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Research Article

New Results for Generalized Gronwall Inequalities and Their Applications

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In succession to our earlier work, we further provide some new generalized Gronwall inequalities and apply these inequalities to the study of qualitative estimations of solutions to certain fractional differential equations.

1. Introduction

It is well known that the Gronwall inequality contributes significantly to research on many differential and integral equations. An increasing number of generalizations of this inequality have been made in recent years to derive qualitative properties of solutions to various fractional differential equations. One remarkable result was obtained by Ye et al. in 2007 and is presented below.

Theorem 1 (see [1, Theorem 1]). *For any $t \in [0, T)$,*

$$u(t) \le a(t) + b(t) \int_0^t (t-s)^{\beta-1} u(s)\, ds, \tag{1}$$

where all functions are nonnegative and continuous. The constant $\beta > 0$. b is a bounded and monotonically increasing function of $[0, T)$; then

$$u(t) \le a(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(b(t)\Gamma(\beta))^n}{\Gamma(n\beta)} (t-s)^{n\beta-1} a(s)\right] ds, \quad t \in [0, T). \tag{2}$$

This result continuously extends and improves and has been widely used in many studies (e.g., see [1–8]). Among them, our previous research [5] provides the following generalizations to deal with complex fractional differential equations.

Theorem 2 (see [5, Theorem 1.4]). *For any $t \in [0, T)$,*

$$u(t) \le a(t) + \sum_{i=1}^{n} b_i(t) \int_0^t (t-s)^{\beta_i-1} u(s)\, ds, \tag{3}$$

where all functions are nonnegative and continuous. The constants $\beta_i > 0$. b_i $(i = 1, 2, \dots, n)$ are bounded and monotonically increasing functions of $[0, T)$. Then

$$\begin{aligned} u(t) \le{}& a(t) \\ &+ \sum_{k=1}^{\infty} \left(\sum_{1', 2', \dots, k'=1}^{n} \frac{\prod_{i=1}^{k} [b_{i'}(t)\Gamma(\beta_{i'})]}{\Gamma\left(\sum_{i=1}^{k} \beta_{i'}\right)} \right. \\ &\left. \times \int_0^t (t-s)^{\sum_{i=1}^{k} \beta_{i'}-1} a(s)\, ds \right), \\ & t \in [0, T). \end{aligned} \tag{4}$$

Theorem 3 (see [5, Theorem 1.5]). *For any $t \in [1, T)$,*

$$u(t) \le a(t) + \sum_{i=1}^{n} b_i(t) \int_1^t \left(\ln \frac{t}{s}\right)^{\beta_i-1} \frac{u(s)}{s} ds, \tag{5}$$

where all functions are nonnegative and continuous. The constants $\beta_i > 0$. b_i $(i = 1, 2, \ldots, n)$ *are bounded and monotonically increasing functions of* $[1, T)$. *Then*

$$u(t) \le a(t) + \sum_{k=1}^{\infty} \left(\sum_{1', 2', \ldots, k'=1}^{n} \frac{\prod_{i=1}^{k} [b_{i'}(t) \Gamma(\beta_{i'})]}{\Gamma(\sum_{i=1}^{k} \beta_{i'})} \times \int_1^t \left[\left(\ln \frac{t}{s} \right)^{\sum_{i=1}^{k} \beta_{i'} - 1} a(s) \right] \frac{ds}{s} \right), \quad t \in [1, T). \tag{6}$$

In this paper, we aim to discuss further issues by using the aforementioned conclusions and suitable analytical techniques according to the above facts. We will establish several new classes of generalized Gronwall inequalities in the next section. In the last section, we will qualitatively analyze certain fractional differential equations to highlight the applications of the inequalities.

2. Main Results and Proofs

We introduce the useful Young's inequality with $\epsilon > 0$ (see [9, page 622]); that is, for any $a, b > 0$ and $1 < p, q < +\infty$, $1/p + 1/q = 1$,

$$ab \le \epsilon a^p + C(\epsilon) b^q, \tag{7}$$

where $C(\epsilon) = (\epsilon p)^{-q/p} q^{-1}$.

The first main result and its proof procedure are presented as follows.

Theorem 4. *For any* $t \in [0, T)$,

$$u(t) \le a(t) + \sum_{i=1}^{n} b_i(t) \int_0^t (t-s)^{\beta_i - 1} c_i(s) u^{\lambda_i}(s)\, ds, \tag{8}$$

where all functions are nonnegative and continuous. For any $i \in \{1, 2, \ldots, n\}$, *the constants* $\beta_i > 0$ *and* $0 < \lambda_i < 1$, *and the function* $b_i(t)$ *is bounded and monotonically increasing on* $[0, T)$. *Then*

$$u(t) \le \widetilde{a}(t) + \sum_{k=1}^{\infty} \left(\sum_{1', 2', \ldots, k'=1}^{n} \frac{\prod_{i=1}^{k} [\widetilde{b}_{i'}(t) \Gamma(\beta_{i'})]}{\Gamma(\sum_{i=1}^{k} \beta_{i'})} \times \int_0^t (t-s)^{\sum_{i=1}^{k} \beta_{i'} - 1} \widetilde{a}(s)\, ds \right), \quad t \in [0, T), \tag{9}$$

where

$$\widetilde{a}(t) = a(t) + \sum_{i=1}^{n} (1 - \lambda_i) \left(\frac{\lambda_i}{\varepsilon} \right)^{\lambda_i/(1-\lambda_i)} b_i(t) \times \int_0^t (t-s)^{\beta_i - 1} [c_i(s)]^{1/(1-\lambda_i)} ds; \quad \widetilde{b}_i(t) = \varepsilon \cdot b_i(t). \tag{10}$$

Here ε *is an arbitrary given positive number.*

Proof. By Young's inequality,

$$\begin{aligned} c_i(s) u^{\lambda_i}(s) &\le \varepsilon \cdot \left[u^{\lambda_i}(s) \right]^{1/\lambda_i} + \left(\frac{\lambda_i}{\varepsilon} \right)^{\lambda_i/(1-\lambda_i)} \cdot (1 - \lambda_i) \cdot [c_i(s)]^{1/(1-\lambda_i)} \\ &= \varepsilon \cdot u(s) + \left(\frac{\lambda_i}{\varepsilon} \right)^{\lambda_i/(1-\lambda_i)} \cdot (1 - \lambda_i) \cdot [c_i(s)]^{1/(1-\lambda_i)}, \end{aligned} \tag{11}$$

which implies that, for any $t \in [0, T)$,

$$\begin{aligned} u(t) &\le a(t) + \sum_{i=1}^{n} b_i(t) \int_0^t (t-s)^{\beta_i - 1} c_i(s) u^{\lambda_i}(s)\, ds \\ &\le a(t) + \sum_{i=1}^{n} b_i(t) \times \int_0^t (t-s)^{\beta_i - 1} \left\{ \varepsilon \cdot u(s) + \left(\frac{\lambda_i}{\varepsilon} \right)^{\lambda_i/(1-\lambda_i)} \cdot (1 - \lambda_i) \cdot [c_i(s)]^{1/(1-\lambda_i)} \right\} ds \\ &= \left\{ a(t) + \sum_{i=1}^{n} (1 - \lambda_i) \left(\frac{\lambda_i}{\varepsilon} \right)^{\lambda_i/(1-\lambda_i)} b_i(t) \times \int_0^t (t-s)^{\beta_i - 1} [c_i(s)]^{1/(1-\lambda_i)} ds \right\} \\ &\quad + \sum_{i=1}^{n} \varepsilon \cdot b_i(t) \int_0^t (t-s)^{\beta_i - 1} u(s)\, ds \\ &= \widetilde{a}(t) + \sum_{i=1}^{n} \widetilde{b}_i(t) \int_0^t (t-s)^{\beta_i - 1} u(s)\, ds. \end{aligned} \tag{12}$$

We immediately derived estimation (9) by using Theorem 2. This estimation completes the proof of Theorem 4. □

Our second result can be proved in the same manner by applying Young's inequality and Theorem 3. The proof procedure is similar to that of Theorem 4 and is omitted here.

Theorem 5. *For any $t \in [1, T)$,*

$$u(t) \leq a(t) + \sum_{i=1}^{n} b_i(t) \int_1^t \left(\ln \frac{t}{s}\right)^{\beta_i - 1} \frac{c_i(s) u^{\lambda_i}(s)}{s} ds, \quad (13)$$

where all functions are nonnegative and continuous. For any $i \in \{1, 2, \ldots, n\}$, the constants $\beta_i > 0$ and $0 < \lambda_i < 1$, and the function $b_i(t)$ is bounded and monotonically increasing on $[1, T)$. Then

$$\begin{aligned} u(t) \leq{}& \bar{a}(t) \\ &+ \sum_{k=1}^{\infty} \left(\sum_{1', 2', \ldots, k'=1}^{n} \frac{\prod_{i=1}^{k} \left[\bar{b}_{i'}(t) \Gamma(\beta_{i'})\right]}{\Gamma\left(\sum_{i=1}^{k} \beta_{i'}\right)} \right. \\ &\left. \times \int_1^t \left(\ln \frac{t}{s}\right)^{\sum_{i=1}^{k} \beta_{i'} - 1} \frac{\bar{a}(s)}{s} ds \right), \\ & t \in [1, T), \end{aligned} \quad (14)$$

where

$$\begin{aligned} \bar{a}(t) &= a(t) + \sum_{i=1}^{n} (1 - \lambda_i) \left(\frac{\lambda_i}{\varepsilon}\right)^{\lambda_i/(1-\lambda_i)} b_i(t) \\ &\quad \times \int_1^t \left(\ln \frac{t}{s}\right)^{\beta_i - 1} \left[c_i(s)\right]^{1/(1-\lambda_i)} \frac{ds}{s}; \\ \bar{b}_i(t) &= \varepsilon \cdot b_i(t). \end{aligned} \quad (15)$$

Here ε is an arbitrary given positive number.

Evidently, the following two corollaries are directly obtained by using Theorems 4 and 5 when choosing $\beta_i = 1$ for any $i \in \{1, 2, \ldots, n\}$.

Corollary 6. *For any $t \in [0, T)$,*

$$u(t) \leq a(t) + \sum_{i=1}^{n} b_i(t) \int_0^t c_i(s) u^{\lambda_i}(s) ds, \quad (16)$$

where all functions are nonnegative and continuous. For any $i \in \{1, 2, \ldots, n\}$, the constant $0 < \lambda_i < 1$, and the function $b_i(t)$ is bounded and monotonically increasing on $[0, T)$. Then

$$\begin{aligned} u(t) \leq{}& \tilde{a}(t) \\ &+ \sum_{k=1}^{\infty} \left(\sum_{1', 2', \ldots, k'=1}^{n} \frac{\prod_{i=1}^{k} \tilde{b}_{i'}(t)}{k!} \right. \\ &\left. \times \int_0^t (t - s)^{k-1} \tilde{a}(s) ds \right), \\ & t \in [0, T), \end{aligned} \quad (17)$$

where

$$\begin{aligned} \tilde{a}(t) &= a(t) + \sum_{i=1}^{n} (1 - \lambda_i) \left(\frac{\lambda_i}{\varepsilon}\right)^{\lambda_i/(1-\lambda_i)} b_i(t) \\ &\quad \times \int_0^t \left[c_i(s)\right]^{1/(1-\lambda_i)} ds; \\ \tilde{b}_i(t) &= \varepsilon \cdot b_i(t). \end{aligned} \quad (18)$$

Here ε is an arbitrary given positive number.

Corollary 7. *For any $t \in [1, T)$,*

$$u(t) \leq a(t) + \sum_{i=1}^{n} b_i(t) \int_1^t \frac{c_i(s) u^{\lambda_i}(s)}{s} ds, \quad (19)$$

where all functions are nonnegative and continuous. For any $i \in \{1, 2, \ldots, n\}$, the constant $0 < \lambda_i < 1$, and the function $b_i(t)$ is bounded and monotonically increasing on $[1, T)$. Then

$$\begin{aligned} u(t) \leq{}& \bar{a}(t) \\ &+ \sum_{k=1}^{\infty} \left(\sum_{1', 2', \ldots, k'=1}^{n} \frac{\prod_{i=1}^{k} \bar{b}_{i'}(t)}{k!} \right. \\ &\left. \times \int_1^t \left[\left(\ln \frac{t}{s}\right)^{k-1} \bar{a}(s)\right] \frac{ds}{s} \right), \\ & t \in [1, T), \end{aligned} \quad (20)$$

where

$$\begin{aligned} \bar{a}(t) &= a(t) + \sum_{i=1}^{n} (1 - \lambda_i) \left(\frac{\lambda_i}{\varepsilon}\right)^{\lambda_i/(1-\lambda_i)} b_i(t) \\ &\quad \times \int_1^t \left[c_i(s)\right]^{1/(1-\lambda_i)} \frac{ds}{s}; \\ \bar{b}_i(t) &= \varepsilon \cdot b_i(t). \end{aligned} \quad (21)$$

Here ε is an arbitrary given positive number.

Our next task is estimating the nonnegative and continuous function $u(t)$, which satisfies, for any $t \in [0, T)$,

$$\begin{aligned} u(t) \leq{}& a(t) + b(t) \int_0^t c(s) u(s) ds \\ &+ \sum_{i=1}^{n} b_i(t) \int_0^t (t - s)^{\beta_i - 1} c_i(s) u^{\lambda_i}(s) ds, \end{aligned} \quad (22)$$

where all the functions are nonnegative and continuous. For any $i \in \{1, 2, \ldots, n\}$, the constants $\beta_i > 0$ and $0 < \lambda_i < 1$, and the functions $b(t)$ and $b_i(t)$ are bounded and monotonically increasing on $[0, T)$. Put

$$M_i(t) = \max_{0 \leq s \leq t} \left\{b_i(s) c(s)\right\}. \quad (23)$$

Suppose that

$$A(t) = a(t) + b(t)\int_0^t c(s)\, e^{\int_s^t c(w)b(w)dw} \cdot a(s)\, ds;$$

$$B_i(t) = b_i(t) + \frac{t \cdot b(t) \cdot M_i(t)}{\beta_i} e^{\int_0^t c(w)b(w)dw}, \quad i = 1, 2, \ldots, n. \tag{24}$$

Obviously, $B_i(t) \geq 0$ $(i = 1, 2, \ldots, n)$ are bounded and monotonically increasing functions. Moreover, we assumed that

$$\widetilde{A}(t) = A(t) + \sum_{i=1}^{n} (1-\lambda_i)\left(\frac{\lambda_i}{\varepsilon}\right)^{\lambda_i/(1-\lambda_i)} B_i(t) \times \int_0^t (t-s)^{\beta_i - 1}\left[c_i(s)\right]^{1/(1-\lambda_i)} ds; \tag{25}$$

$$\widetilde{B}_i(t) = \varepsilon B_i(t), \quad i = 1, 2, \ldots, n.$$

Here ε is an arbitrary given positive number. The third main result is given as follows.

Theorem 8. *For any $t \in [0, T)$, the nonnegative and continuous function $u(t)$ satisfies the inequality* (22)*; then*

$$u(t) \leq \widetilde{A}(t) + \sum_{k=1}^{\infty}\left(\sum_{1',2',\ldots,k'=1}^{n} \frac{\prod_{i=1}^{k}\left[\widetilde{B}_{i'}(t)\Gamma(\beta_{i'})\right]}{\Gamma\left(\sum_{i=1}^{k}\beta_{i'}\right)} \times \int_0^t (t-s)^{\sum_{i=1}^{k}\beta_{i'}-1}\, \widetilde{A}(s)\, ds\right), \quad t \in [0, T), \tag{26}$$

where the expressions $\widetilde{A}(t)$ and $\widetilde{B}_i(t)$ are described in (25).

Proof. Suppose that

$$I(t) = a(t) + \sum_{i=1}^{n} b_i(t)\int_0^t (t-s)^{\beta_i-1} c_i(s)\, u^{\lambda_i}(s)\, ds. \tag{27}$$

Then, (22) transforms into the following form:

$$u(t) \leq I(t) + b(t)\int_0^t c(s)\, u(s)\, ds. \tag{28}$$

Therefore,

$$c(t)\, u(t) \leq c(t)\, I(t) + c(t)\, b(t)\int_0^t c(s)\, u(s)\, ds. \tag{29}$$

Letting $u_1(t) = c(t)u(t)$, $I_1(t) = c(t)I(t)$, and $J_1(t) = c(t)b(t)$ obtains

$$u_1(t) \leq I_1(t) + J_1(t)\int_0^t u_1(s)\, ds. \tag{30}$$

By using the classical Gronwall inequality (see [10, page 15]), we have

$$u_1(t) \leq I_1(t) + J_1(t)\int_0^t I_1(s)\, e^{\int_s^t J_1(w)dw} ds; \tag{31}$$

that is,

$$\begin{aligned} u(t) &\leq I(t) + b(t)\int_0^t c(s)\, I(s)\, e^{\int_s^t c(w)b(w)dw} ds \\ &= \left[a(t) + b(t)\int_0^t c(s)\, a(s)\, e^{\int_s^t c(w)b(w)dw} ds\right] \\ &\quad + \sum_{i=1}^{n} b_i(t)\int_0^t (t-s)^{\beta_i-1} c_i(s)\, u^{\lambda_i}(s)\, ds \\ &\quad + b(t)\sum_{i=1}^{n}\int_0^t\int_0^s c(s)\, e^{\int_s^t c(w)b(w)dw} \\ &\qquad \times b_i(s)\,(s-v)^{\beta_i-1} c_i(v)\, u^{\lambda_i}(v)\, dv\, ds, \end{aligned} \tag{32}$$

given that

$$\begin{aligned} &\sum_{i=1}^{n}\int_0^t\int_0^s b_i(s)\, c(s)\, e^{\int_s^t c(w)b(w)dw} \\ &\qquad \times (s-v)^{\beta_i-1} c_i(v)\, u^{\lambda_i}(v)\, dv\, ds \\ &= \sum_{i=1}^{n}\int_0^t\int_v^t b_i(s)\, c(s)\, e^{\int_s^t c(w)b(w)dw} \\ &\qquad \times (s-v)^{\beta_i-1} c_i(v)\, u^{\lambda_i}(v)\, ds\, dv \\ &\leq e^{\int_0^t c(w)b(w)dw} \\ &\quad \times \sum_{i=1}^{n} M_i(t)\int_0^t\int_v^t (s-v)^{\beta_i-1} c_i(v)\, u^{\lambda_i}(v)\, ds\, dv \\ &= e^{\int_0^t c(w)b(w)dw}\sum_{i=1}^{n}\frac{1}{\beta_i} M_i(t) \\ &\qquad \times \int_0^t (t-v)^{\beta_i-1}(t-v)\, c_i(v)\, u^{\lambda_i}(v)\, dv \\ &\leq e^{\int_0^t c(w)b(w)dw}\sum_{i=1}^{n}\frac{1}{\beta_i} t M_i(t) \\ &\qquad \times \int_0^t (t-v)^{\beta_i-1} c_i(v)\, u^{\lambda_i}(v)\, dv, \end{aligned} \tag{33}$$

where the function $M_i(t)$ is defined in (23). Combining (24), (32), and (33) yields the following inequality:

$$u(t) \leq A(t) + \sum_{i=1}^{n} B_i(t)\int_0^t (t-s)^{\beta_i-1} c_i(s)\, u^{\lambda_i}(s)\, ds. \tag{34}$$

The estimation (26) is obtained according to Theorem 4. This process completes the proof of Theorem 8. □

In the same manner, the final result in this section can be obtained by applying the conclusion of Theorem 5.

Theorem 9. *For any $t \in [1, T)$,*

$$u(t) \le a(t) + b(t)\int_1^t c(s)\,u(s)\,\frac{ds}{s} + \sum_{i=1}^{n} b_i(t)\int_1^t \left(\ln\frac{t}{s}\right)^{\beta_i-1} c_i(s)\,u^{\lambda_i}(s)\,\frac{ds}{s}, \tag{35}$$

where all functions are nonnegative and continuous. For any $i \in \{1,2,\ldots,n\}$, the constants $\beta_i > 0$ and $0 < \lambda_i < 1$, and the functions $b(t)$ and $b_i(t)$ are bounded and monotonically increasing on $[1, T)$. Then

$$\begin{aligned} u(t) \le{}& \overline{A}(t) + \sum_{k=1}^{\infty}\left(\sum_{1',2',\ldots,k'=1}^{n} \frac{\prod_{i=1}^{k}\left[\overline{B}_{i'}(t)\,\Gamma(\beta_{i'})\right]}{\Gamma\left(\sum_{i=1}^{k}\beta_{i'}\right)} \times \int_1^t \left(\ln\frac{t}{s}\right)^{\sum_{i=1}^{k}\beta_{i'}-1}\frac{\overline{A}(s)}{s}ds\right), \\ & t \in [1, T), \end{aligned} \tag{36}$$

where

$$\begin{aligned} \overline{A}(t) &= A'(t) + \sum_{i=1}^{n}(1-\lambda_i)\left(\frac{\lambda_i}{\varepsilon}\right)^{\lambda_i/(1-\lambda_i)} B_i'(t) \times \int_1^t \left(\ln\frac{t}{s}\right)^{\beta_i-1}\left[c_i(s)\right]^{1/(1-\lambda_i)}\frac{ds}{s}; \\ \overline{B}_i(t) &= \varepsilon\cdot B_i'(t). \end{aligned} \tag{37}$$

Here

$$\begin{aligned} A'(t) &= a(t) + b(t)\int_1^t c(s)\,e^{\int_s^t c(w)b(w)(dw/w)}\cdot a(s)\,\frac{ds}{s}; \\ B_i'(t) &= b_i(t) + \frac{\ln t\cdot b(t)\cdot M_i'(t)}{\beta_i}e^{\int_1^t c(w)b(w)(dw/w)}, \\ & i = 1,2,\ldots,n. \end{aligned} \tag{38}$$

Also $M_i'(t) = \max_{1\le s\le t}\{b_i(s)c(s)\}$, and ε is an arbitrary given positive number.

The proof procedure of Theorem 9 is relatively similar to that of Theorem 8. Hence, the procedure will not be presented in this paper.

3. Applications

In this section, we apply the main results in Section 2 to provide qualitative conclusions for solutions of certain fractional differential equations. First, the definitions and some properties of the Riemann-Liouville fractional derivative and integral need to be recalled.

Definition 10 (see [11–16]). For any $0 < \beta < 1$ and a continuous function w, the βth Riemann-Liouville type fractional order derivative $D_R^{\beta}w$ and the corresponding fractional integral operator $I_R^{\beta}w$ are defined by

$$\begin{aligned} I_R^{\beta}w(t) &= \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}w(s)\,ds; \\ D_R^{\beta}w(t) &= \frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\int_0^t (t-s)^{-\beta}w(s)\,ds. \end{aligned} \tag{39}$$

Lemma 11 (see [13, Lemma 2.2]). *For any $0 < \beta < 1$ and a continuous function w,*

$$I_R^{\beta}D_R^{\beta}w(t) = w(t) + k\cdot t^{\beta-1}, \tag{40}$$

where k is a certain constant in $\mathbb{R}$.

Lemma 12 (see [17, Page 14]). *For any $\alpha, \beta > 0$ and a continuous function w,*

$$\begin{aligned} I_R^{\alpha}I_R^{\beta}w(t) &= I_R^{\alpha+\beta}w(t); \\ I_R^{\alpha}t^{\beta} &= \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}t^{\alpha+\beta}. \end{aligned} \tag{41}$$

Given the aforementioned preliminary knowledge, we consider the following initial value problem:

$$\begin{aligned} & \sum_{i=1}^{n} D_R^{\beta_{n+1}-\beta_i}\left[c_i(t)\,u^{\lambda_i}(t)\right] + D_R^{\beta_{n+1}}\left[u^{\lambda_0}(t)\right] = f(t, u(t)); \\ & \left.\left\{\sum_{i=1}^{n} I_R^{1+\beta_i-\beta_{n+1}}\left[c_i(t)\,u^{\lambda_i}(t)\right] + I_R^{1-\beta_{n+1}}\left[u^{\lambda_0}(t)\right]\right\}\right|_{t=0} = \delta, \end{aligned} \tag{42}$$

where all functions are continuous. Moreover $c_i(t) > 0$ and the constants $\lambda_i, \beta_i > 0$ $(i = 1,2,\ldots,n)$. Consider $\lambda_0 > 0$, $\delta \in \mathbb{R}$, and $1 > \beta_{n+1} > \max_{1\le i\le n}\{\beta_i\}$.

Theorem 13. *Suppose that, for any $t \in [0, T]$ and $y, z \in \mathbb{R}$,*

$$\left|f(t,y) - f(t,z)\right| \le c_{n+1}(t)\left|y^{\lambda_{n+1}} - z^{\lambda_{n+1}}\right|, \tag{43}$$

where the function $c_{n+1}(t) > 0$ is continuous and the constant $\lambda_{n+1} \in (0,1)$.

(i) *If* $\lambda_0 > \max_{1\le i\le n+1}\{\lambda_i\}$, *then for any solution* $u(t)$ *of the problem* (42),

$$|u(t)| \le \left\{ \widetilde{A}_0(t) + \sum_{k=1}^{\infty} \left(\sum_{1',2',\dots,k'=1}^{n+1} \frac{\varepsilon^k}{\Gamma\left(\sum_{i=1}^{k} \beta_{i'}\right)} \times \int_0^t (t-s)^{\sum_{i=1}^{k}\beta_{i'}-1} \widetilde{A}_0(s)\, ds \right) \right\}^{1/\lambda_0}, \quad t\in[0,T], \tag{44}$$

where

$$\widetilde{A}_0(t) = \frac{|\delta|\, t^{\beta_{n+1}-1}}{\Gamma(\beta_{n+1})} + \frac{1}{\Gamma(\beta_{n+1})}\int_0^t (t-s)^{\beta_{n+1}-1} \left|f(s,0)\right| ds + \sum_{i=1}^{n+1} \frac{1}{\Gamma(\beta_i)}\left(1-\frac{\lambda_i}{\lambda_0}\right)\left(\frac{\lambda_i}{\varepsilon\lambda_0}\right)^{\lambda_i/(\lambda_0-\lambda_i)} \times \int_0^t (t-s)^{\beta_i-1}\left[c_i(s)\right]^{\lambda_0/(\lambda_0-\lambda_i)} ds. \tag{45}$$

Here ε *is an arbitrary given positive number.*

(ii) *If* $\lambda_0 \le \min_{1\le i\le n+1}\{\lambda_i\}$, *then the continuous solution of problem* (42) *is unique.*

Proof. Since $1 > \beta_{n+1} > \max_{1\le i\le n}\{\beta_i\}$, we obtain the following by using Lemmas 11 and 12:

$$\begin{aligned} I_R^{\beta_{n+1}} f(t,u(t)) &= I_R^{\beta_{n+1}} D_R^{\beta_{n+1}}\left[u^{\lambda_0}(t)\right] + \sum_{i=1}^{n} I_R^{\beta_{n+1}} D_R^{\beta_{n+1}-\beta_i}\left[c_i(t)\, u^{\lambda_i}(t)\right] \\ &= I_R^{\beta_{n+1}} D_R^{\beta_{n+1}}\left[u^{\lambda_0}(t)\right] + \sum_{i=1}^{n} I_R^{\beta_i}\left(I_R^{\beta_{n+1}-\beta_i} D_R^{\beta_{n+1}-\beta_i}\left[c_i(t)\, u^{\lambda_i}(t)\right]\right) \\ &= u^{\lambda_0}(t) + k_{n+1} t^{\beta_{n+1}-1} + \sum_{i=1}^{n} I_R^{\beta_i}\left(c_i(t)\, u^{\lambda_i}(t) + k_i t^{\beta_{n+1}-\beta_i-1}\right) \\ &= u^{\lambda_0}(t) + \sum_{i=1}^{n} I_R^{\beta_i}\left[c_i(t)\, u^{\lambda_i}(t)\right] + \frac{t^{\beta_{n+1}-1}}{\Gamma(\beta_{n+1})}\left\{k_{n+1}\Gamma(\beta_{n+1}) + \sum_{i=1}^{n} k_i \Gamma(\beta_{n+1}-\beta_i)\right\}, \end{aligned} \tag{46}$$

where k_i, $i = 1,2,\dots,n+1$, are some constants. Therefore,

$$\begin{aligned} 0 &= \left\{\int_0^t f(s,u(s))\, ds\right\}\Big|_{t=0} \\ &= \left\{I_R^{1-\beta_{n+1}} I_R^{\beta_{n+1}} f(t,u(t))\right\}\Big|_{t=0} \\ &= \left\{\sum_{i=1}^{n} I_R^{1+\beta_i-\beta_{n+1}}\left[c_i(t)\, u^{\lambda_i}(t)\right] + I_R^{1-\beta_{n+1}}\left[u^{\lambda_0}(t)\right]\right\}\Bigg|_{t=0} + k_{n+1}\Gamma(\beta_{n+1}) + \sum_{i=1}^{n} k_i\Gamma(\beta_{n+1}-\beta_i) \\ &= \delta + k_{n+1}\Gamma(\beta_{n+1}) + \sum_{i=1}^{n} k_i\Gamma(\beta_{n+1}-\beta_i); \end{aligned} \tag{47}$$

that is,

$$k_{n+1}\Gamma(\beta_{n+1}) + \sum_{i=1}^{n} k_i\Gamma(\beta_{n+1}-\beta_i) = -\delta. \tag{48}$$

Integrating this equality into (46) obtains

$$\begin{aligned} u^{\lambda_0}(t) &= I_R^{\beta_{n+1}} f(t,u(t)) - \sum_{i=1}^{n} I_R^{\beta_i}\left[c_i(t)\, u^{\lambda_i}(t)\right] + \frac{\delta t^{\beta_{n+1}-1}}{\Gamma(\beta_{n+1})} \\ &= \frac{1}{\Gamma(\beta_{n+1})}\int_0^t (t-s)^{\beta_{n+1}-1} f(s,u(s))\, ds + \frac{\delta t^{\beta_{n+1}-1}}{\Gamma(\beta_{n+1})} - \sum_{i=1}^{n} \frac{1}{\Gamma(\beta_i)}\int_0^t (t-s)^{\beta_i-1} c_i(s)\, u^{\lambda_i}(s)\, ds. \end{aligned} \tag{49}$$

Let $U(t) = u^{\lambda_0}(t)$, and applying (49), given the fact that

$$\begin{aligned} \left|f(s,u(s))\right| &\le \left|f(s,0)\right| + \left|f(s,u(s)) - f(s,0)\right| \\ &\le \left|f(s,0)\right| + c_{n+1}(s)\, |u(s)|^{\lambda_{n+1}}, \end{aligned} \tag{50}$$

obtains the following:

$$\begin{aligned}
|U(t)| &= |u(t)|^{\lambda_0} \\
&\leq \frac{1}{\Gamma(\beta_{n+1})}\int_0^t (t-s)^{\beta_{n+1}-1}\left|f(s,u(s))\right| ds + \frac{|\delta|\, t^{\beta_{n+1}-1}}{\Gamma(\beta_{n+1})} \\
&\quad + \sum_{i=1}^{n}\frac{1}{\Gamma(\beta_i)}\int_0^t (t-s)^{\beta_i-1} c_i(s)\,|u(s)|^{\lambda_i} ds \\
&\leq \frac{1}{\Gamma(\beta_{n+1})}\int_0^t (t-s)^{\beta_{n+1}-1}\left|f(s,0)\right| ds + \frac{|\delta|\, t^{\beta_{n+1}-1}}{\Gamma(\beta_{n+1})} \\
&\quad + \sum_{i=1}^{n+1}\frac{1}{\Gamma(\beta_i)}\int_0^t (t-s)^{\beta_i-1} c_i(s)\,|u(s)|^{\lambda_i} ds \\
&= \left\{\frac{1}{\Gamma(\beta_{n+1})}\int_0^t (t-s)^{\beta_{n+1}-1}\left|f(s,0)\right| ds + \frac{|\delta|\, t^{\beta_{n+1}-1}}{\Gamma(\beta_{n+1})}\right\} \\
&\quad + \sum_{i=1}^{n+1}\frac{1}{\Gamma(\beta_i)}\int_0^t (t-s)^{\beta_i-1} c_i(s)\,|U(s)|^{\lambda_i/\lambda_0} ds.
\end{aligned} \tag{51}$$

If $\lambda_0 > \max_{1\leq i\leq n+1}\{\lambda_i\}$, then, for any $i \in \{1,2,\ldots,n+1\}$, $0 < \lambda_i/\lambda_0 < 1$. According to Theorem 4, for any $t \in [0,T]$,

$$\begin{aligned}
|U(t)| &\leq \widetilde{A}_0(t) \\
&\quad + \sum_{k=1}^{\infty}\left(\sum_{1',2',\ldots,k'=1}^{n+1}\frac{\varepsilon^k}{\Gamma\left(\sum_{i=1}^{k}\beta_{i'}\right)}\right. \\
&\qquad \left.\times \int_0^t (t-s)^{\sum_{i=1}^{k}\beta_{i'}-1}\widetilde{A}_0(s)\,ds\right),
\end{aligned} \tag{52}$$

where the expression of $\widetilde{A}_0(t)$ is shown in (45). Hence, the conclusion of (i) is derived.

In proving (ii), we assume that problem (42) has two continuous solutions u and v. Combining with the fact that $c_i(t) \in C[0,T]$ for any $1 \leq i \leq n+1$ and the boundedness of the continuous function on a closed interval, there exists a finite number M which satisfies that, for any $t \in [0,T]$,

$$M > \max\left\{|u(t)|, |v(t)|, \max_{1\leq i\leq n+1}|c_i(t)|\right\}. \tag{53}$$

Cauchy's mean value theorem provides

$$\begin{aligned}
\left|u^{\lambda_i}(t) - v^{\lambda_i}(t)\right| &= \left|u^{\lambda_0}(t) - v^{\lambda_0}(t)\right| \cdot \left|\frac{\lambda_i \cdot \xi_i^{\lambda_i-1}}{\lambda_0 \cdot \xi_i^{\lambda_0-1}}\right| \\
&= \frac{\lambda_i}{\lambda_0}\left|u^{\lambda_0}(t) - v^{\lambda_0}(t)\right| \cdot \left|\xi_i\right|^{\lambda_i-\lambda_0},
\end{aligned} \tag{54}$$

where ξ_i, $i = 1,2,\ldots,n+1$, are the numbers between $u(t)$ and $v(t)$. The following estimation is deduced by applying (53) and the hypothesis of $\lambda_0 \leq \min_{1\leq i\leq n+1}\{\lambda_i\}$ in (ii):

$$\left|u^{\lambda_i}(t) - v^{\lambda_i}(t)\right| \leq \frac{\lambda_i}{\lambda_0}\left|u^{\lambda_0}(t) - v^{\lambda_0}(t)\right| \cdot M^{\lambda_i-\lambda_0}, \tag{55}$$

holds for any $t \in [0,T]$ and $i = 1,2,\ldots,n+1$. Therefore, (49), (53), and (55) obtain

$$\begin{aligned}
&\left|u^{\lambda_0}(t) - v^{\lambda_0}(t)\right| \\
&\quad = \left|\frac{1}{\Gamma(\beta_{n+1})}\int_0^t (t-s)^{\beta_{n+1}-1}\right. \\
&\qquad \times \left[f(s,u(s)) - f(s,v(s))\right] ds \\
&\qquad - \sum_{i=1}^{n}\frac{1}{\Gamma(\beta_i)}\int_0^t (t-s)^{\beta_i-1} c_i(s) \\
&\qquad \left.\times \left[u^{\lambda_i}(s) - v^{\lambda_i}(s)\right] ds\right| \\
&\quad \leq \sum_{i=1}^{n+1}\frac{1}{\Gamma(\beta_i)}\int_0^t (t-s)^{\beta_i-1} \\
&\qquad \times c_i(s)\left|u^{\lambda_i}(s) - v^{\lambda_i}(s)\right| ds \\
&\quad \leq \sum_{i=1}^{n+1}\frac{\lambda_i M^{1+\lambda_i-\lambda_0}}{\lambda_0\Gamma(\beta_i)}\int_0^t (t-s)^{\beta_i-1} \\
&\qquad \times \left|u^{\lambda_0}(s) - v^{\lambda_0}(s)\right| ds.
\end{aligned} \tag{56}$$

According to Theorem 2,

$$\left|u^{\lambda_0}(t) - v^{\lambda_0}(t)\right| \leq 0, \tag{57}$$

which means that

$$u(t) = v(t), \quad t \in [0,T]. \tag{58}$$

This completes the proof of (ii). □

Moreover, we can also address the following initial value problem with the Hadamard type fractional derivative:

$$\begin{aligned}
&\sum_{i=1}^{n} D_H^{\alpha_{n+1}-\alpha_i}\left[d_i(t)\,u^{\gamma_i}(t)\right] + D_H^{\alpha_{n+1}}\left[u^{\gamma_0}(t)\right] = g(t,u(t)); \\
&\left\{\sum_{i=1}^{n} I_H^{1+\alpha_i-\alpha_{n+1}}\left[d_i(t)\,u^{\gamma_i}(t)\right] + I_H^{1-\alpha_{n+1}}\left[u^{\gamma_0}(t)\right]\right\}\bigg|_{t=1} = \eta,
\end{aligned} \tag{59}$$

where all functions are continuous. $d_i(t) > 0$ and the constants $\gamma_i, \alpha_i > 0$ $(i = 1,2,\ldots,n)$. Also $\gamma_0 > 0$, $\eta \in \mathbb{R}$, and $1 > \alpha_{n+1} > \max_{1\leq i\leq n}\{\alpha_i\}$. For any $\alpha \in (0,1)$ and a continuous function w, the operators D_H^{α} and I_H^{α} are presented below (see [18, page 110]):

$$\begin{aligned}
D_H^{\alpha}w(t) &= \frac{1}{\Gamma(1-\alpha)}\left(t\frac{d}{dt}\right)\int_1^t \left(\ln\frac{t}{s}\right)^{-\alpha}\frac{w(s)}{s}ds; \\
I_H^{\alpha}w(t) &= \frac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{w(s)}{s}ds.
\end{aligned} \tag{60}$$

We are able to deduce the following result by Theorems 3 and 5.

Theorem 14. *Suppose that, for any $t \in [1,T]$ and $y,z \in \mathbb{R}$, $|g(t,y) - g(t,z)| \le d_{n+1}(t)|y^{\gamma_{n+1}} - z^{\gamma_{n+1}}|$, where the function $d_{n+1}(t) > 0$ is continuous and the constant $\gamma_{n+1} \in (0,1)$.*

(i) *If $\gamma_0 > \max_{1\le i\le n+1}\{\gamma_i\}$, then, for any solution $u(t)$ of problem* (59),

$$|u(t)| \le \left\{ \overline{A}_0(t) + \sum_{k=1}^{\infty} \left(\sum_{1',2',\ldots,k'=1}^{n+1} \frac{\varepsilon^k}{\Gamma\left(\sum_{i=1}^{k} \alpha_{i'}\right)} \times \int_1^t \left(\ln\frac{t}{s}\right)^{\sum_{i=1}^{k}} \overline{A}_0(s)\frac{ds}{s} \right) \right\}^{1/\gamma_0}, \quad t \in [1,T], \tag{61}$$

where

$$\begin{aligned} \overline{A}_0(t) &= \frac{|\eta| (\ln t)^{\alpha_{n+1}-1}}{\Gamma(\alpha_{n+1})} + \frac{1}{\Gamma(\alpha_{n+1})} \int_1^t \left(\ln\frac{t}{s}\right)^{\alpha_{n+1}-1} |g(s,0)| \frac{ds}{s} \\ &\quad + \sum_{i=1}^{n+1} \frac{1}{\Gamma(\alpha_i)} \left(1 - \frac{\gamma_i}{\gamma_0}\right) \left(\frac{\gamma_i}{\varepsilon\gamma_0}\right)^{\gamma_i/(\gamma_0-\gamma_i)} \times \int_1^t \left(\ln\frac{t}{s}\right)^{\alpha_i - 1} [d_i(s)]^{\gamma_0/(\gamma_0-\gamma_i)} \frac{ds}{s}. \end{aligned} \tag{62}$$

Here ε is an arbitrary given positive number.

(ii) *If $\gamma_0 \le \min_{1\le i\le n+1}\{\gamma_i\}$, then the continuous solution of problem* (59) *is unique.*

The proof procedure is similar to that of Theorem 13. Thus, the procedure is omitted here.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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