Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 157498, 9 pages
http://dx.doi.org/10.1155/2014/157498

Research Article

Strong Convergence of the Split-Step Theta Method for Stochastic Delay Differential Equations with Nonglobally Lipschitz Continuous Coefficients

Chao Yue and Chengming Huang

School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan 430074, China

Correspondence should be addressed to Chengming Huang; chengming_huang@hotmail.com

Received 27 April 2014; Accepted 6 August 2014; Published 20 August 2014

Academic Editor: Jaeyoung Chung

This paper is concerned with the convergence analysis of numerical methods for stochastic delay differential equations. We consider the split-step theta method for nonlinear nonautonomous equations and prove the strong convergence of the numerical solution under a local Lipschitz condition and a coupled condition on the drift and diffusion coefficients. In particular, these conditions admit that the diffusion coefficient is highly nonlinear. Furthermore, the obtained results are supported by numerical experiments.

1. Introduction

Stochastic delay differential equations (SDDEs) play an important role in modeling some real-world phenomena in many scientific areas, such as economics [1], biology [2, 3], and medicine [4, 5]. However, many SDDEs arising in applications cannot be solved analytically; hence one needs to develop effective numerical methods to solve them.

In recent years, the numerical solution of SDDEs has attracted much attention and a number of numerical methods have been constructed (see, e.g., [6–8]). An important topic in this context is the investigation of the convergence of numerical methods and a number of interesting results have been found (see, e.g., [9–13]). In the analysis of strong convergence, a widely used assumption is that the drift and diffusion coefficients satisfy global Lipschitz and linear growth conditions [9–11]. In order to weaken this assumption, Mao and Sabanis [14] proved strong convergence of Euler-Maruyama type methods with local Lipschitz conditions and the bounded pth moments ($p > 2$) for solving SDDEs. Wang and Gan [12] showed that the improved split-step backward Euler method is convergent in the mean square sense under the condition that the diffusion coefficient $g(x, y)$ is globally Lipschitz, and the drift coefficient $f(x, y)$ satisfies a one-sided Lipschitz condition in the nondelay variable x and a global Lipschitz condition in the delay variable y. Bao and Yuan [15] proved the convergence rate of the Euler-Maruyama (EM) scheme for a class of SDDEs, where the corresponding coefficients may be highly nonlinear with respect to the delay variables. The strong convergence was also studied in [16, 17]. Nevertheless, all the above results are derived for SDDEs of which the diffusion coefficient with respect to the nondelay variables satisfies a linear growth or global Lipschitz condition. For example, they cannot be applied to some highly nonlinear problems such as

$$dx(t) = \left(-x(t) - x^3(t) + x(t-1)\right) dt + x^2(t)\, dW(t). \tag{1}$$

In this paper, we study the strong convergence of the split-step theta method [8] under a local Lipschitz condition and a coupled condition on the drift and diffusion coefficients. These conditions admit that the diffusion coefficient with respect to the nondelay variables is highly nonlinear; that is, it does not necessarily satisfy a linear growth or global Lipschitz condition.

The structure of this paper is organized as follows. First, the existence of a unique solution of SDDEs under weaker conditions is recalled in Section 2. Then, some moment properties of the split-step theta method (3) are investigated in Section 3, while its strong convergence is derived in Section 4. Finally, some numerical results to support our theorems are presented.

2. Existence and Uniqueness of Solution

Throughout this paper, we denote both the Euclidean vector norm and the Frobenius matrix norm by $\|\cdot\|$ and the complete probability space by $\{\Omega, \mathcal{F}, \mathbb{P}\}$. $\{\mathcal{F}_t\}_t \geq 0$ is increasing and continuous, and $\{\mathcal{F}_0\}$ contains all $\mathbb{P}$-null sets. Let $f : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $g : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^{d\times q}$. In this paper, we consider the numerical solution of the SDDEs in Itô's sense:

$$
\begin{aligned}
dx(t) &= f(t, x(t), x(t-\tau))\,dt \\
&\quad + g(t, x(t), x(t-\tau))\,dW(t), \quad t \in (0, T], \qquad (2) \\
x(t) &= \phi(t) \quad t \in [-\tau, 0],
\end{aligned}
$$

where $\tau > 0$ and $\phi(t)$ is $\mathcal{F}_0$-measurable, $C([-\tau, 0]; \mathbb{R}^d)$-valued random variable satisfying $\sup_{-\tau\leq t\leq 0}\mathbb{E}[\|\phi(t)\|^2] \leq M < +\infty$. Here $W(t)$ denotes a standard q-dimensional Brownian motion defined on the probability space.

Let h be a time stepsize satisfying $h = \tau/m$ with a positive integer m. Then the split-step theta method (SST) introduced in [8] for problem (2) reads

$$
\begin{aligned}
Y_n &= y_n + \theta h f(t_n + \theta h, Y_n, Y_{n-m}), \\
y_{n+1} &= y_n + h f(t_n + \theta h, Y_n, Y_{n-m}) \\
&\quad + g(t_n + \theta h, Y_n, Y_{n-m})\,\Delta W_n,
\end{aligned} \qquad (3)
$$

where $t_n = nh$, y_n is an approximation to $x(t_n)$, $\theta \in [0, 1]$ is a fixed parameter, $\Delta W_n = W(t_{n+1}) - W(t_n)$, and $Y_n = \phi(t_n + \theta h)$ for $n < 0$. For the given stepsize h, let N be the greatest integer satisfying $Nh \leq T$. For simplicity, we directly assume $Nh = T$ in the following.

Huang [8] studied the exponential mean square stability of the SST method (3) under the following condition:

$$
\begin{aligned}
&x^T Q f(t, x, y) + \frac{1}{2}\operatorname{trace}\left[g^T(t, x, y) Q g(t, x, y)\right] \\
&\quad \leq \tilde{\alpha} x^T Q x + \tilde{\beta} y^T Q y,
\end{aligned} \qquad (4)
$$

where Q is a real symmetric, positive definite matrix. It is proved that when $\tilde{\alpha} + \tilde{\beta} < 0$ and $\theta > 0.5$, SST method (3) is exponentially mean square stable for all positive stepsizes. In this paper, we further study its strong convergence property under weaker conditions.

To ensure the existence of a unique solution on $[-\tau, T]$ to SDDEs (2), we introduce the following assumption.

Assumption 1. The functions $f(t, x, y)$ and $g(t, x, y)$ in (2) are continuous in t, x, and y and satisfy the nonglobal Lipschitz condition. More precisely, there exist positive constants R, L_R, α, and β such that

$$
\begin{aligned}
&\|f(t, x_1, y_1) - f(t, x_2, y_2)\|^2 + \|g(t, x_1, x_2) - g(t, x_2, y_2)\|^2 \\
&\quad \leq L_R \left(\|x_1 - x_2\|^2 + \|y_1 - y_2\|^2\right),
\end{aligned} \qquad (5)
$$

$$
\langle x, f(t, x, y)\rangle + \frac{1}{2}\|g(t, x, y)\|^2 \leq \alpha + \beta\left(\|x\|^2 + \|y\|^2\right), \qquad (6)
$$

where $\|x_1\| \vee \|x_2\| \vee \|y_1\| \vee \|y_2\| \leq R$, $x_1, x_2, y_1, y_2, x, y \in \mathbb{R}^d$, and $t \in (0, T]$.

From Theorem 1.2 in [18], it follows that for any given initial value $\{\phi(t) : -\tau \leq t \leq 0\} \in C([-\tau, 0]; \mathbb{R}^d)$ there exists a unique solution $x(t)$ to SDDEs (2).

Now, we state the following lemma which will play an important role in proving the strong convergence of the SST method (3).

Lemma 2. *Under Assumption 1, the solution of the SDDEs* (2) *on* $[0, T]$ *has the properties that*

$$
\mathbb{E}\left[\|x(T)\|^2\right] \leq \left(\mathbb{E}\left[\|\phi(0)\|^2\right] + 2\alpha T\right)\exp(2\beta T), \qquad (7)
$$

$$
\mathbb{P}(\rho_R \leq T) \leq \frac{\left(\mathbb{E}\left[\|\phi(0)\|^2\right] + 2\alpha T + M\tau\right)\exp(2\beta T) + M}{R^2}, \qquad (8)
$$

where

$$
\rho_R = \inf\{t \geq 0 : \|x(t)\| \vee \|x(t-\tau)\| \geq R\}. \qquad (9)
$$

Proof. Let $V(x, t) = \|x\|^2$. Then by using Itô formula, we infer that

$$
\begin{aligned}
\|x(t)\|^2 &= \|\phi(0)\|^2 + 2\int_0^t \langle x(s), f(s, x(s), x(s-\tau))\rangle\,ds \\
&\quad + \int_0^t \|g(s, x(s), x(s-\tau))\|^2 ds \\
&\quad + 2\int_0^t \langle x(s), g(s, x(s), x(s-\tau))\rangle\,dW(s).
\end{aligned} \qquad (10)
$$

According to Assumption 1 and taking mathematical expectation on both sides of (10), we have

$$
\begin{aligned}
&\mathbb{E}\left[\|x(t\wedge\rho_R)\|^2\right] \\
&\quad \leq \mathbb{E}\left[\|\phi(0)\|^2\right] + 2\alpha t \\
&\qquad + \mathbb{E}\left[\int_0^{t\wedge\rho_R} 2\beta\left(\|x(s)\|^2 + \|x(s-\tau)\|^2\right) ds\right] \\
&\quad \leq \mathbb{E}\left[\|\phi(0)\|^2\right] + 2\alpha t \\
&\qquad + \mathbb{E}\left[\int_0^{t\wedge\rho_R} 4\beta\|x(s)\|^2 ds + \int_0^{\tau} \|\phi(s-\tau)\|^2 ds\right] \\
&\quad \leq \mathbb{E}\left[\|\phi(0)\|^2\right] + 2\alpha t + M\tau + \int_0^t 4\beta\mathbb{E}\left[\|x(s\wedge\rho_R)\|^2\right] ds.
\end{aligned} \qquad (11)
$$

Using the Gronwall inequality yields

$$\mathbb{E}\left[\|x(T\wedge\rho_R)\|^2\right] \le \left(\mathbb{E}\left[\|\phi(0)\|^2\right] + 2\alpha T + M\tau\right)\exp(4\beta T), \tag{12}$$

$$\begin{aligned}\mathbb{P}(\rho_R \le T) &\le \mathbb{E}\left[\frac{\|x(T\wedge\rho_R)\|^2 + \|x(T\wedge\rho_R-\tau)\|^2}{R^2} 1_{\{\rho_R\le T\}}\right] \\ &\le \frac{\left(\mathbb{E}\left[\|\phi(0)\|^2\right] + 2\alpha T + M\tau\right)\exp(4\beta T) + M}{R^2}.\end{aligned} \tag{13}$$

Applying Fatou's lemma to (12), we obtain

$$\mathbb{E}\left[\|x(T)\|^2\right] \le \left(\mathbb{E}\left[\|\phi(0)\|^2\right] + 2\alpha T + M\tau\right)\exp(4\beta T). \tag{14}$$

The proof is completed. □

3. Moment Properties of SST

Before proving the strong convergence of the SST method (3), it is necessary to show that the SST method (3) has a unique solution. So we introduce the following assumption and lemma.

Assumption 3. There exists a positive constant L, such that

$$\langle x_1 - x_2, f(t, x_1, y) - f(t, x_2, y)\rangle \le L\|x_1 - x_2\|^2, \tag{15}$$

for $x_1, x_2, y \in \mathbb{R}^d$ and $t \in [0, T]$.

From [19] we easily obtain that the SST method (3) has a unique solution under $0 < \theta L h < 1$. We now show that, under Assumptions 1 and 3, the 2nd moment of numerical solution y_n and Y_n is bounded.

Lemma 4. *Assume that $f(t, x(t), x(t-\tau))$ and $g(t, x(t), x(t-\tau))$ in (2) satisfy Assumptions 1 and 3; then for $1/2 \le \theta \le 1$ and $h < h^* < \min\{1/\theta L, 1/2\theta\beta\}$, the following moment bounds hold:*

$$\begin{aligned}&\sup_{0\le t\le T}\mathbb{E}\left[\|x(t)\|^2\right] \vee \sup_{h\le h^*}\sup_{0\le nh\le T}\mathbb{E}\left[\|y_n\|^2\right] \\ &\vee \sup_{h\le h^*}\sup_{0\le nh\le T}\mathbb{E}\left[\|Y_{n-m}\|^2\right] \vee \sup_{h\le h^*}\sup_{0\le nh\le T}\mathbb{E}\left[\|Y_n\|^2\right] \le A < \infty,\end{aligned} \tag{16}$$

where A is a positive constant independent of N.

Proof. First, by Lemma 2, we know that $\sup_{0\le t\le T}\mathbb{E}[\|x(t)\|^2]$ is bounded. Denoting

$$F(Y_{n+1}) := Y_{n+1} - \theta h f(t_{n+1} + \theta h, Y_{n+1}, Y_{n+1-m}) \tag{17}$$

and then inserting (17) into (3), we have

$$\begin{aligned}\|F(Y_{n+1})\|^2 &= \|F(Y_n)\|^2 + 2h\langle Y_n, f(t_n, Y_n, Y_{n-m})\rangle \\ &\quad + \|g(t_n + \theta h, Y_n, Y_{n-m})\Delta W_n\|^2 \\ &\quad + (1-2\theta)h^2\|f(t_n + \theta h, Y_n, Y_{n-m})\|^2 + \mathbb{M}_n,\end{aligned} \tag{18}$$

where $\mathbb{M}_n = 2\langle F(Y_n), g(t_n + \theta h, Y_n, Y_{n-m})\Delta W_n\rangle + 2\langle hf(t_n + \theta h, Y_n, Y_{n-m}), g(t_n + \theta h, Y_n, Y_{n-m})\Delta W_n\rangle$. By recursive calculation, we obtain

$$\begin{aligned}&\|F(Y_{n+1})\|^2 \\ &\quad = \|F(Y_0)\|^2 \\ &\qquad + \sum_{j=0}^{n}\Big(2h\left\langle Y_j, f\left(t_j + \theta h, Y_j, Y_{j-m}\right)\right\rangle \\ &\qquad\qquad + \left\|g\left(t_j + \theta h, Y_j, Y_{j-m}\right)\Delta W_j\right\|^2\Big) \\ &\qquad + (1-2\theta)h^2\sum_{j=0}^{n}\left\|f(t_j + \theta h, Y_j, Y_{j-m})\right\|^2 + \sum_{j=0}^{n}\mathbb{M}_j.\end{aligned} \tag{19}$$

Noting that $Y_n \in \mathscr{F}_{t_n}$, ΔW_n is independent of $\mathscr{F}_{t_n}$; we have $\mathbb{E}[\|\Delta W_n\|^2] = h$ and $\mathbb{E}[\sum_{j=0}^{n}\mathbb{M}_j] = 0$. Hence, taking the mathematical expectation on both sides of (19) and then substituting (6) into (19) lead us to

$$\begin{aligned}\mathbb{E}\left[\|F(Y_{n+1})\|^2\right] &\le \mathbb{E}\left[\|F(Y_0)\|^2\right] \\ &\quad + 2h\sum_{j=0}^{n}\mathbb{E}\left[\alpha + \beta\left(\|Y_j\|^2 + \|Y_{j-m}\|^2\right)\right].\end{aligned} \tag{20}$$

Next, using $h < h^* < \min\{1/\theta L, 1/2\theta\beta\}$ and regrouping (20), we have

$$\begin{aligned}&\mathbb{E}\left[\|Y_{n+1}\|^2\right] \\ &\quad \le \alpha_1\Bigg(\mathbb{E}\left[\|F(Y_0)\|^2\right] + 2\alpha T + 2h\beta\sum_{j=0}^{n}\mathbb{E}\left[\|Y_{j-m}\|^2\right] \\ &\qquad + 2h\beta\sum_{j=0}^{n}\mathbb{E}\left[\|Y_j\|^2\right]\Bigg) \\ &\quad \le \alpha_1\left(\mathbb{E}\left[\|F(Y_0)\|^2\right] + 2\alpha T + 2\beta M\tau\right) \\ &\qquad + 4h\alpha_1\beta\sum_{j=0}^{n}\mathbb{E}\left[\|Y_j\|^2\right],\end{aligned} \tag{21}$$

where $\alpha_1 = 1/(1 - 2\theta\beta h^*)$. By virtue of recursive calculation from $n = 0$ to $N-1$, we know that $\mathbb{E}[\|Y_N\|^2]$ is bounded. Applying the discrete Gronwall's inequality, we have

$$\begin{aligned}&\mathbb{E}\left[\|Y_{n+1}\|^2\right] \\ &\quad \le \left(\alpha_1\mathbb{E}\left[\|F(Y_0)\|^2\right] + 2\alpha_1\alpha T + 2\alpha_1\beta M\tau\right)\exp(4\alpha_1\beta T).\end{aligned} \tag{22}$$

Therefore, there exists a positive constant M_1, such that

$$\sup_{h\le h^*}\sup_{0\le nh\le T}\mathbb{E}\left[\|Y_n\|^2\right] \vee \sup_{h\le h^*}\sup_{0\le nh\le T}\mathbb{E}\left[\|Y_{n-m}\|^2\right] \le M_1. \tag{23}$$

By using (3), (20), and (23), we infer

$$\begin{aligned}\mathbb{E}\left[\left\|y_{n+1}\right\|^2\right] &= \mathbb{E}\left[\left\|F\left(Y_{n+1}\right)\right\|^2\right] \\ &\leq \mathbb{E}\left[\left\|F\left(Y_0\right)\right\|^2\right] + 2T\alpha + 4T\beta M_1.\end{aligned} \tag{24}$$

Hence, by using Lemma 2, (23), and (24), there exists a positive constant A which is independent of N, such that

$$\begin{aligned}&\sup_{0\leq t\leq T}\mathbb{E}\left[\|x(t)\|^2\right] \vee \sup_{h\leq h^*}\sup_{0\leq nh\leq T}\mathbb{E}\left[\left\|y_n\right\|^2\right] \\ &\vee \sup_{h\leq h^*}\sup_{0\leq nh\leq T}\mathbb{E}\left[\left\|Y_{n-m}\right\|^2\right] \vee \sup_{h\leq h^*}\sup_{0\leq nh\leq T}\mathbb{E}\left[\left\|Y_n\right\|^2\right] \leq A.\end{aligned} \tag{25}$$

The proof is completed. □

4. Strong Convergence

In this paper, it is convenient to use continuous-time approximation solution. First, we denote

$$\begin{aligned}y(s) &:= \sum_{n=0}^{N-1} 1_{\{t_n\leq s<t_{n+1}\}} y_n + 1_{\{s=t_N\}} y_N, \\ Y(s) &:= \sum_{n=0}^{N-1} 1_{\{t_n\leq s<t_{n+1}\}} Y_n + 1_{\{s=t_N\}} Y_N, \\ \tilde{s} &:= \left[\frac{s}{h}\right] h + \theta h,\end{aligned} \tag{26}$$

and $[s/h]$ is the largest integer of s/h. Then, we define continuous version $\overline{y}(t)$ of y_n in (3) as follows:

$$\overline{y}(t) := \begin{cases}\phi(t), & t \in [-\tau, 0], \\ y_n + (t - t_n) \\ \quad \times f\left(t_n + \theta h +, Y_n, Y_{n-m}\right) \\ \quad + g\left(t_n + \theta h, Y_n, Y_{n-m}\right)\Delta W_n(t), & t \in \left[t_n, t_{n+1}\right),\ n \geq 0,\end{cases} \tag{27}$$

where $\Delta W_n(t) := W(t) - W(t_n)$. For convenience, rewriting (27) in integral form

$$\begin{aligned}\overline{y}(t) = \phi(0) &+ \int_0^t f(\tilde{s}, Y(s), Y(s-\tau))\, ds \\ &+ \int_0^t g(\tilde{s}, Y(s), Y(s-\tau))\, dW(s),\end{aligned} \tag{28}$$

where $y_0 = \phi(0)$, $t \in [t_n, t_{n+1})$, and $Y(s-\tau) = \phi(\tilde{s} - \tau)$ for $s-\tau < 0$. From (27) and (28), we easily verified that $\overline{y}(t_n) = y_n$.

Assumption 5. There exist a positive constant K_3 and a positive integer m_1, such that

$$\begin{aligned}&\|f(t,x,y) - f(s,x,y)\|^2 \vee \|g(t,x,y) - g(s,x,y)\|^2 \\ &\quad \leq K_3\left(1 + \|x\|^{2m_1} + \|y\|^2\right)|t-s|^2,\end{aligned} \tag{29}$$

for $x, y \in \mathbb{R}^d$ and $s, t \in [0, T]$.

We now establish the following lemma, which will play a key role in proving the convergence of the SST method (3).

Lemma 6. *Let Assumptions 1 and 3 hold. Then, for any given $\varepsilon > 0$, there exists a positive integer N_0 such that for every $R \geq N_0$, we can find a $\overline{h} = \overline{h}(R)$ so that whenever $h < \min\{1/\theta L, 1/2\theta\beta, \overline{h}\}$,*

$$\mathbb{P}\left(\tau_R < T\right) \leq \varepsilon, \tag{30}$$

where $\tau_R = \inf\{t \geq 0 : \|Y(t)\| \vee \|Y(t-\tau)\| \vee \|\overline{y}(t)\| \geq R\}$.

Proof. Let $s \in [0, T\wedge\tau_R)$ and $V(\overline{y}(t)) = \|\overline{y}(t)\|^2$. Then, by Itô formula, we derive

$$\begin{aligned}&\left\|\overline{y}\left(T\wedge\tau_R\right)\right\|^2 \\ &\quad \leq 2\int_0^{T\wedge\tau_R} \langle \overline{y}(s), g(s, Y(s), Y(s-\tau))\rangle\, dW(s) \\ &\qquad + 2\int_0^{T\wedge\tau_R} \Big(\langle Y(s), f(s, Y(s), Y(s-\tau))\rangle \\ &\qquad\qquad + \frac{1}{2}\|g(s, Y(s), Y(s-\tau))\|^2\Big)\, ds \\ &\qquad + 2\int_0^{T\wedge\tau_R} \left(\|\overline{y}(s) - Y(s)\|\, \|f(s, Y(s), Y(s-\tau))\|\right) ds \\ &\qquad + \|\phi(0)\|^2.\end{aligned} \tag{31}$$

Applying Assumption 1 and taking the mathematical expectation on both sides of (31) lead us to

$$\begin{aligned}&\mathbb{E}\left[\left\|\overline{y}\left(T\wedge\tau_R\right)\right\|^2\right] \\ &\leq 2\mathbb{E}\left[\int_0^{T\wedge\tau_R}\left(\alpha + \beta\|Y(s)\|^2 + \beta\|Y(s-\tau)\|^2\right) ds\right] \\ &\quad + 2\mathbb{E}\left[\int_0^{T\wedge\tau_R}\left(\|\overline{y}(s) - Y(s)\|\, \|f(s, Y(s), Y(s-\tau))\|\right) ds\right] \\ &\quad + \mathbb{E}\left[\|\phi(0)\|^2\right].\end{aligned} \tag{32}$$

By using Lemma 4 and the Hölder inequality we have

$$\begin{aligned}&\mathbb{E}\left[\left\|\overline{y}\left(T\wedge\tau_R\right)\right\|^2\right] \\ &\quad \leq 2\left(\mathbb{E}\left[\int_0^{T\wedge\tau_R}\|\overline{y}(s) - Y(s)\|^2 ds\right]\right)^{1/2} \\ &\qquad \times \left(\mathbb{E}\left[\int_0^{T\wedge\tau_R}\|f(s, Y(s), Y(s-\tau))\|^2 ds\right]\right)^{1/2} \\ &\qquad + \mathbb{E}\left[\|\phi(0)\|^2\right] + 2\alpha T + 4\beta A T.\end{aligned} \tag{33}$$

Next, we will bound the first term on the right-hand side of (33). According to Assumption 1 and Lemma 4, for $\|Y(s)\| \vee \|Y(s-\tau)\| \vee \|\overline{y}(s)\| \leq R$, we have

$$\begin{aligned}
&\|f(s,Y(s),Y(s-\tau))\|^2 \\
&\quad \leq 2\left(\|f(s,Y(s),Y(s-\tau))-f(s,0,0)\|^2+\|f(s,0,0)\|^2\right) \\
&\quad \leq 2L_R\left(\|Y(s)\|^2+\|Y(s-\tau)\|^2\right)+2\|f(s,0,0)\|^2, \\
&\|g(s,Y(s),Y(s-\tau))\|^2 \\
&\quad \leq 2\left(\|g(s,Y(s),Y(s-\tau))-g(s,0,0)\|^2+\|g(s,0,0)\|^2\right) \\
&\quad \leq 2L_R\left(\|Y(s)\|^2+\|Y(s-\tau)\|^2\right)+2\|g(s,0,0)\|^2,
\end{aligned} \tag{34}$$

and then there exists a positive constant $C_1(R)$, such that

$$\begin{aligned}
&\left(\mathbb{E}\left[\sup_{0\leq s<T\wedge\tau_R}\|f(s,Y(s),Y(s-\tau))\|^2\right]\right)^{1/2} \\
&\quad \vee\left(\mathbb{E}\left[\sup_{0\leq s<T\wedge\tau_R}\|g(s,Y(s),Y(s-\tau))\|^2\right]\right)^{1/2} \leq C_1(R),
\end{aligned} \tag{35}$$

where $C_1(R)$ depends on R, M, A, and $C = \max_{0\leq s\leq T}\{\|f(s,0,0)\|^2 \vee \|g(s,0,0)\|^2\}$. For $s\in[t_n,t_{n+1})\subset[0,T\wedge\tau_R)$, using (27), (35), and Lemma 4, we obtain that

$$\begin{aligned}
\overline{y}(s)-Y(s) &= (s-t_n-\theta h)f(\tilde{s},Y(s),Y(s-\tau)) \\
&\quad + g(\tilde{s},Y(s),Y(s-\tau))\Delta W_n(s),
\end{aligned} \tag{36}$$

$$\begin{aligned}
&\left(\mathbb{E}\left[\sup_{0\leq s\leq T\wedge\tau_R}\|\overline{y}(s)-Y(s)\|^2\right]\right)^{1/2} \\
&\quad \leq \left(2h^2\mathbb{E}\left[\sup_{0\leq s<T\wedge\tau_R}\|f(\tilde{s},Y(s),Y(s-\tau))\|^2\right]\right. \\
&\qquad \left. + 2h\mathbb{E}\left[\sup_{0\leq s<T\wedge\tau_R}\|g(\tilde{s},Y(s),Y(s-\tau))\|^2\right]\right)^{1/2} \\
&\quad \leq C_2(R)h^{1/2},
\end{aligned} \tag{37}$$

where $C_2(R)$ is a positive constant which depends on $C_1(R)$. Substituting (35) and (37) into (33), we have

$$\begin{aligned}
&\mathbb{E}\left[\|\overline{y}(T\wedge\tau_R)\|^2\right] \\
&\quad \leq \mathbb{E}\left[\|\phi(0)\|^2\right]+2\alpha T+4\beta AT+2C_1(R)C_2(R)h^{1/2}T,
\end{aligned} \tag{38}$$

$$\begin{aligned}
&\mathbb{P}(\tau_R<T) \\
&\quad \leq \frac{\mathbb{E}\left[\|\phi(0)\|^2\right]+2\alpha T+4\beta AT+2C_1(R)C_2(R)h^{1/2}T+2A}{R^2}.
\end{aligned} \tag{39}$$

Now, for any given $\varepsilon > 0$, we choose N_0 such that for any $R \geq N_0$

$$\frac{\mathbb{E}\left[\|\phi(0)\|^2\right]+2\alpha T+4\beta AT+2A}{R^2} \leq \frac{\varepsilon}{2}. \tag{40}$$

Then, we can choose $\overline{h} = \overline{h}(R)$, such that for any $h < \min\{1/\theta L, 1/2\theta\beta, \overline{h}\}$

$$\frac{2C_1(R)C_2(R)h^{1/2}T}{R^2} \leq \frac{\varepsilon}{2}. \tag{41}$$

Therefore $\mathbb{P}(\tau_R < T) \leq \varepsilon$. □

We are now ready to prove the strong convergence of the SST method (3).

Theorem 7. *Under Assumptions 1, 3, and 5, the continuous-time approximate solution $\overline{y}(t)$ with $1/2 \leq \theta \leq 1$ and $r \in [1,2)$ will converge to the true solution of SDDEs (2); that is,*

$$\mathbb{E}\left[\|\overline{y}(T)-x(T)\|^r\right] \longrightarrow 0, \quad \text{as } h \longrightarrow 0. \tag{42}$$

Proof. We divided our proof into three steps for readability. First, we define $\sigma_R = \rho_R \wedge \tau_R$ and $e(t) := \overline{y}(t) - x(t)$. By applying Young's inequality ($x^r y \leq (\delta_1 r/2)x^2 + ((2-r)/2\delta_1^{r/(2-r)})y^{2/(2-r)}$, $\forall x, y, \delta_1 > 0$), we obtain that for $\delta_1 > 0$

$$\begin{aligned}
&\mathbb{E}\left[\|e(T)\|^r\right] \\
&\quad = \mathbb{E}\left[\|e(T)\|^r 1_{\{\tau_R>T,\rho_R>T\}}\right]+\mathbb{E}\left[\|e(T)\|^r 1_{\{\tau_R\leq T,\text{or}\,\rho_R\leq T\}}\right] \\
&\quad \leq \mathbb{E}\left[\sup_{0\leq t\leq T}\|e(t\wedge\sigma_R)\|^r 1_{\{\sigma_R>T\}}\right]+\frac{\delta_1 r}{2}\mathbb{E}\left[\|e(T)\|^2\right] \\
&\qquad + \frac{2-r}{2\delta_1^{2/(2-r)}}\mathbb{P}\left(\tau_R\leq T,\text{or}\,\rho_R\leq T\right).
\end{aligned} \tag{43}$$

Second, to bound the first term on the right-hand side of (43), it is enough to show that $\mathbb{E}[\sup_{0\le t\le T}\|e(t\wedge\sigma_R)\|^2 1_{\{\sigma_R>T\}}]$ is bounded due to the Lyapunov inequality. By using Burkholder-Davis-Gundy inequality, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le T}\left\|\overline{y}\left(t\wedge\sigma_R\right)-x\left(t\wedge\sigma_R\right)\right\|^2\right]\\
&=\mathbb{E}\left[\sup_{0\le t\le T}\left\|\int_0^{t\wedge\sigma_R}\left(f\left(\tilde{s},Y\left(s\right),Y\left(s-\tau\right)\right)\right.\right.\right.\\
&\qquad\left.-f\left(s,x\left(s\right),x\left(s-\tau\right)\right)\right)ds\\
&\qquad+\int_0^{t\wedge\sigma_R}\left(g\left(\tilde{s},Y\left(s\right),Y\left(s-\tau\right)\right)\right.\\
&\qquad\left.\left.\left.-g\left(s,x\left(s\right),x\left(s-\tau\right)\right)\right)d\omega(s)\right\|^2\right]\\
&\le 2T\mathbb{E}\left[\int_0^{T\wedge\sigma_R}\left\|f\left(\tilde{s},Y\left(s\right),Y\left(s-\tau\right)\right)\right.\right.\\
&\qquad\left.\left.-f\left(s,x\left(s\right),x\left(s-\tau\right)\right)\right\|^2ds\right]\\
&\quad+8\mathbb{E}\left[\int_0^{T\wedge\sigma_R}\left\|g\left(\tilde{s},Y\left(s\right),Y\left(s-\tau\right)\right)\right.\right.\\
&\qquad\left.\left.-g\left(s,x\left(s\right),x\left(s-\tau\right)\right)\right\|^2ds\right].
\end{aligned}\tag{44}$$

Furthermore, using Assumption 5, Lemma 4, Assumption 1, and (37), we get

$$\begin{aligned}
&\mathbb{E}\left[\int_0^{T\wedge\sigma_R}\left\|f\left(\tilde{s},Y\left(s\right),Y\left(s-\tau\right)\right)-f\left(s,x\left(s\right),x\left(s-\tau\right)\right)\right\|^2ds\right]\\
&\le 3\mathbb{E}\left[\int_0^{T\wedge\sigma_R}\left\|f\left(\tilde{s},Y\left(s\right),Y\left(s-\tau\right)\right)\right.\right.\\
&\qquad\left.\left.-f\left(s,Y\left(s\right),Y\left(s-\tau\right)\right)\right\|^2ds\right]\\
&\quad+3\mathbb{E}\left[\int_0^{T\wedge\sigma_R}\left\|f\left(s,Y\left(s\right),Y\left(s-\tau\right)\right)\right.\right.\\
&\qquad\left.\left.-f\left(s,\overline{y}\left(s\right),\overline{y}\left(s-\tau\right)\right)\right\|^2ds\right]\\
&\quad+3\mathbb{E}\left[\int_0^{T\wedge\sigma_R}\left\|f\left(s,\overline{y}\left(s\right),\overline{y}\left(s-\tau\right)\right)\right.\right.\\
&\qquad\left.\left.-f\left(s,x\left(s\right),x\left(s-\tau\right)\right)\right\|^2ds\right]\\
&\le 3K_3h^2\left(1+A+C_3\left(R\right)\right)T+6L_RC_2^2\left(R\right)hT\\
&\quad+6L_R\int_0^T\mathbb{E}\left[\sup_{0\le t\le s}\left\|\overline{y}\left(t\wedge\sigma_R\right)-x\left(t\wedge\sigma_R\right)\right\|^2\right]ds,
\end{aligned}\tag{45}$$

where $C_3(R)=\mathbb{E}[\sup_{0\le s<T\wedge\sigma_R}\|Y(s)\|^{2m_1}]$ is a positive constant dependent of R. Similarly, we have

$$\begin{aligned}
&\mathbb{E}\left[\int_0^{T\wedge\sigma_R}\left\|g\left(\tilde{s},Y\left(s\right),Y\left(s-\tau\right)\right)-g\left(s,x\left(s\right),x\left(s-\tau\right)\right)\right\|^2ds\right]\\
&\le 3K_3h^2\left(1+A+C_3\left(R\right)\right)T+6L_RC_2^2\left(R\right)Th\\
&\quad+6L_R\int_0^T\mathbb{E}\left[\sup_{0\le t\le s}\left\|\overline{y}\left(t\wedge\sigma_R\right)-x\left(t\wedge\sigma_R\right)\right\|^2\right]ds.
\end{aligned}\tag{46}$$

By substituting (45) and (46) into (44), we obtain

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le T}\left\|\overline{y}\left(t\wedge\sigma_R\right)-x\left(t\wedge\sigma_R\right)\right\|^2\right]\\
&\le\left(12TL_R+48L_R\right)\int_0^T\mathbb{E}\left[\sup_{0\le t\le s}\left\|\overline{y}\left(t\wedge\sigma_R\right)-x\left(t\wedge\sigma_R\right)\right\|^2\right]ds\\
&\quad+\left(2T+8\right)\left(3K_3h\left(1+A+C_3\left(R\right)\right)T+6L_RC_2^2\left(R\right)T\right)h.
\end{aligned}\tag{47}$$

Applying the Gronwall inequality and the Lyapunov inequality leads us to

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le T}\left\|\overline{y}\left(t\wedge\sigma_R\right)-x\left(t\wedge\sigma_R\right)\right\|^2\right]\\
&\le h\left(2T+8\right)\left(3K_3h\left(1+A+C_3\left(R\right)\right)T+6L_RC_2^2\left(R\right)T\right)\\
&\quad\times\exp^{12T^2L_R+48TL_R},
\end{aligned}\tag{48}$$

and then

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le T}\left\|\overline{y}\left(t\wedge\sigma_R\right)-x\left(t\wedge\sigma_R\right)\right\|^r\right]\\
&\le\left(h\left(2T+8\right)\left(3K_3h\left(1+A+C_3\left(R\right)\right)T+6L_RC_2^2\left(R\right)T\right)\right.\\
&\quad\left.\times\exp^{12T^2L_R+48TL_R}\right)^{r/2}.
\end{aligned}\tag{49}$$

Table 1: The means of absolute errors ϵ with $t_N = 2$ of SST (3) for solving SDDE (54).

h	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
$\theta = 1/2$	0.0105521	0.0066357	0.0041003	0.0025833	0.0017500
$\theta = 3/4$	0.0096784	0.0059020	0.0037484	0.0024271	0.0017922
$\theta = 1$	0.0091289	0.0056502	0.0037745	0.0025027	0.0017006

Table 2: The means of absolute errors ϵ with $t_N = 2$ of SST (3) for solving SDDE (55).

h	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
$\theta = 1/2$	0.0056293	0.0037120	0.0024775	0.0017213	0.0011661
$\theta = 3/4$	0.0059738	0.0038085	0.0025652	0.0017131	0.0012165
$\theta = 1$	0.0062445	0.0038978	0.0025618	0.0017181	0.0011714

Finally, combining (49) and (43), we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le T}\|e(t)\|^r\right] \\
&\le \Big(h(2T+8)\left(3K_3h\left(1+A+C_3(R)\right)T+6L_RC_2^2(R)T\right) \\
&\qquad \times \exp^{12T^2L_R+48TL_R}\Big)^{r/2} \\
&\qquad + \frac{\delta_1 r}{2}\mathbb{E}\left[\|e(T)\|^2\right]+\frac{2-r}{2\delta^{2/(2-r)}}\mathbb{P}\left(\tau_R\le T,\text{or }\rho_R\le T\right) \\
&\le \Big(h(2T+8)\left(3K_3h\left(1+A+C_3(R)\right)T+6L_RC_2^2(R)T\right) \\
&\qquad \times \exp^{12T^2L_R+48TL_R}\Big)^{r/2} \\
&\qquad + \frac{\delta_1 r}{2}\mathbb{E}\left[\|e(T)\|^2\right]+\frac{2-r}{2\delta_1^{2/(2-r)}}\mathbb{P}\left(\tau_R\le T\right) \\
&\qquad + \frac{2-r}{2\delta_1^{2/(2-r)}}\mathbb{P}\left(\rho_R\le T\right).
\end{aligned} \tag{50}$$

For any given $\varepsilon > 0$, by Lemmas 2 and 4, we can choose δ_1 such that

$$\frac{\delta_1 r}{2}\mathbb{E}\left[\|e(T)\|^2\right]\le\delta_1 r\mathbb{E}\left[\|x(T)\|^2+\|\overline{y}(T)\|^2\right]\le\frac{\varepsilon}{3}. \tag{51}$$

Now, by (8) and (39), there exists N_0 such that for $R \ge N_0$

$$\begin{aligned}
&\frac{2-r}{2\delta_1^{2/(2-r)}}\mathbb{P}\left(\rho_R\le T\right) \\
&\qquad + \frac{2-r}{2\delta_1^{2/(2-r)}}\left(\frac{\mathbb{E}\left[\|\phi(0)\|^2\right]+2\alpha T+4\beta AT+2A}{R^2}\right)\le\frac{\varepsilon}{3}.
\end{aligned} \tag{52}$$

Furthermore, by Lemma 6, we choose h sufficiently small such that

$$\begin{aligned}
&\Big(h(2T+8)\left(3K_3h\left(1+A+C_3(R)\right)T+6L_RC_2^2(R)T\right) \\
&\qquad \times \exp^{12T^2L_R+48TL_R}\Big)^{r/2} \\
&\qquad + \frac{2-r}{2\delta_1^{2/(2-r)}}\left(\frac{2C_1(R)C_2(R)h^{1/2}T}{R^2}\right)\le\frac{\varepsilon}{3}.
\end{aligned} \tag{53}$$

The proof is completed. □

5. Numerical Results

In this section we consider the following numerical experiments that confirm the conclusions obtained in the previous sections.

For the first example, we consider the nonautonomous SDDEs

$$dx(t) = \left(-t^2x(t)-x^3(t)+x(t-0.5)\right)dt+x^2(t)\,dW(t),\qquad t\in[0,2], \tag{54}$$

with initial value $x(t) = 1$, $t \in [-0.5, 0]$.

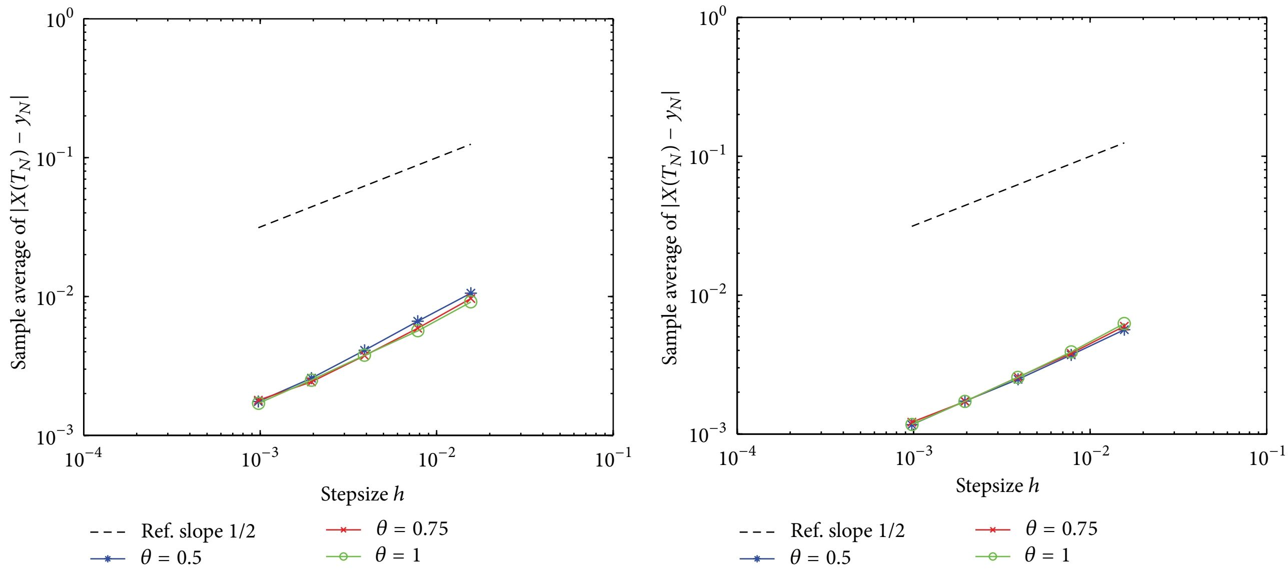
For the second example, we consider the SDDEs [8]

$$dx(t) = \left(-2x(t)-x^3(t)+x(t-1)\right)dt+x^2(t)\,dW(t),\qquad t\in[0,2], \tag{55}$$

with initial value $x(t) = 1$, $t \in [-1, 0]$.

It is easy to show that SDDEs (54) and (55) satisfy Assumptions 1, 3, and 5. Following the idea of [20] and denoting by $y_N^{(i)}$ the numerical approximation to $x(t_N^{(i)})$ at end point t_N in the ith simulation of all M_2 simulations, we approximate means of absolute errors at terminal time t_N by

$$\epsilon = \frac{1}{M_2}\sum_{i=1}^{M_2}\left|y_N^{(i)}-x\left(t_N^{(i)}\right)\right|. \tag{56}$$



(a) (b)

FIGURE 1: $\log \epsilon$ with $t_N = 2$ versus $\log h$ for SST method (3) solving SDDEs (54) (a) and SDDEs (55) (b).

It is difficult to obtain the analytic form of the exact solution to SDDEs (54) and (55). Recall that Theorem 7 guarantees that the SST method (3) strongly converges to the exact solution. Therefore, it is reasonable to identify numerical solution obtained by the SST method (3) ($\theta = 1$) using the very small stepsize $h = 2^{-15}$ as the "exact" solution. With the "exact" solution at hand, we can follow to obtain numerical solution by the SST method (3) using different stepsizes $h = 2^{-6}, 2^{-7}, 2^{-8}, 2^{-9}, 2^{-10}$ on the same discretized path. We generate $M_2 = 5000$ different discretized Brownian paths over $[0, 2]$ and apply the formula (56) to obtain the absolute errors. Errors ϵ at $t_N = 2$ for the SST method (3) solving SDDEs (54) and (55) with different stepsizes h are listed in Tables 1 and 2, respectively. In Figure 1, we plot the means of absolute errors ϵ against h on a log-log scale.

From Figure 1 and Tables 1 and 2, we observe that errors ϵ of numerical approximations decrease as the stepsize h decreases. This is in accordance with our convergence results in the preceding section.

6. Conclusion

In this work, we carried out a strong convergence analysis on the SST method for SDDEs under a local Lipschitz condition and a coupled condition on the drift and diffusion coefficients. Different from most of the existing convergence results for SDDEs, our results can be applied to equations of which the diffusion coefficient with respect to the nondelay variables is highly nonlinear. Both theoretical analysis and numerical tests show that the SST method is efficient for the numerical solution of SDDEs.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors thank the referees and the editors for their valuable comments and suggestions. This work was supported by NSF of China (nos. 91130003 and 11371157) and the Fundamental Research Funds for the Central Universities (no. 2013TS137).

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