

Research Article

Integral Majorization Theorem for Invex Functions

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We obtain some general inequalities and establish integral inequalities of the majorization type for invex functions. We give applications to relative invex functions.

1. Introduction

Let $x, y : [a, b] \rightarrow \mathbb{R}$ be two decreasing real functions. Then the function x is said to *majorize* y if

$$\int_a^s x(t) dt \geq \int_a^s y(t) dt \quad \text{for } s \in [a, b], \quad (1)$$

$$\int_a^b x(t) dt = \int_a^b y(t) dt$$

(see [1, p. 417] and [2, p. 324]).

The following result is known as *majorization theorem* for integrals (see [1, p. 417] and [2, p. 325]).

Theorem 1 (see [1, 2]). *Let $x, y : [a, b] \rightarrow I$ be two decreasing real functions, where $I \subset \mathbb{R}$ is an interval. The function x majorizes y if and only if the inequality*

$$\int_a^b \phi(y(t)) dt \leq \int_a^b \phi(x(t)) dt \quad (2)$$

holds for all continuous convex functions $\phi : I \rightarrow \mathbb{R}$ such that the integrals exist.

Some authors have investigated the weighted versions of (2) (see [1, 3, 4]).

In our main results we will use the following definition of invex function.

Definition 2. Let $\phi : I \rightarrow \mathbb{R}$ be a differentiable function on the interval I , and let $\eta : I \times I \rightarrow \mathbb{R}$ be a function of two variables. The function ϕ is said to be η -invex if, for all $x, y \in I$,

$$\phi(x) \geq \phi(y) + \phi'(y)\eta(x, y); \quad (3)$$

see [5, pp. 1]. ϕ is called invex if ϕ is η -invex for some η .

Clearly, each differentiable convex function $\phi : I \rightarrow \mathbb{R}$ is an η -invex function with $\eta(x, y) = x - y$ for $x, y \in I$. It is known that a differentiable function is invex if and only if each stationary point is a global minimum point [5]. This fact was the motivation to introduce invex functions in optimization theory [6].

Let $\Theta : \mathbb{R}^2 \rightarrow \mathbb{R}$ be an arbitrary function vanishing at points of the form $(0, b)$, $b \in \mathbb{R}$. It is easy to verify that if a differentiable function ϕ satisfies the condition

$$\Theta(\phi'(y), \eta_0(x, y)) \geq 0 \quad (4)$$

implies $\phi(x) - \phi(y) \geq 0$ for $x, y \in I$

for some functions η_0 , then ϕ is invex.

In particular each pseudoconvex function is invex [5, pp. 3-4]. In fact, it is sufficient to consider $\Theta(a, b) = ab$ for $a, b \in \mathbb{R}$ and $\eta_0(x, y) = x - y$ for $x, y \in I$.

In the multidimensional case ($n \geq 1$), we have the following definition.

Definition 3. Let $\langle \cdot, \cdot \rangle$ be an inner product on \mathbb{R}^n . Let $\eta : I^n \times I^n \rightarrow \mathbb{R}^n$ be a function of two variables. One has the following.

- (i) A differentiable function $F : I^n \rightarrow \mathbb{R}$ is said to be η -invex if, for all $x, y \in I^n$,

$$F(x) - F(y) \geq \langle \nabla F(y), \eta(x, y) \rangle, \tag{5}$$

where ∇ denotes the gradient [5, pp. 1].

- (ii) A differentiable function $F : I^n \rightarrow \mathbb{R}$ is said to be η -pseudo-invex if, for all $x, y \in I^n$,

$$\langle \nabla F(y), \eta(x, y) \rangle \geq 0 \text{ implies } F(x) - F(y) \geq 0 \tag{6}$$

(see [6]).

- (iii) A differentiable function $F : I^n \rightarrow \mathbb{R}$ is said to be η -quasi-invex if, for all $x, y \in I^n$,

$$F(x) - F(y) \leq 0 \text{ implies } \langle \nabla F(y), \eta(x, y) \rangle \leq 0 \tag{7}$$

(see [6]).

A differentiable real function $F : I^n \rightarrow \mathbb{R}$ is said to be invex (resp., pseudoinvex, quasi-invex), if F is η -invex (resp., η -pseudo-invex, η -quasi-invex) for some functions $\eta : I^n \times I^n \rightarrow \mathbb{R}^n$.

For applications of invex functions in optimization and mathematical programming, see [5–12] and for some recent results of majorization discrete results for invex functions see [13].

In this paper, we extend integral version of majorization theorem from convex functions to invex ones. We also give some applications to relative invex functions.

2. Main Results

In the following theorem we obtain an inequality which we will use in our other results.

Theorem 4. Let $\phi : I \rightarrow \mathbb{R}$ be an η -invex function on the interval I , where $\eta : I \times I \rightarrow \mathbb{R}$ is a continuous function, and let $f, g, p : [a, b] \rightarrow I$ be integrable functions with $p(t) \geq 0$ for $t \in [a, b]$. Then

$$\begin{aligned} \int_a^b p(t) \phi(f(t)) dt - \int_a^b p(t) \phi(g(t)) dt \\ \geq \int_a^b p(t) \phi'(g(t)) \eta(f(t), g(t)) dt. \end{aligned} \tag{8}$$

Proof. If we take $x \rightarrow f(t), y \rightarrow g(t)$ in (3), we obtain

$$\phi(f(t)) - \phi(g(t)) \geq \phi'(g(t)) \eta(f(t), g(t)). \tag{9}$$

Multiplying (9) by $p(t) \geq 0$ and integrating with respect to t , we deduce (8). \square

The following weighted integral majorization theorem holds.

Theorem 5. Let $\phi : I \rightarrow \mathbb{R}$ be an η -invex function on the interval I , where $\eta : I \times I \rightarrow \mathbb{R}$ is a continuous function, and let $f, g, p : [a, b] \rightarrow I$ be integrable functions with $p(t) > 0$ for $t \in [a, b]$. Moreover if $\phi'(g(t))$ and $\eta(f(t), g(t))$ are increasing (decreasing) on $[a, b]$ and

$$\int_a^b p(t) \eta(f(t), g(t)) dt = 0, \tag{10}$$

then

$$\int_a^b p(t) \phi(f(t)) dt \geq \int_a^b p(t) \phi(g(t)) dt. \tag{11}$$

Proof. We know that the Chebyshev’s inequality is

$$\begin{aligned} \frac{1}{\int_a^b p(t) dt} \int_a^b p(t) F(t) G(t) dt \\ \geq \frac{1}{\int_a^b p(t) dt} \int_a^b p(t) F(t) dt \frac{1}{\int_a^b p(t) dt} \int_a^b p(t) G(t) dt, \end{aligned} \tag{12}$$

where $F, G : [a, b] \rightarrow \mathbb{R}$ are the functions of same monotonicity and $p : [a, b] \rightarrow \mathbb{R}^+$ is any integrable function.

By assumption the functions $\phi'(g(t)), \eta(f(t), g(t)) : [a, b] \rightarrow \mathbb{R}$ have the same monotonicity. Therefore, by applying Chebyshev’s inequality (12) in the right hand side of (8), we have

$$\begin{aligned} \int_a^b p(t) \phi'(g(t)) \eta(f(t), g(t)) dt \\ \geq \frac{1}{\int_a^b p(t) dt} \int_a^b p(t) \phi'(g(t)) dt \\ \times \int_a^b p(t) \eta(f(t), g(t)) dt. \end{aligned} \tag{13}$$

Also (10) holds, so from (8) and (13) we have

$$\begin{aligned} \int_a^b p(t) \phi(f(t)) dt - \int_a^b p(t) \phi(g(t)) dt \\ \geq \int_a^b p(t) \phi'(g(t)) \eta(f(t), g(t)) dt \\ \geq \frac{1}{\int_a^b p(t) dt} \int_a^b p(t) \phi'(g(t)) dt \\ \times \int_a^b p(t) \eta(f(t), g(t)) dt = 0; \end{aligned} \tag{14}$$

then we deduce the desired result (11). \square

Remark 6. In Theorem 5 the assumption (10) is a strong condition for p, η . This can be relaxed if a monotonicity property for the η -invex function ϕ is assumed.

Theorem 7. Let all the assumptions of Theorem 5 hold, but instead of (10) one has the assumption that

$$\int_a^b p(t) \eta(f(t), g(t)) dt \geq 0, \tag{15}$$

and $\phi : I \rightarrow \mathbb{R}$ is an η -invex increasing function on the interval I ; then (11) holds.

Proof. The proof is similar to the proof of Theorem 5 but by (15) and monotonicity of ϕ we have

$$\int_a^b p(t) \phi'(g(t)) dt \geq 0, \quad \int_a^b p(t) \eta(f(t), g(t)) dt \geq 0. \tag{16}$$

Using these in (13) we have

$$\int_a^b p(t) \phi'(g(t)) \eta(f(t), g(t)) dt \geq 0. \tag{17}$$

□

Theorem 8. Let $\phi : I \rightarrow \mathbb{R}$ be an η -invex function on the interval I , where $\eta : I \times I \rightarrow \mathbb{R}$ is a continuous function, and let $f, g, p : [a, b] \rightarrow I$ be integrable functions with $p(t) \geq 0$ for $t \in [a, b]$. One has the following.

(i) If $\phi' \circ g$ is increasing on $[a, b]$, $\int_a^x p(t) \eta(f(t), g(t)) dt \leq 0$ for all $x \in [a, b]$ and $\int_a^b p(t) \eta(f(t), g(t)) dt = 0$, then

$$\int_a^b p(t) \phi(f(t)) dt \geq \int_a^b p(t) \phi(g(t)) dt. \tag{18}$$

(ii) If $\phi' \circ f$ is decreasing on $[a, b]$, $\int_a^x p(t) \eta(g(t), f(t)) dt \geq 0$ for all $x \in [a, b]$ and $\int_a^b p(t) \eta(g(t), f(t)) dt = 0$, then

$$\int_a^b p(t) \phi(f(t)) dt \leq \int_a^b p(t) \phi(g(t)) dt. \tag{19}$$

Proof. Let set $G(x) = \int_a^x p(t) \eta(f(t), g(t)) dt$; then $G(x) \leq 0$ for all $x \in [a, b]$, and $G(a) = G(b) = 0$. If $\phi' \circ g$ is increasing on $[a, b]$, then

$$\begin{aligned} & \int_a^b p(t) \phi(f(t)) dt - \int_a^b p(t) \phi(g(t)) dt \\ & \geq \int_a^b p(t) \phi'(g(t)) \eta(f(t), g(t)) dt \\ & = \int_a^b \phi'(g(t)) dG(t) \\ & = \phi'(g(t)) G(t) \Big|_a^b - \int_a^b G(t) d\phi'(f(t)) \\ & = - \int_a^b G(t) d\phi'(g(t)) \geq 0. \end{aligned} \tag{20}$$

This proves (18).

Similarly, setting $F(x) = \int_a^x p(t) \eta(g(t), f(t)) dt$, then $F(x) \geq 0$ for all $x \in [a, b]$, and $F(a) = F(b) = 0$.

Now if $\phi' \circ f$ is decreasing on $[a, b]$, then by using Theorem 4 we have

$$\begin{aligned} & \int_a^b p(t) \phi(g(t)) dt - \int_a^b p(t) \phi(f(t)) dt \\ & \geq \int_a^b p(t) \phi'(f(t)) \eta(g(t), f(t)) dt \\ & = \int_a^b \phi'(f(t)) dF(t) \\ & = \phi'(f(t)) F(t) \Big|_a^b - \int_a^b F(t) d\phi'(f(t)) \\ & = - \int_a^b F(t) d\phi'(f(t)) \geq 0. \end{aligned} \tag{21}$$

This proves (19). □

The following extension of majorization theorem for relative invex function can be given.

Theorem 9. Let $f, g, p : [a, b] \rightarrow \mathbb{R}$ be integrable functions with p being positive function. Suppose $\phi, \psi : \mathbb{R} \rightarrow \mathbb{R}$ are such that ψ is a strictly increasing function and $\phi \circ \psi^{-1}$ is η -invex function on \mathbb{R} , where $\eta : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function. One has the following.

(i) If $(\phi \circ \psi^{-1})' \circ \psi \circ g$ and $\eta(\psi(f(t)), \psi(g(t)))$ are increasing (decreasing) on $[a, b]$, and $\int_a^b p(t) \eta(\psi(f(t)), \psi(g(t))) dt = 0$, then

$$\int_a^b p(t) \phi(f(t)) dt \geq \int_a^b p(t) \phi(g(t)) dt. \tag{22}$$

(ii) If $(\phi \circ \psi^{-1})' \circ \psi \circ g$ is increasing on $[a, b]$, $\int_a^x p(t) \eta(\psi(f(t)), \psi(g(t))) dt \leq 0$ for all $x \in [a, b]$, and $\int_a^b p(t) \eta(\psi(f(t)), \psi(g(t))) dt = 0$, then

$$\int_a^b p(t) \phi(f(t)) dt \geq \int_a^b p(t) \phi(g(t)) dt. \tag{23}$$

(iii) If $(\phi \circ \psi^{-1})' \circ \psi \circ f$ is decreasing on $[a, b]$, $\int_a^x p(t) \eta(\psi(g(t)), \psi(f(t))) dt \geq 0$ for all $x \in [a, b]$, and $\int_a^b p(t) \eta(\psi(g(t)), \psi(f(t))) dt = 0$, then

$$\int_a^b p(t) \phi(g(t)) dt \leq \int_a^b p(t) \phi(f(t)) dt. \tag{24}$$

Proof. The proof of this theorem is similar to the proof of Theorems 5 and 8. □

Remark 10. By using $\eta(x, y) = x - y$ in Theorems 4, 5, 7, 8, and 9 we recover the integral majorization results for convex functions given in [1, 4, 14].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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