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Research Article

Existence of Positive Solutions to Nonlinear Fractional Boundary Value Problem with Changing Sign Nonlinearity and Advanced Arguments

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We discuss the existence of positive solutions to a class of fractional boundary value problem with changing sign nonlinearity and advanced arguments $D^{\alpha}x(t)+\mu h(t)f(x(a(t)))=0, t\in(0,1), 2<\alpha\le 3, \mu>0, x(0)=x'(0)=0, x(1)=\beta x(\eta)+\lambda[x], \beta>0$, and $\eta\in(0,1)$, where D^{α} is the standard Riemann-Liouville derivative, $f:[0,\infty)\to[0,\infty)$ is continuous, $f(0)>0, h:[0,1]\to(-\infty,+\infty)$, and $a(t)$ is the advanced argument. Our analysis relies on a nonlinear alternative of Leray-Schauder type. An example is given to illustrate our results.

1. Introduction

Fractional differential equations (FDEs) have been of great interest for the past three decades. It is caused both by the intensive development of the theory of fractional calculus itself and by the applications of such constructions in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, and so forth (see [1, 2]). Therefore, the theory of FDEs has been developed very quickly. There has been a significant development in fractional differential equations in recent years; see [1–30].

In [5], the author studied existence of positive solutions in case of the nonlinear fractional differential equation as follows:

$$\begin{gathered} D^{s}u=\lambda a(t)f(u), \quad 0<t<1, \\ u(0)=0, \end{gathered} \tag{1}$$

where $0<s<1$, D^{s} is the standard *Riemann-Liouville* fractional derivative, $f:[0,\infty)\to[0,\infty)$ is continuous, and $a:[0,1]\to R$. In [10], the author applied the *Avery-Peterson* fixed point theorem to obtain sufficient conditions of the existence of multiple solutions to the following problem:

$$\begin{gathered} x'''(t)+h(t)f(x(\alpha(t)))=0, \quad t\in(0,1), \\ x(0)=x''(0)=0, \\ x(1)=\beta x(\eta)+\lambda[x], \quad \beta>0,\ \eta\in(0,1), \end{gathered} \tag{2}$$

where $f:[0,\infty)\to[0,\infty)$ is continuous and $h(t)$ is a nonnegative continuous function defined on $[0,1]$.

Motivated by [5, 10], in this paper, we consider the existence of positive solution of the following boundary value problem for nonlinear fractional differential equation with changing sign nonlinearity and advanced arguments:

$$\begin{gathered} D^{\alpha}x(t)+\mu h(t)f(x(a(t)))=0, \\ t\in(0,1), \quad 2<\alpha\le 3, \quad \mu>0, \\ x(0)=x'(0)=0, \\ x(1)=\beta x(\eta)+\lambda[x], \quad \beta>0,\ \eta\in(0,1), \end{gathered} \tag{3}$$

where λ denotes a linear functional on $C[0,1]$ given by $\lambda[x]=\int_0^1 x(t)d\Lambda(t)$ involving a Stieltjes integral with a suitable function Λ of bounded variation. It is important to indicate that we did not assume that $\lambda[x]$ is positive to all positive x. The measure $d\Lambda$ can be a signed measure.

Put $J = [0,1]$; let us introduce the following assumptions:

(H_1) $f : [0,\infty) \to [0,\infty)$ is continuous, and $f(0) > 0$;

(H_2) $a \in C(J,J)$, and $t \le a(t)$ on J;

(H_3) $h : [0,1] \to (-\infty,+\infty)$ may change sign; h is not identically zero on any subinterval on J;

(H_4) $0 < \beta\eta^{\alpha-1} + \lambda[p] < 1$, where $p(t) = t^{\alpha-1}$.

2. Basic Definitions and Preliminaries

In this section, we present some preliminaries and lemmas that are useful to the proof of our main results. For convenience, we also present the necessary definitions from fractional calculus theory here. These definitions can be found in the recent literature.

Definition 1. The fractional integral of order $\alpha > 0$ of a function $x : (0,+\infty) \to R$ is given by

$$I_{0+}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(t)\, ds, \tag{4}$$

provided that the right-hand side is pointwise defined on $(0,+\infty)$.

Definition 2. The fractional derivative of order $\alpha > 0$ of a continuous function $x : (0,+\infty) \to R$ is given by

$$D_{0+}^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} x(t)\, ds, \tag{5}$$

where $n = [\alpha]+1$ and $[\alpha]$ denotes the integral part of number α, provided that the right-hand side is pointwise defined on $(0,+\infty)$.

Lemma 3. *Let $\alpha > 0$, $x \in C(0,1) \cap L(0,1)$; then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} x(t) = x(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \tag{6}$$

where $c_i \in R$ ($i = 1,2,\ldots,n$), n being the smallest integer greater than or equal to α.

Consider the following boundary value problem:

$$\begin{gathered} D^{\alpha} x(t) + y(t) = 0, \quad t \in (0,1),\ 2 < \alpha \le 3, \\ x(0) = x'(0) = 0, \\ x(1) = \beta x(\eta) + \lambda[x], \quad \beta > 0,\ \eta \in (0,1). \end{gathered} \tag{7}$$

Lemma 4. *Assume that $\beta\eta^{\alpha-1} \ne 1$ and $y \in C(J,R)$; then problem (7) has the unique solution given by the following formula:*

$$\begin{aligned} x(t) = {} & \frac{t^{\alpha-1}}{1-\beta\eta^{\alpha-1}} \lambda[x] + \frac{\beta t^{\alpha-1}}{1-\beta\eta^{\alpha-1}} \int_0^1 k(\eta,s) y(s)\, ds \\ & + \int_0^1 k(t,s) y(s)\, ds, \end{aligned} \tag{8}$$

where

$$k(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} [t(1-s)]^{\alpha-1}, & 0 \le t \le s \le 1, \\ [t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1. \end{cases} \tag{9}$$

Theorem 5. *Let X be a Banach space with $C \subset X$ closed and convex. Assume that U is a relatively open subset of C with $0 \in C$ and $A : \overline{U} \to C$ is a continuous, compact map. Then either*

(i) *A has a fixed point in $\overline{U}$ or*

(ii) *there exist $u \in \partial U$ and $\tau \in (0,1)$ with $u = \tau A u$.*

3. Existence of Positive Solutions

Let us denote by $X = C[0,1]$ the Banach space of all continuous real functions on $[0,1]$ endowed with the sup norm and let K be the cone:

$$K = \{x \in X, x(t) \ge 0, t \in J\}. \tag{10}$$

Lemma 6. *Let assumptions (H_1)–(H_4) hold. Moreover, we assume that assumptions (H_5)-(H_6) hold with*

(H_5) *$\int_0^1 d\Lambda(t) \ge 0$, $\int_0^1 t^{\alpha-1} d\Lambda(t) \ge 0$ and $\kappa(s) = \int_0^1 k(t,s) d\Lambda(t) \ge 0$,*

(H_6) *$h : [0,1] \to (-\infty,+\infty)$ is continuous, $h(0) \ne 0$, and there is $\sigma > 1$ such that*

$$\begin{aligned} & \frac{t^{\alpha-1}}{\Delta-\rho} \left(\frac{\beta\rho}{\Delta} \int_0^1 k(\eta,s) h^+(s)\, ds + \int_0^1 \kappa(s) h^+(s)\, ds \right) \\ & \quad + \int_0^1 k(t,s) h^+(s)\, ds + \frac{\beta t^{\alpha-1}}{\Delta} \int_0^1 k(\eta,s) h^+(s)\, ds \\ & \ge \sigma \left[\frac{t^{\alpha-1}}{\Delta-\rho} \left(\frac{\beta\rho}{\Delta} \int_0^1 k(\eta,s) h^-(s)\, ds + \int_0^1 \kappa(s) h^-(s)\, ds \right) \right. \\ & \quad \left. + \int_0^1 k(t,s) h^-(s)\, ds + \frac{\beta t^{\alpha-1}}{\Delta} \int_0^1 k(\eta,s) h^-(s)\, ds \right], \end{aligned} \tag{11}$$

where $\Delta = 1 - \beta\eta^{\alpha-1}$, $\rho = \lambda[p]$, $h^+(t) = \max\{0, h(t)\}$ and $h^-(t) = \max\{0, -h(t)\}$. Then, for every $0 < \delta < 1$, there exists a positive number $\overline{\mu}$ such that, for $0 < \mu < \overline{\mu}$, the nonlinear fractional differential equation,

$$\begin{gathered} D^{\alpha} x(t) + \mu h^+(t) f(x(a(t))) = 0, \\ t \in (0,1), \quad 2 < \alpha \le 3, \quad \mu > 0, \\ x(0) = x'(0) = 0, \\ x(1) = \beta x(\eta) + \lambda[x], \quad \beta > 0,\ \eta \in (0,1), \end{gathered} \tag{12}$$

has a positive solution $\overline{x}_{\mu}$ with $\|\overline{x}_{\mu}\| \to 0$ as $\mu \to 0$ and

$$\overline{x}_{\mu}(t) \ge \mu\delta f(0) m(t), \tag{13}$$

where

$$m(t) = \frac{t^{\alpha-1}}{\Delta-\rho}\left(\frac{\beta\rho}{\Delta}\int_0^1 k(\eta,s)h^+(s)\,ds + \int_0^1 \kappa(s)h^+(s)\,ds\right) + \frac{\beta t^{\alpha-1}}{\Delta}\int_0^1 k(\eta,s)h^+(s)\,ds + \int_0^1 k(t,s)h^+(s)\,ds. \tag{14}$$

Proof. It is easy to know from (9), (H_5), and (H_6) that $m(t) > 0$, $t \in (0,1]$. By Lemma 4, (12) has a unique solution in X:

$$x(t) = \frac{t^{\alpha-1}}{1-\beta\eta^{\alpha-1}}\lambda[x] + \frac{\beta t^{\alpha-1}}{1-\beta\eta^{\alpha-1}}\mu\int_0^1 k(\eta,s)h^+(s)f(x(a(s)))\,ds + \mu\int_0^1 k(t,s)h^+(s)f(x(a(s)))\,ds. \tag{15}$$

For $x \in C(J,R_+)$, we define two operators T and S by

$$Tx(t) = \frac{t^{\alpha-1}}{\Delta}\lambda[x] + \mu Fx(t), \qquad Sx(t) = \frac{t^{\alpha-1}}{\Delta-\rho}\mu\lambda[Fx] + \mu Fx(t), \tag{16}$$

where

$$Fx(t) = \frac{\beta t^{\alpha-1}}{\Delta}\int_0^1 k(\eta,s)h^+(s)f(x(a(s)))\,ds + \int_0^1 k(t,s)h^+(s)f(x(a(s)))\,ds,$$
$$\lambda[Fx] = \frac{\beta\rho}{\Delta}\int_0^1 k(\eta,s)h^+(s)f(x(a(s)))\,ds + \int_0^1 \kappa(s)h^+(s)f(x(a(s)))\,ds. \tag{17}$$

It is easy to show that $T: K \to K$ and $S: K \to K$ are completely continuous. We claim that operators T and S have the same fixed points in K. In fact, let $x = Sx$; then

$$\lambda[x] = \frac{\rho}{\Delta-\rho}\mu\lambda[Fx] + \mu\lambda[Fx] = \frac{\Delta}{\Delta-\rho}\mu\lambda[Fx]. \tag{18}$$

So

$$x(t) = Sx(t) = \frac{t^{\alpha-1}}{\Delta-\rho}\mu\lambda[Fx] + \mu Fx(t) = \frac{t^{\alpha-1}}{\Delta}\lambda[x] + \mu Fx(t) = Tx(t). \tag{19}$$

Let $x = Tx$; then $\lambda[x] = (\rho/\Delta)\lambda[x] + \mu\lambda[Fx]$. So $\lambda[x] = (\Delta/(\Delta-\rho))\mu\lambda[Fx]$, and hence

$$x(t) = Tx(t) = \frac{t^{\alpha-1}}{\Delta}\lambda[x] + \mu Fx(t) = \frac{t^{\alpha-1}}{\Delta-\rho}\mu\lambda[Fx] + \mu Fx(t) = Sx(t). \tag{20}$$

This shows that fixed points of S are solutions of (12). We will apply the nonlinear alternative of Leray-Schauder type to prove that S has at least one fixed point for small μ.

Let $\epsilon > 0$ be such that

$$f(x(a(t))) \geq \delta f(0), \quad 0 \leq x(a(t)) \leq \epsilon, \quad \forall t \in [0,1]. \tag{21}$$

Suppose that $0 < \mu < \epsilon/2\|m\|\overline{f}(\epsilon) := \overline{\mu}$, where $\overline{f}(t) = \max_{0\leq s\leq t} f(s)$; then

$$\overline{f}(\|x\|) = \max_{0\leq|x(a(t))|\leq\|x\|} f(x(a(t))), \quad \forall t \in [0,1]. \tag{22}$$

Since $\lim_{t\to 0^+}(\overline{f}(t)/t) = +\infty$, $\overline{f}(\epsilon)/\epsilon < 1/2\mu\|m\|$, there exists a unique $R_\mu \in (0,\epsilon)$ such that

$$\frac{\overline{f}(R_\mu)}{R_\mu} = \frac{1}{2\mu\|m\|}. \tag{23}$$

Let $x \in K$ and $\tau \in (0,1)$ be such that $x = \tau Sx$. We claim that $\|x\| \neq R_\mu$. In fact,

$$\begin{aligned} x(t) &= \frac{\tau t^{\alpha-1}}{\Delta-\rho}\mu\left(\frac{\beta\rho}{\Delta}\int_0^1 k(\eta,s)h^+(s)f(x(a(s)))\,ds + \int_0^1 \kappa(s)h^+(s)f(x(a(s)))\,ds\right) \\ &\quad + \tau\mu\left(\frac{\beta t^{\alpha-1}}{\Delta}\int_0^1 k(\eta,s)h^+(s)f(x(a(s)))\,ds + \int_0^1 k(t,s)h^+(s)f(x(a(s)))\,ds\right) \\ &\leq \frac{t^{\alpha-1}}{\Delta-\rho}\mu\overline{f}(\|x\|) \times\left(\int_0^1 k(\eta,s)h^+(s)\,ds + \int_0^1 \kappa(s)h^+(s)\,ds\right) \\ &\quad + \mu\overline{f}(\|x\|) \times\left(\frac{\beta t^{\alpha-1}}{\Delta}\int_0^1 k(\eta,s)h^+(s)\,ds + \int_0^1 k(t,s)h^+(s)\,ds\right) \\ &= \mu\overline{f}(\|x\|)m(t) \leq \mu\overline{f}(\|x\|)\|m\|. \end{aligned} \tag{24}$$

That is, $\overline{f}(\|x\|)/\|x\| \geq 1/\mu\|m\|$, which implies that $\|x\| \neq R_\mu$. Let $U = \{x \in K : \|x\| < R_\mu\}$. By Theorem 5, S has a fixed point $\overline{x}_\mu \in \overline{U}$. Moreover, combining (21) with the expression of operator S, we obtain that

$$\overline{x}_\mu(t) \geq \mu\delta f(0)m(t), \quad \forall t \in (0,1]. \tag{25}$$

Hence (12) has a positive solution $\overline{x}_\mu(t)$. Note that $R_\mu \to 0$ as $\mu \to 0$; we get that $\|\overline{x}_\mu\| \to 0$ as $\mu \to 0$. □

Theorem 7. *Suppose that* (H_1)–(H_6) *hold. Then there exists a positive number* $\mu^* > 0$ *such that* (3) *has at least one positive solution for* $\mu \in (0, \mu^*)$.

Proof. Let

$$\omega(t) = \frac{t^{\alpha-1}}{\Delta-\rho}\left(\frac{\beta\rho}{\Delta}\int_0^1 k(\eta,s)h^-(s)\,ds + \int_0^1 \kappa(s)h^-(s)\,ds\right) + \frac{\beta t^{\alpha-1}}{\Delta}\int_0^1 k(\eta,s)h^-(s)\,ds + \int_0^1 k(t,s)h^-(s)\,ds. \tag{26}$$

Then $\omega(t) \geq 0$ for each $t \in (0,1]$. We have $m(t) \geq \sigma\omega(t), \sigma > 1$. Choose $c \in (0,1)$ such that $\sigma c > 1$. There is $b > 0$ such that $f(x(a(t))) \leq \sigma c f(0)$ for $x \in [0,b]$; then

$$\omega(t) f(x(a(t))) \leq cm(t) f(0) \quad \text{for } t \in (0,1],\ x \in [0,b]. \tag{27}$$

Fix $\delta \in (c,1)$, and let $\mu^* > 0$ be such that

$$\left\|\overline{x}_\mu\right\| + \mu\delta f(0)\|m\| \leq b, \quad \mu \in (0,\mu^*), \tag{28}$$

where $\overline{x}_\mu$ is given by Lemma 6, and

$$\left|f(x_1(a(t))) - f(x_2(a(t)))\right| \leq f(0)\frac{\delta-c}{2}, \tag{29}$$

for $x_1, x_2 \in [0,b]$ with $|x_1 - x_2| \leq \mu^*\delta f(0)\|m\|$.

Let $\mu \in (0,\mu^*)$. We look for a solution x_μ of the form $\overline{x}_\mu + v_\mu$, where $\overline{x}_\mu$ is the solution of (12), given by Lemma 6. Thus v_μ solves the following equation:

$$\begin{gathered} D^\alpha v_\mu = \mu h^+(t)\left(f_1' - f_2'\right) - \mu h^-(t) f_1', \\ v_\mu(0) = v_\mu'(0) = 0, \\ v_\mu(1) = \beta v_\mu(\eta) + \lambda\left[v_\mu\right], \end{gathered} \tag{30}$$

where $f_1' = f(\overline{x}_\mu(a(t)) + v_\mu(a(t))), f_2' = f(\overline{x}_\mu(a(t)))$.

Now, we need to prove the existence of v_μ. Consider the following equation:

$$\begin{gathered} D^\alpha v = \mu h^+(t)\left(f_1 - f_2\right) - \mu h^-(t) f_1, \\ v(0) = v'(0) = 0, \\ v(1) = \beta v(\eta) + \lambda[v], \end{gathered} \tag{31}$$

where

$$f_1 = f\left(\overline{x}_\mu(a(t)) + v(a(t))\right), \qquad f_2 = f\left(\overline{x}_\mu(a(t))\right). \tag{32}$$

Obviously, (31) is equivalent to the operator equation:

$$\begin{aligned} Sv(t) &= \frac{t^{\alpha-1}}{\Delta-\rho}\mu\left(\frac{\beta\rho}{\Delta}\int_0^1 k(\eta,s)h^+(s)(f_1-f_2)\,ds + \int_0^1 \kappa(s)h^+(s)(f_1-f_2)\,ds\right) \\ &\quad + \mu\left(\frac{\beta t^{\alpha-1}}{\Delta}\int_0^1 k(\eta,s)h^+(s)(f_1-f_2)\,ds + \int_0^1 k(t,s)h^+(s)(f_1-f_2)\,ds\right) \\ &\quad - \frac{t^{\alpha-1}}{\Delta-\rho}\mu\left(\frac{\beta\rho}{\Delta}\int_0^1 k(\eta,s)h^-(s)f_1\,ds + \int_0^1 \kappa(s)h^-(s)f_1\,ds\right) \\ &\quad - \mu\left(\frac{\beta t^{\alpha-1}}{\Delta}\int_0^1 k(\eta,s)h^-(s)f_1\,ds + \int_0^1 k(t,s)h^-(s)f_1\,ds\right). \end{aligned} \tag{33}$$

It is easy to show that operator $S : X \to X$ is completely continuous. Let $v \in X$ and $\tau \in (0,1)$ such that $v = \tau S v$. That is,

$$\begin{aligned} v(t) &= \frac{\tau t^{\alpha-1}}{\Delta-\rho}\mu\left(\frac{\beta\rho}{\Delta}\int_0^1 k(\eta,s)h^+(s)(f_1-f_2)\,ds + \int_0^1 \kappa(s)h^+(s)(f_1-f_2)\,ds\right) \\ &\quad + \tau\mu\left(\frac{\beta t^{\alpha-1}}{\Delta}\int_0^1 k(\eta,s)h^+(s)(f_1-f_2)\,ds + \int_0^1 k(t,s)h^+(s)(f_1-f_2)\,ds\right) \\ &\quad - \frac{\tau t^{\alpha-1}}{\Delta-\rho}\mu\left(\frac{\beta\rho}{\Delta}\int_0^1 k(\eta,s)h^-(s)f_1\,ds + \int_0^1 \kappa(s)h^-(s)f_1\,ds\right) \\ &\quad - \tau\mu\left(\frac{\beta t^{\alpha-1}}{\Delta}\int_0^1 k(\eta,s)h^-(s)f_1\,ds + \int_0^1 k(t,s)h^-(s)f_1\,ds\right). \end{aligned} \tag{34}$$

We claim that $\|v\| \neq \mu\delta f(0)\|m\|$. Suppose on the contrary that $\|v\| = \mu\delta f(0)\|m\|$. Then, by (28) and (29), we get

$$\begin{gathered}\left\|\overline{x}_\mu + v\right\| \leq \left\|\overline{x}_\mu\right\| + \|v\| \leq b, \\ \left|f_1 - f_2\right| \leq f(0)\frac{\delta - c}{2}.\end{gathered} \tag{35}$$

From (27), we get

$$\omega(t) f(x(a(t))) \leq cm(t) f(0), \quad t \in (0,1]. \tag{36}$$

Using (34)–(36), for each $t \in (0,1]$, we obtain that

$$\begin{aligned}
&|v(t)| \\
&\leq \frac{t^{\alpha-1}}{\Delta-\rho}\mu\left(\frac{\beta\rho}{\Delta}\int_0^1 k(\eta,s)h^+(s)\left|f_1-f_2\right|ds \right.\\
&\qquad \left. + \int_0^1 \kappa(s)h^+(s)\left|f_1-f_2\right|ds\right) \\
&+ \mu\left(\frac{\beta t^{\alpha-1}}{\Delta}\int_0^1 k(\eta,s)h^+(s)\left|f_1-f_2\right|ds \right.\\
&\qquad \left. + \int_0^1 k(t,s)h^+(s)\left|f_1-f_2\right|ds\right) \\
&+ \frac{t^{\alpha-1}}{\Delta-\rho}\mu\left(\frac{\beta\rho}{\Delta}\int_0^1 k(\eta,s)h^-(s)f_1 ds + \int_0^1 \kappa(s)h^-(s)f_1 ds\right) \\
&+ \mu\left(\frac{\beta t^{\alpha-1}}{\Delta}\int_0^1 k(\eta,s)h^-(s)f_1 ds + \int_0^1 k(t,s)h^-(s)f_1 ds\right) \\
&\leq \frac{t^{\alpha-1}}{\Delta-\rho}\mu f(0)\frac{\delta-c}{2} \\
&\quad \times\left(\frac{\beta\rho}{\Delta}\int_0^1 k(\eta,s)h^+(s)ds + \int_0^1 \kappa(s)h^+(s)ds\right) \\
&+ \mu f(0)\frac{\delta-c}{2}\left(\frac{\beta t^{\alpha-1}}{\Delta}\int_0^1 k(\eta,s)h^+(s)ds + \int_0^1 k(t,s)h^+(s)ds\right) \\
&+ \frac{t^{\alpha-1}}{\Delta-\rho}\mu\overline{f}(b)\left(\frac{\beta\rho}{\Delta}\int_0^1 k(\eta,s)h^-(s)ds + \int_0^1 \kappa(s)h^-(s)ds\right) \\
&+ \mu\overline{f}(b)\left(\frac{\beta t^{\alpha-1}}{\Delta}\int_0^1 k(\eta,s)h^-(s)ds + \int_0^1 k(t,s)h^-(s)ds\right) \\
&= \mu f(0)\frac{\delta-c}{2}m(t) + \mu\overline{f}(b)\omega(t) \\
&\leq \mu f(0)\frac{\delta-c}{2}m(t) + \mu c f(0)m(t) \\
&= \mu f(0)\frac{\delta+c}{2}m(t).
\end{aligned} \tag{37}$$

In particular,

$$\|v\| \leq \mu f(0)\frac{\delta+c}{2}\|m\| < \mu f(0)\delta\|m\|, \tag{38}$$

which is a contradiction. And so the claim is proved. Let $U = \{x \in X : \|x\| < \mu\delta f(0)\|m\|\}$. By Theorem 5, S has a fixed point $v_\mu \in \overline{U}$. Consequently, $\|v_\mu\| \leq \mu\delta f(0)\|m\|$. This proves that there exists v_μ this is the solution of (30). Hence v_μ satisfies (37) and Lemma 6; then we get

$$\begin{aligned}
x_\mu(t) &\geq \overline{x}_\mu(t) - \left|v_\mu(t)\right| \geq \mu\delta f(0)m(t) - \mu f(0)\frac{\delta+c}{2}m(t) \\
&= \mu f(0)\frac{\delta-c}{2}m(t) > 0;
\end{aligned} \tag{39}$$

that is, x_μ is a positive solution of (3). So the proof of Theorem 7 is complete. □

4. An Example

In this section, we give an example to illustrate the result of this paper. Consider the following nonlinear fractional differential equation:

$$\begin{gathered}
D^{5/2}x(t) - \mu\left(\frac{4}{5}-t\right)\left(x^{(4)}\left(\sqrt{t}\right) + \sin^2 x\left(\sqrt{t}\right) + \frac{1}{10}\right) = 0, \\
x(0) = x'(0) = 0, \\
x(1) = \frac{1}{2}x\left(\frac{1}{2}\right) + \int_0^1 x(t)(3t-1)\,dt.
\end{gathered} \tag{40}$$

Let $f(x(a(t))) = x^{(4)}(\sqrt{t}) + \sin^2 x(\sqrt{t}) + 1/10, a(t) = \sqrt{t}$ and $h(t) = 4/5 - t$. Obviously, all assumptions (H_1)–(H_3) hold. In the following, we will verify that assumptions (H_4)–(H_6) hold also.

(i) It is obvious that

$$\begin{aligned}
\beta\eta^{\alpha-1} + \lambda[p] &= \beta\eta^{\alpha-1} + \int_0^1 t^{\alpha-1}(3t-1)\,dt = \beta\eta^{\alpha-1} \\
&+ \frac{2\alpha-1}{\alpha(\alpha+1)} = \frac{1}{4\sqrt{2}} + \frac{16}{35}
\end{aligned} \tag{41}$$

implies (H_4).

(ii) By direct calculation, we have

$$\int_0^1 (3t-1)\,dt = \frac{1}{2}, \qquad \int_0^1 t^{\alpha-1}(3t-1)\,dt = \frac{16}{35},$$

$$\begin{aligned}\kappa(s) &= \int_0^1 k(t,s)(3t-1)\,dt \\ &= \frac{1}{\alpha(\alpha+1)}(1-s)^{\alpha-1}s(3s+2\alpha-4) \\ &= \frac{4}{35}(1-s)^{3/2}s(3s+1) \geq 0,\end{aligned} \tag{42}$$

so assumption (H_5) holds.

(iii) Finally, we check assumption (H_6). It means that there exists $\epsilon > 0$ such that $m(t) \geq (1+\epsilon)\omega(t), t \in (0,1]$. Note that

$$h^+(t) = \max\{0, h(t)\} = \begin{cases} \frac{4}{5} - t, & 0 \leq t \leq \frac{4}{5}, \\ 0, & \frac{4}{5} < t \leq 1, \end{cases}$$
$$h^-(t) = \max\{0, -h(t)\} = \begin{cases} 0, & 0 \leq t \leq \frac{4}{5}, \\ t - \frac{4}{5}, & \frac{4}{5} < t \leq 1. \end{cases} \tag{43}$$

We now verify that there exists $\epsilon_1 > 0$ such that

$$\begin{aligned}&\frac{t^{\alpha-1}}{\Delta-\rho}\frac{\beta\rho}{\Delta}\int_0^1 k(\eta,s)h^+(s)\,ds \\ &\quad \geq (1+\epsilon_1)\frac{t^{\alpha-1}}{\Delta-\rho}\frac{\beta\rho}{\Delta}\int_0^1 k(\eta,s)h^-(s)\,ds, \quad t \in (0,1];\end{aligned} \tag{44}$$

that is,

$$\int_0^1 k\left(\frac{1}{2},s\right)\left(\frac{4}{5}-s\right)ds \geq \epsilon_1 \int_{4/5}^1 k\left(\frac{1}{2},s\right)\left(s-\frac{4}{5}\right)ds, \quad t \in (0,1]. \tag{45}$$

By simple calculation, we get

$$\begin{aligned}\int_0^1 k\left(\frac{1}{2},s\right)\left(\frac{4}{5}-s\right)ds &= \frac{13\sqrt{2}}{700}, \\ \int_{4/5}^1 k\left(\frac{1}{2},s\right)\left(s-\frac{4}{5}\right)ds &= \frac{2}{7\times 5^4\times\sqrt{10}}.\end{aligned} \tag{46}$$

Setting $\epsilon_1 \in (0, (13\times 5^2\times\sqrt{5})/4)$, then inequality (44) holds. Similarly, there exists $\epsilon_2, \epsilon_3, \epsilon_4 > 0$ such that

$$\begin{aligned}&\frac{t^{\alpha-1}}{\Delta-\rho}\int_0^1 \kappa(s)h^+(s)\,ds \\ &\quad \geq (1+\epsilon_2)\frac{t^{\alpha-1}}{\Delta-\rho}\int_0^1 \kappa(s)h^-(s)\,ds, \quad t \in (0,1], \\ &\frac{\beta t^{\alpha-1}}{\Delta}\int_0^1 k(\eta,s)h^+(s)\,ds \\ &\quad \geq (1+\epsilon_3)\frac{\beta t^{\alpha-1}}{\Delta}\int_0^1 k(\eta,s)h^-(s)\,ds, \quad t \in (0,1], \\ &\int_0^1 k(t,s)h^+(s)\,ds \\ &\quad \geq (1+\epsilon_4)\int_0^1 k(t,s)h^-(s)\,ds, \quad t \in (0,1].\end{aligned} \tag{47}$$

Let $\epsilon = \min\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$. By (44)–(47), we obtain that there exists $\epsilon > 0$ such that

$$m(t) \geq (1+\epsilon)\,\omega(t), \quad t \in (0,1]. \tag{48}$$

Thus assumption (H_6) holds. By applying Theorem 7, we know that there exists a number $\mu^* > 0$ such that (40) has at least one positive solution for $\mu \in (0, \mu^*)$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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