = 0$, and $\lim_{m\to\infty} 2^{-tm}\varphi(2^{tm}x_1,\ldots,2^{tm}x_n) = 0$ for all $x_1, x_2, \ldots, x_n \in X\setminus\{0\}$. Hence, it follows from (11) and the definition of F that

$$\begin{aligned} &DF(x_1,x_2,\ldots,x_n) \\ &\quad = \lim_{m\to\infty}\frac{1}{2}\Big(4^{-sm}\big(Df(2^{sm}x_1,\ldots,2^{sm}x_n) \\ &\qquad + Df(-2^{sm}x_1,\ldots,-2^{sm}x_n)\big) \\ &\qquad + 2^{-tm}\big(Df(2^{tm}x_1,\ldots,2^{tm}x_n) \\ &\qquad - Df(-2^{tm}x_1,\ldots,-2^{tm}x_n)\big)\Big) = 0 \end{aligned} \tag{16}$$

for all $x_1, x_2, \ldots, x_n \in X\setminus\{0\}$.

Now, let $F' : X \to Y$ be another mapping satisfying (8) for all $x_1, x_2, \ldots, x_n \in X\setminus\{0\}$ and (12) with $F'(0) = 0$. Using Lemma 1, (12), and $F'(0) = 0 = F(0)$, we obtain

$$\begin{aligned} &\left\|F(x) - F'(x)\right\| \\ &\quad \le \left\|J_m F(x) - J_m F'(x)\right\| \\ &\quad \le \frac{4^{-sm}}{2}\Big(\Big\|(f-F)(2^{sm}x) - \frac{n+2}{3n}f(0)\Big\| \\ &\qquad + \Big\|(f-F')(2^{sm}x) - \frac{n+2}{3n}f(0)\Big\| \\ &\qquad + \Big\|(f-F)(-2^{sm}x) - \frac{n+2}{3n}f(0)\Big\| \\ &\qquad + \Big\|(f-F')(-2^{sm}x) - \frac{n+2}{3n}f(0)\Big\|\Big) \\ &\quad + \frac{2^{-tm}}{2}\Big(\Big\|(f-F)(2^{tm}x) - \frac{n+2}{3n}f(0)\Big\| \\ &\qquad + \Big\|(f-F')(2^{tm}x) - \frac{n+2}{3n}f(0)\Big\| \\ &\qquad + \Big\|(f-F)(-2^{tm}x) - \frac{n+2}{3n}f(0)\Big\| \\ &\qquad + \Big\|(f-F')(-2^{tm}x) - \frac{n+2}{3n}f(0)\Big\|\Big) \end{aligned} \tag{17}$$

for all $x \in X\setminus\{0\}$ and all positive integers m. It follows from (12) and (17) that

$$\begin{aligned} &\left\|F(x) - F'(x)\right\| \\ &\quad \le \sum_{j=0}^{\infty} 2\left(4^{-sm}\Phi_j(2^{sm}x) + 2^{-tm}\Phi_j(2^{tm}x)\right) \end{aligned} \tag{18}$$

for all $x \in X\setminus\{0\}$ and all positive integers m. We can easily show that the terms on the right-hand side of the inequality (18) tend to 0 as $m \to \infty$ for the cases $(s,t) = (1,1)$ and $(s,t) = (-1,-1)$. For the case $(s,t) = (1,-1)$, we have

$$\begin{aligned} &n(n-2)\sum_{j=0}^{\infty}\left(4^{-m}\Phi_j(2^m x) + 2^m\Phi_j(2^{-m}x)\right) \\ &= \sum_{j=0}^{\infty}\frac{2\varphi\left(\overline{2^{j+m}x}\right) + 2\varphi\left(\overline{-2^{j+m}x}\right)}{4^{j+m+1}} \\ &\quad + \frac{2^j n}{2^{2m+1}}\left(\sum_{j=0}^{m/2-1} + \sum_{j=m/2}^{m-1} + \sum_{j=m}^{\infty}\right) \\ &\quad \times\left(\varphi\left(\overline{\frac{2^m x}{2^{j+1}}}\right) + \varphi\left(\overline{\frac{-2^m x}{2^{j+1}}}\right)\right) \\ &\quad + \frac{2^m}{2^{2j+1}}\left(\sum_{j=0}^{m/2-1} + \sum_{j=m/2}^{m-1} + \sum_{j=m}^{\infty}\right) \\ &\quad \times\left(\varphi\left(\overline{\frac{2^j x}{2^m}}\right) + \varphi\left(\overline{\frac{-2^j x}{2^m}}\right)\right) \\ &\quad + \sum_{j=0}^{\infty}\frac{2^{j+m}n}{2}\left(\varphi\left(\overline{2^{-j-m-1}x}\right) + \varphi\left(\overline{-2^{-j-m-1}x}\right)\right) \\ &\le \left(\frac{1}{2}\sum_{j=m}^{\infty} + n\sum_{j=m/2}^{m-1} + \frac{n}{2^{m/2}}\sum_{j=0}^{m/2-1}\right) \\ &\quad \times\left(\frac{\varphi\left(\overline{2^j x}\right) + \varphi\left(\overline{-2^j x}\right)}{4^j}\right) \\ &\quad + \left(\frac{n}{2^m}\sum_{j=1}^{\infty} + \sum_{j=m/2+1}^{m} + \frac{1}{2^{m/2}}\sum_{j=1}^{m/2}\right) \\ &\quad \times\left(2^j\left(\varphi\left(\overline{\frac{x}{2^j}}\right) + \varphi\left(\overline{\frac{-x}{2^j}}\right)\right)\right) \\ &\quad + \frac{1}{2^m}\sum_{j=0}^{\infty}\frac{\varphi\left(\overline{2^j x}\right) + \varphi\left(\overline{-2^j x}\right)}{4^j} \\ &\quad + \frac{n}{4}\sum_{j=m+1}^{\infty}2^j\left(\varphi\left(\overline{\frac{x}{2^j}}\right) + \varphi\left(\overline{\frac{-x}{2^j}}\right)\right) \end{aligned} \tag{19}$$

for all $x \in X\setminus\{0\}$ and all positive even integers m. So, we also show that the terms on the right-hand side of the inequality (18) tend to 0 as $m \to \infty$ for the cases $(s,t) = (1,-1)$. Using the equality $F(0) = 0 = F'(0)$, we can conclude that $F(x) = F'(x)$ for all $x \in X$. This proves the uniqueness of F. □

Corollary 3. *Let $p \neq 1, 2$ be a real number. Suppose that $f : X \to Y$ is a mapping such that*

$$\left\|Df(x_1,x_2,\ldots,x_n)\right\| \le \left\|x_1\right\|^p + \left\|x_2\right\|^p + \cdots + \left\|x_n\right\|^p \tag{20}$$

for all $x_1, x_2, \ldots, x_n \in X \setminus \{0\}$ (with $f(0) = 0$ if $p > 2$). Then, there exists a unique mapping F satisfying (8) for all $x_1, x_2, \ldots, x_n \in X \setminus \{0\}$ and

$$\begin{aligned} &\|f(x) - F(x)\| \\ &\quad \le \left(\frac{4}{|2^p - 4|} + \frac{n}{|2^p - 2|}\right)\frac{\|x\|^p}{n-2}, \quad \text{if } p > 0, \\ &\left\|f(x) - \frac{n+2}{3n} f(0) - F(x)\right\| \\ &\quad \le \left(\frac{4}{|2^p - 4|} + \frac{n}{|2^p - 2|}\right)\frac{1}{n-2}, \quad \text{if } p = 0, \end{aligned} \tag{21}$$

$$f(x) = F(x), \quad \text{if } p < 0 \tag{22}$$

for all $x \in X \setminus \{0\}$ with $F(0) = 0$.

Proof. Put $\varphi(x_1, x_2, \ldots, x_n) = \|x_1\|^p + \|x_2\|^p + \cdots + \|x_n\|^p$ for all $x_1, x_2, \ldots, x_n \in X \setminus \{0\}$. By Theorem 2, there exists a unique mapping F satisfying (8) for all $x_1, x_2, \ldots, x_n \in X \setminus \{0\}$ and

$$\begin{aligned} &\left\|f(x) - \frac{n+2}{3n} f(0) - F(x)\right\| \\ &\quad \le \left(\frac{4}{|2^p - 4|} + \frac{n}{|2^p - 2|}\right)\frac{\|x\|^p}{n-2} \end{aligned} \tag{23}$$

for all $x \in X \setminus \{0\}$ with $F(0) = 0$. From these, we get the inequalities

$$\begin{aligned} &\frac{(n-1)(n^2-4)}{6n}\|f(0)\| \\ &\quad \le \|(Df - DF)(kx, kx, \ldots, kx)\| \\ &\qquad + \left\|(F - f)(nkx) - \frac{n+2}{3n} f(0)\right\| \\ &\qquad + n(n-2)\left\|(F - f)(kx) - \frac{n+2}{3n} f(0)\right\| \\ &\qquad + \frac{n(n-1)}{2}\left\|(f - F)(2kx) - \frac{n+2}{3n} f(0)\right\| \\ &\quad \le \left(n + \left(n^p + n(n-2) + \frac{n(n-1)2^p}{2}\right)\right. \\ &\qquad \left. \times \left(\frac{4}{|2^p - 4|} + \frac{n}{|2^p - 2|}\right)\right) k^p \|x\|^p \end{aligned} \tag{24}$$

for all $x \in X \setminus \{0\}$ and all positive real numbers k. Taking the limit as $k \to \infty$ or $k \to 0$ in the above inequality, we have $f(0) = 0$ if $p \neq 0$. Hence, if $p \neq 0, 1, 2$, then the inequality

$$\|f(x) - F(x)\| \le \left(\frac{4}{|2^p - 4|} + \frac{n}{|2^p - 2|}\right)\frac{\|x\|^p}{n-2} \tag{25}$$

for all $x \in X \setminus \{0\}$ follows from (23). If $p < 0$, then we get the inequalities

$$\begin{aligned} &\|f(x) - F(x)\| \\ &\quad \le \frac{1}{n-1}\Big(\|(Df - DF)((-k+1)x, kx, \ldots, kx)\| \\ &\qquad + \|(f - F)(((n-2)k+1)x)\| \\ &\qquad + (n-1)(n-2)\|(f - F)(kx)\| \\ &\qquad + (n-2)\|(f - F)((-k+1)x)\| \\ &\qquad + \frac{(n-1)(n-2)}{2}\|(f - F)(2kx)\|\Big) \\ &\quad \le \left(k^p + \frac{(2k)^p}{2} + \frac{(-k+1)^p}{n-1} + \frac{((n-2)k+1)^p}{(n-1)(n-2)}\right) \\ &\qquad \times \left(\frac{4}{|2^p - 4|} + \frac{n}{|2^p - 2|}\right)\|x\|^p \\ &\qquad + \left(\frac{(-k+1)^p}{n-1} + k^p\right)\|x\|^p \end{aligned} \tag{26}$$

for all $x \in X \setminus \{0\}$ and all positive integers k. Taking the limit as $k \to \infty$ in the above inequality, we get $F(x) = f(x)$ for all $x \in X \setminus \{0\}$. Since $f(0) = 0 = F(0)$, the equality $f(x) = F(x)$ holds for all $x \in X$. The result follows from this, (23), and (25). □

Lemma 4. *If $f : X \to Y$ is a mapping satisfying (8) for all $x_1, x_2, \ldots, x_n \in X \setminus \{0\}$ with $f(0) = 0$ and $f(tx)$ is continuous in t for each fixed x, then f is represented by*

$$f(rx) = \left(\frac{f(x) + f(-x)}{2}\right) r^2 + \left(\frac{f(x) - f(-x)}{2}\right) r \tag{27}$$

for all $x \in X$ and all $r \in \mathbb{R}$.

Proof. We will prove the equality

$$f(mx) = \left(\frac{f(x) + f(-x)}{2}\right) m^2 + \left(\frac{f(x) - f(-x)}{2}\right) m \tag{28}$$

for all integers m. First, we will use the induction on m to prove the equality (28) for all nonnegative integers m. Note that $f(0) = 0$. We can easily prove it for the cases $m = 0, 1$. For the case $m = 2$, we can show that

$$\begin{aligned} f(2x) &= -\frac{2}{n(n-2)}\left(Df(\overline{x}) + Df(\overline{-x})\right) \\ &\qquad - \frac{1}{2(n-2)}\left(Df(\overline{x}) - Df(\overline{-x})\right) + 3f(x) + f(-x) \\ &= \left(\frac{f(x) + f(-x)}{2}\right) 2^2 + \left(\frac{f(x) - f(-x)}{2}\right) 2 \end{aligned} \tag{29}$$

for all $x \in X$. Assume that (28) holds for all $x \in X$ and all nonnegative integers k $(\leq m)$. Then, we obtain

$$\begin{aligned} f((m+1)x) &= -Df\Big(mx, \overbrace{x,\ldots,x}^{n/2}, \overbrace{-x,\ldots,-x}^{n/2-1}\Big) + 2f(mx) \\ &\quad - f((m-1)x) - \frac{n-4}{4} f(-2x) - \frac{n}{4} f(2x) \\ &\quad + nf(x) + (n-2) f(-x) \\ &= \Big(2m^2 - (m-1)^2 - \frac{n-4}{4}\cdot(-2)^2 - \frac{n}{4}\cdot 2^2 \\ &\quad + n\cdot 1^2 + (n-2)\cdot(-1)^2\Big) \\ &\quad \times \left(\frac{f(x)+f(-x)}{2}\right) + \left(\frac{f(x)-f(-x)}{2}\right) \\ &\quad \times \Big(2m - (m-1) - \frac{n-4}{4}\cdot(-2) \\ &\quad -\frac{n}{4}\cdot 2 + n\cdot 1 + (n-2)(-1)\Big) \\ &= \left(\frac{f(x)+f(-x)}{2}\right)(m+1)^2 \\ &\quad + \left(\frac{f(x)-f(-x)}{2}\right)(m+1) \end{aligned} \tag{30}$$

which completes (28) for all nonnegative integers m. Using the similar method, we also can prove the equality (28) for all negative integers m. By (28), we get the equalities

$$\begin{aligned} \frac{f(mx)+f(-mx)}{2} &= \left(\frac{f(x)+f(-x)}{2}\right)m^2, \\ \frac{f(mx)-f(-mx)}{2} &= \left(\frac{f(x)-f(-x)}{2}\right)m, \\ \frac{f(x/m)+f(-x/m)}{2} &= \left(\frac{f(x)+f(-x)}{2}\right)\frac{1}{m^2}, \\ \frac{f(x/m)-f(-x/m)}{2} &= \frac{f(x)-f(-x)}{2m} \end{aligned} \tag{31}$$

for all $x \in X$ and all integers $m \neq 0$. Hence,

$$\begin{aligned} f\left(\frac{p}{q}x\right) &= \frac{f((p/q)x)+f(-(p/q)x)}{2} \\ &\quad + \frac{f((p/q)x)-f(-(p/q)x)}{2} \\ &= \left(\frac{f(x/q)+f(-x/q)}{2}\right)p^2 \\ &\quad + \left(\frac{f(x/q)-f(-x/q)}{2}\right)p \\ &= \left(\frac{f(x)+f(-x)}{2}\right)\frac{p^2}{q^2} + \left(\frac{f(x)-f(-x)}{2}\right)\frac{p}{q} \end{aligned} \tag{32}$$

for all $x \in X$ and all integers $p, q(\neq 0)$. If $r \in \mathbb{R}$, then there exists a rational sequence $\{r_m\}$ satisfying $\lim_{m\to\infty} r_m = r$. Since $f(tx)$ is continuous in t for each fixed x, we have

$$\begin{aligned} f(rx) &= \lim_{m\to\infty} f(r_m x) \\ &= \lim_{m\to\infty}\left(\frac{f(x)+f(-x)}{2}\right)r_m^2 + \left(\frac{f(x)-f(-x)}{2}\right)r_m \\ &= \left(\frac{f(x)+f(-x)}{2}\right)r^2 + \left(\frac{f(x)-f(-x)}{2}\right)r \end{aligned} \tag{33}$$

for all $x \in X$. □

3. Stability of the Functional Equation (2) (n Is Odd)

Let (s,t), φ, $Df(x_1, x_2, \ldots, x_n)$, and $\tau_{k,m}$ be as in Section 2. In this section, let n be an odd integer greater than 2. For convenience, we use the following abbreviations in this section for a given mapping $f : X \to Y$:

$$\begin{aligned} J_m f(x) &= \frac{1}{2}\Big(4^{-sm}\Big(f(2^{sm}x) + f(-2^{sm}x) \\ &\quad - \frac{2(n+1)}{3(n-1)} f(0)\Big) \\ &\quad + 2^{-tm}\left(f\left(2^{tm}x\right) - f\left(-2^{tm}x\right)\right)\Big), \\ \overline{x} &= \Big(\overbrace{x,\ldots,x}^{(n+1)/2}, \overbrace{-x,\ldots,-x}^{(n-1)/2}\Big), \end{aligned} \tag{34}$$

for $x \in X$. From these, we get

$$\begin{aligned} &J_m f(x) - J_{m+1} f(x) \\ &= \frac{4^{\tau_{-s,m}+1}}{2(n-1)(n-1)}\left(Df\left(\overline{2^{\tau_{s,m}}x}\right) + Df\left(\overline{-2^{\tau_{s,m}}x}\right)\right)s \\ &\quad + \frac{2^{\tau_{-t,m}}}{n-1}\left(Df\left(\overline{2^{\tau_{t,m}}x}\right) - Df\left(\overline{-2^{\tau_{t,m}}x}\right)\right)t \end{aligned} \tag{35}$$

for all $x \in X$.

Using (35) and a similar method in the proof of Lemma 1, we get the following lemma.

Lemma 5. *If $f : X \to Y$ is a mapping satisfying* (8) *for all $x_1, x_2, \dots, x_n \in X \setminus \{0\}$, then*

$$J_m f(x) = f(x) - \frac{n+1}{3(n-1)} f(0) \tag{36}$$

for all $x \in X \setminus \{0\}$.

From (35), Lemma 5, and similar methods used in Theorem 2, we get the following theorem.

Theorem 6. *If $f : X \to Y$ is a unique mapping satisfying* (11) *for all $x_1, x_2, \dots, x_n \in X \setminus \{0\}$ with $\lim_{m\to\infty} J_m f(0) = 0$, then there exists a unique mapping $F : X \to Y$ satisfying* (8) *for all $x_1, x_2, \dots, x_n \in X \setminus \{0\}$ and*

$$\left\| f(x) - \frac{n+1}{3(n-1)} f(0) - F(x) \right\| \le \sum_{j=0}^{\infty} \Phi_j(x) \tag{37}$$

for all $x \in X \setminus \{0\}$ with $F(0) = 0$, where Φ_j are the mappings defined by

$$\begin{aligned}\Phi_j(x) := {} & \frac{4^{\tau_{-s,m}+1}}{2(n-1)(n-1)} \left(\varphi\left(\overline{2^{\tau_{s,m}} x}\right) + \varphi\left(\overline{-2^{\tau_{s,m}} x}\right)\right) \\ & + \frac{2^{\tau_{-t,m}}}{n-1} \left(\varphi\left(\overline{2^{\tau_{t,m}} x}\right) + \varphi\left(\overline{-2^{\tau_{t,m}} x}\right)\right)\end{aligned} \tag{38}$$

for all $x \in X \setminus \{0\}$.

From Theorem 6 and similar methods used in Corollary 3, we get the following corollary.

Corollary 7. *Let $p \neq 1, 2$ be a real number. Suppose that $f : X \to Y$ is a mapping satisfying* (20) *for all $x_1, x_2, \dots, x_n \in X \setminus \{0\}$ (with $f(0) = 0$ if $p > 2$). Then, there exists a unique mapping F satisfying* (8) *for all $x_1, x_2, \dots, x_n \in X \setminus \{0\}$ and*

$$\begin{aligned} & \| f(x) - F(x) \| \\ & \quad \le \left(\frac{4}{(n-1)|2^p - 4|} + \frac{2}{|2^p - 2|} \right) \frac{n \| x \|^p}{n-1}, \\ & \qquad \text{if } p > 0, \\ & \left\| f(x) - \frac{n+1}{3(n-1)} f(0) - F(x) \right\| \\ & \quad \le \left(\frac{4}{(n-1)|2^p - 4|} + \frac{2}{|2^p - 2|} \right) \frac{n}{n-1}, \\ & \qquad \text{if } p = 0, \\ & f(x) = F(x), \quad \text{if } p < 0, \end{aligned} \tag{39}$$

for all $x \in X \setminus \{0\}$ with $F(0) = 0$.

Proof. Put $\varphi(x_1, x_2, \dots, x_n) = \|x_1\|^p + \|x_2\|^p + \cdots + \|x_n\|^p$ for all $x_1, x_2, \dots, x_n \in X \setminus \{0\}$. By Theorem 6, there exists a unique mapping F satisfying (8) for all $x_1, x_2, \dots, x_n \in X \setminus \{0\}$ and

$$\begin{aligned} & \left\| f(x) - \frac{n+1}{3(n-1)} f(0) - F(x) \right\| \\ & \quad \le \left(\frac{4}{(n-1)|2^p - 4|} + \frac{2}{|2^p - 2|} \right) \frac{n\|x\|^p}{n-1} \end{aligned} \tag{40}$$

for all $x \in X \setminus \{0\}$ with $F(0) = 0$. From these, we get the inequalities

$$\begin{aligned} & \frac{(n+1)(n-2)}{6} \|f(0)\| \\ & \quad \le \left\| (F - f)(nkx) - \frac{n+1}{3(n-1)} f(0) \right\| \\ & \qquad + n(n-2) \left\| (F - f)(kx) - \frac{n+1}{3(n-1)} f(0) \right\| \\ & \qquad + \frac{n(n-1)}{2} \left\| (f - F)(2kx) - \frac{n+1}{3(n-1)} f(0) \right\| \\ & \qquad + \| (Df - DF)(kx, kx, \dots, kx) \| \\ & \quad \le \left(\left(n^p + n(n-2) + \frac{n(n-1) 2^p}{2} \right) \right. \\ & \qquad \left. \times \left(\frac{4}{(n-1)|2^p - 4|} + \frac{2}{|2^p - 2|} \right) \frac{n}{n-1} + n \right) k^p \|x\|^p \end{aligned} \tag{41}$$

for all $x \in X \setminus \{0\}$ and all positive real numbers k. Taking the limit as $k \to \infty$ or $k \to 0$ in the above inequality, we have $f(0) = 0$ if $p \neq 0$. Hence, if $p \neq 0, 1, 2$, then the inequality

$$\| f(x) - F(x) \| \le \left(\frac{4}{(n-1)|2^p - 4|} + \frac{2}{|2^p - 2|} \right) \frac{n\|x\|^p}{n-1} \tag{42}$$

for all $x \in X \setminus \{0\}$ follows from (40). If $p < 0$, then we get the inequalities

$$\begin{aligned} & \| f(x) - F(x) \| \\ & \quad \le \frac{1}{n-1} \Big(\| (Df - DF)((-k+1)x, kx, \dots, kx) \| \\ & \qquad + \| (f - F)(((n-2)k + 1)x) \| \\ & \qquad + (n-1)(n-2) \| (f - F)(kx) \| \\ & \qquad + (n-2) \| (f - F)((-k+1)x) \| \\ & \qquad + \frac{(n-1)(n-2)}{2} \| (f - F)(2kx) \| \Big) \end{aligned}$$

$$\leq (n-2)\left(k^p + \frac{(2k)^p}{2} + \frac{(-k+1)^p}{n-1} + \frac{((n-2)k+1)^p}{(n-1)(n-2)}\right) \times \left(\frac{4}{|2^p-4|(n-1)} + \frac{2}{|2^p-2|}\right)\frac{n\|x\|^p}{n-1} + \left(\frac{(-k+1)^p}{n-1} + k^p\right)\|x\|^p \tag{43}$$

for all $x \in X \setminus \{0\}$ and all positive integers k. Taking the limit as $k \to \infty$ in the above inequality, we get $F(x) = f(x)$ for all $x \in X \setminus \{0\}$. Since $f(0) = 0 = F(0)$, the equality $f(x) = F(x)$ holds for all $x \in X$. The result follows from this, (40), and (42). □

From similar methods used in Lemma 4, we get the following lemma.

Lemma 8. *If $f : X \to Y$ is a mapping satisfying (8) for all $x_1, x_2, \ldots, x_n \in X \setminus \{0\}$ with $f(0) = 0$ and $f(tx)$ is continuous in t for each fixed x, then f is represented by*

$$f(rx) = \left(\frac{f(x)+f(-x)}{2}\right)r^2 + \left(\frac{f(x)-f(-x)}{2}\right)r \tag{44}$$

for all $x \in X$ and all $r \in \mathbb{R}$.

Proof. We will use the induction on m to prove (44) for all nonnegative integers m. Note that $f(0) = 0$. We can easily prove it for the cases $m = 0, 1$. For the case $m = 2$, we can show that

$$f(2x) = -\frac{2}{(n-1)^2}\left(Df(\overline{x}) + Df(\overline{-x})\right) - \frac{1}{n-1}\left(Df(\overline{x}) - Df(\overline{-x})\right) + 3f(x) + f(-x) = \left(\frac{f(x)+f(-x)}{2}\right)2^2 + \left(\frac{f(x)-f(-x)}{2}\right)2 \tag{45}$$

for all $x \in X$. Assume that (44) holds for all $x \in X$ and all nonnegative integers k $(\leq m)$. Then, we obtain

$$\begin{aligned}
&f((m+1)x)\\
&= -\frac{2}{n-1}Df\Big(mx, \overbrace{x,\ldots,x}^{(n-1)/2}, \overbrace{-x,\ldots,-x}^{(n-1)/2}\Big) + 2f(mx)\\
&\quad - f((m-1)x) - \frac{n-3}{4}f(-2x) - \frac{n-3}{4}f(2x)\\
&\quad + (n-2)f(x) + (n-2)f(-x)\\
&= \left(2m^2 - (m-1)^2 - \frac{n-3}{4}\cdot\left((-2)^2+2^2\right) + (n-2)\left(1+(-1)^2\right)\right)\\
&\quad\times\left(\frac{f(x)+f(-x)}{2}\right) + \left(\frac{f(x)-f(-x)}{2}\right)\\
&\quad\times\Big(2m - (m-1) - \frac{n-3}{4}\\
&\qquad\times((-2)+2) + (n-2)(1+(-1))\Big)\\
&= \left(\frac{f(x)+f(-x)}{2}\right)(m+1)^2 + \left(\frac{f(x)-f(-x)}{2}\right)(m+1)
\end{aligned} \tag{46}$$

which completes the proof of (44). The remainder of the proof is the same in the proof of Lemma 4. □

Corollary 9. *If $f : X \to Y$ is a mapping satisfying (8) for all $x_1, x_2, \ldots, x_n \in X \setminus \{0\}$, then $f(0) = 0$.*

Proof. Put $p = -1$. Then, we have

$$\|Df(x_1, x_2, \ldots, x_n)\| = 0 \leq \|x_1\|^p + \|x_2\|^p + \cdots + \|x_n\|^p \tag{47}$$

for all $x_1, x_2, \ldots, x_n \in X \setminus \{0\}$. By Corollaries 3 and 7, $f(x) = F(x)$ for all $x \in X$ with $F(0) = 0$. So, we get the desired result. □

Corollary 10. *Let $p < 0$ be a real number and $n > 2$ an integer. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a mapping satisfying*

$$|Df(x_1, x_2, \ldots, x_n)| \leq |x_1|^p + |x_2|^p + \cdots + |x_n|^p \tag{48}$$

for all $x_1, x_2, \ldots, x_n \in \mathbb{R} \setminus \{0\}$ and f is continuous. Then, f is represented by

$$f(x) = \left(\frac{f(1)+f(-1)}{2}\right)x^2 + \frac{f(1)-f(-1)}{2}x \tag{49}$$

for all $x \in \mathbb{R}$ and $Df(x_1, x_2, \ldots, x_n) = 0$ for all $x_1, x_2, \ldots, x_n \in \mathbb{R}$.

Proof. If n is even, then the equality (49) follows from Corollary 3 and Lemma 4. If n is odd, then the equality (49) follows from Corollary 7 and Lemma 8. And we can easily show that the function defined by (49) satisfies the functional equation $Df(x_1, x_2, \ldots, x_n) = 0$ for all $x_1, x_2, \ldots, x_n \in \mathbb{R}$. □

4. Another Proof for the Stability of the Functional Equation (2)

Let (s, t), $Df(x_1, x_2, \ldots, x_n)$, $\tau_{k,m}$ be as in Section 2. In this section, Let n be a fixed integer greater than 2 and let $\varphi : X^n \to [0, \infty)$ be a function satisfying the conditions (3) and (4) for all $x_1, x_2, \ldots, x_n \in X$. For convenience, we use the following abbreviations in this section for a given mapping $f : X \to Y$:

$$J_m f(x) = \frac{1}{2}\left(4^{-sm}\left(f(2^{sm}x) + f(-2^{sm}x) + \frac{(n-4)(n+1)}{3} f(0)\right) + 2^{-tm}\left(f\left(2^{tm}x\right) - f\left(-2^{tm}x\right)\right)\right),$$
$$\overline{x} = (x, -x, x, 0, \ldots, 0), \tag{50}$$

for all $x \in X$. From these, we get

$$\begin{aligned} J_m f(x) &- J_{m+1} f(x) \\ &= \frac{4^{\tau_{-s,m}}}{2}\left(Df\left(\overline{2^{\tau_{s,m}}x}\right) + Df\left(\overline{-2^{\tau_{s,m}}x}\right)\right)s \\ &+ \frac{2^{\tau_{-t,m}}}{2}\left(Df\left(\overline{2^{\tau_{t,m}}x}\right) - Df\left(\overline{-2^{\tau_{t,m}}x}\right)\right)t \end{aligned} \tag{51}$$

for all $x \in X$. Using (51) and a similar method in the proof of Lemma 1, we get the following lemma.

Lemma 11. *If $f : X \to Y$ is a quadratic-additive mapping, then*

$$J_m f(x) = f(x) \tag{52}$$

for all $x \in X$.

Theorem 12 (compare with Theorem 3.1 in [15]). *Suppose that $f : X \to Y$ is a mapping such that*

$$\|Df(x_1, x_2, \ldots, x_n)\| \le \varphi(x_1, x_2, \ldots, x_n) \tag{53}$$

for all $x_1, x_2, \ldots, x_n \in X$. Then, there exists a unique quadratic-additive mapping $F : X \to Y$ such that

$$\left\| f(x) + \frac{(n-4)(n+1)}{6} f(0) - F(x)\right\| \le \sum_{j=0}^{\infty} \Phi_j(x) \tag{54}$$

for all $x \in X$, where Φ_j are the mappings defined by

$$\begin{aligned} \Phi_j(x) := &\frac{4^{\tau_{-s,j}}}{2}\left(\varphi\left(\overline{2^{\tau_{s,j}}x}\right) + \varphi\left(\overline{-2^{\tau_{s,j}}x}\right)\right) \\ &+ \frac{2^{\tau_{-t,j}}}{2}\left(\varphi\left(\overline{2^{\tau_{t,j}}x}\right) + \varphi\left(\overline{-2^{\tau_{t,j}}x}\right)\right) \end{aligned} \tag{55}$$

for all $x \in X$.

Proof. Note that

$$\begin{aligned} \|J_m f(0)\| &= \frac{4^{-sm}(n-1)(n-2)}{6}\|f(0)\| \\ &= \frac{4^{-sm}}{3}\|Df(0, 0, \ldots, 0)\| \\ &\le \frac{4^{-sm}}{3}\varphi(0, 0, \ldots, 0) \end{aligned} \tag{56}$$

for all positive integers m. It follows from (3) that $\lim_{m\to\infty} J_m f(0) = 0$. From this, (51), Lemma 11, and similar methods used in Theorem 2, we obtain this theorem. □

Corollary 13 (compare with Corollary 3.3 in [15]). *Let $p \ne 1, 2$ be a positive real number. Suppose that $f : X \to Y$ is a mapping such that*

$$\|Df(x_1, x_2, \ldots, x_n)\| \le \|x_1\|^p + \|x_2\|^p + \cdots + \|x_n\|^p \tag{57}$$

for all $x_1, x_2, \ldots, x_n \in X$. Then, there exists a unique quadratic-additive mapping F such that

$$\|f(x) - F(x)\| \le \left(\frac{3}{|2^p - 4|} + \frac{3}{|2^p - 2|}\right)\|x\|^p \tag{58}$$

for all $x \in X$.

Proof. Since $\|f(0)\| = (2/(n-1)(n-2))\|Df(0, 0, \ldots, 0)\| \le 0$, we get $f(0) = 0$. From Theorem 12 and similar methods used in Corollary 3, we obtain this corollary. □

From Theorem 12 and similar methods used in Corollary 3, we get the following corollary.

Corollary 14 (compare with Corollary 3.2 in [15]). *Suppose that $f : X \to Y$ is a mapping such that*

$$\|Df(x_1, x_2, \ldots, x_n)\| \le \varepsilon \tag{59}$$

for all $x_1, x_2, \ldots, x_n \in X$. Then, there exists a unique quadratic-additive mapping F such that

$$\left\| f(x) + \frac{(n-4)(n+1)}{6} f(0) - F(x)\right\| \le \frac{4\varepsilon}{3} \tag{60}$$

for all $x \in X$.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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