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Research Article

A Concentration Phenomenon for p-Laplacian Equation

Yansheng Zhong

Department of Mathematics, Fujian Normal University, Fuzhou 350117, China

Correspondence should be addressed to Yansheng Zhong; zhyansheng08@163.com

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It is proved that if the bounded function of coefficient Q_n in the following equation $-\operatorname{div}\{|\nabla u|^{p-2}\nabla u\} + V(x)|u|^{p-2}u = Q_n(x)|u|^{q-2}u$, $u(x) = 0$ as $x \in \partial\Omega$. $u(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$ is positive in a region contained in Ω and negative outside the region, the sets $\{Q_n > 0\}$ shrink to a point $x_0 \in \Omega$ as $n \to \infty$, and then the sequence u_n generated by the nontrivial solution of the same equation, corresponding to Q_n, will concentrate at x_0 with respect to $W_0^{1,p}(\Omega)$ and certain $L^s(\Omega)$-norms. In addition, if the sets $\{Q_n > 0\}$ shrink to finite points, the corresponding ground states $\{u_n\}$ only concentrate at one of these points. These conclusions extend the results proved in the work of Ackermann and Szulkin (2013) for case $p = 2$.

1. Introduction

We study a new concentration phenomenon for the following p-Laplacian equations:

$$
\begin{aligned}
&-\operatorname{div}\left\{|\nabla u|^{p-2}\nabla u\right\} + V(x)|u|^{p-2}u = Q_n(x)|u|^{q-2}u,\\
&u(x) = 0 \quad \text{as } x \in \partial\Omega. \\
&u(x) \longrightarrow 0 \quad \text{as } |x| \longrightarrow \infty,
\end{aligned} \tag{1}
$$

where $\Omega \subset \mathbb{R}^N$ is a smooth domain and $V \geqslant 0\,(\in L^\infty(\Omega))$, and $p < q < p^*$, where $p^* := Np/(N-p)$ if $N \geqslant p$ and $p^* := \infty$ if $N < p$. If Ω is unbounded, we assume additionally that $\sigma(-\operatorname{div}(|\nabla\cdot|^{p-2}\nabla\cdot) + V|\cdot|^{p-2}\cdot) \subset (0,\infty)$.

And an assumption of Q_n is as follows.

$(*)$ The set $\{x \mid Q_n(x) > 0\}$ contained in the neighborhood of zero has positive measure, and $|Q_n|_{L^\infty(\Omega)} \leqslant C$ with the constant C is independent of n. Moreover, for each $\epsilon > 0$ there exist constants $\delta_\epsilon(> 0)$ and N_ϵ such that $Q_n \leqslant -\delta_\epsilon$ whenever $x \notin B_\epsilon(0)$ and $n \geqslant N_\epsilon$.

As it is known, $u \equiv 0$ is the only solution to (1) if $Q_n(x) \leqslant 0$ for all $x \in \Omega$. In addition, if $Q_n(x) > 0$ is based on a bounded set of positive measures, it is clear that there exists a solution $u \not\equiv 0$ (see Theorem 1). Hence, without loss of generality, we assume that $0 \in \Omega$ and let $Q = Q_n$ be such that $Q_n > 0$ on the ball $B_{1/n}(0)$ and $Q_n < 0$ on $\Omega \setminus B_{2/n}(0)$ and $u_n \not\equiv 0$ are the solutions to (1) associated with $Q_n(x)$. Accordingly, the question is what happens to u_n as $n \to \infty$. Furthermore, this phenomenon can be found in physics. For instance, considering the materials separately from Q positive or negative (see [1]), it corresponds to investigating the existence of bright ($Q > 0$) or dark ($Q < 0$) solitons.

Equations of these types have been studied extensively in many monographs and lectures (e.g., [2–10] for $p = 2$, [11–18] for general p). In [2], Byeon and Wang considered the standing wave solutions $\psi(x,t) \equiv \exp(-iEt/\hbar)v(x)$ for the nonlinear Schrödinger equation:

$$
i\hbar\frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2}\Delta\psi - V(x)\psi + |\psi|^{p-1}\psi = 0, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^N. \tag{2}
$$

Thus, they needed only to discuss the function v which satisfies

$$
\frac{\hbar^2}{2}\Delta v - (V(x) - E)v + |v|^{p-1}v = 0, \quad x \in \mathbb{R}^N, \tag{3}
$$

and rewrote it in the following form:

$$
\begin{aligned}
&\epsilon^2\Delta v - V(x)v + v^p = 0, \quad v > 0,\ x \in \mathbb{R}^N \\
&\lim_{|x|\to 0} v(x) = 0.
\end{aligned} \tag{4}
$$

By a rescaling, it is transformed to

$$\Delta u - V(\epsilon x)u + u^p = 0, \quad u > 0, \; x \in \mathbb{R}^N$$
$$\lim_{|x|\to 0} u(x) = 0. \tag{5}$$

Let the zero set $\mathcal{Z} \triangleq \{x \in \mathbb{R}^N \mid V(x) = 0\}$ and A be an isolated component of $\mathcal{Z}$, and they distinguished three cases of A to prove the concentration as $\epsilon \to 0$. And then, in [3] by replacing v^p with a fairly general class nonlinearity $f(v)$, they also obtained the concentration. Furthermore, in [4], Byeon and Jeanjean gave the almost optimal condition on f for the concentration. Recently, in [19], different from above with the linearity term $V(\epsilon x)u$, Ackermann and Szulkin considered the concentration phenomenon in the nonlinearity; that is, $-\Delta u + V(x)u = Q_n(x)|u|^{p-2}u$. In contrast, by following the similar strategy as in [19], we first show that the concentration phenomenon also occurs in the general p-Laplacian equation. It seems that this concentration phenomenon was unknown earlier, but to some extent, it answers the question mentioned above.

This paper is organized as follows. In Section 2, we prove that the solutions $\{u_n\}$ to (1) concentrate at the origin in the $W_0^{1,p}(\Omega)$ and the L^q-norm; in Section 3, concentration in the L^s-norms for different s is considered and Section 4 shows that the ground states only concentrate at one of these points when Q_n is positive in a neighbourhood of a finite number of points.

2. Concentration in the $W_0^{1,p}(\Omega)$ and $L^q(\Omega)$

We begin with some notations.

Let $E := W_0^{1,p}(\Omega)$ and

$$\|u\| := \left(\int_\Omega (|\nabla u|^p + V|u|^p)\,dx\right)^{1/p} \tag{6}$$

is an equivalent norm in E (due to $\sigma(-\operatorname{div}(|\nabla\cdot|^{p-2}\nabla\cdot) + V|\cdot|^{p-2}\cdot) \subset (0,\infty)$). Set

$$|u|_{s,A} := \left(\int_A |u|^s dx\right)^{1/s}, \tag{7}$$

$|u|_{\infty,A} = \operatorname{esssup}_A|u|$, and we abbreviate $|u|_{s,A}$ to $|u|_s$ sometimes. Moreover,

$$B_r(a) := \{x \in \mathbb{R}^n : |x-a| < r\} \tag{8}$$

denotes a ball.

Here we offer the existence result for (1).

Theorem 1. *Suppose that Q_n satisfies the assumption $(*)$ above and $q \in (p, p^*)$; then for all sufficiently large n, there is a positive ground state solution $u_n \in E$ to problem (1). Moreover, there exists a constant $\alpha > 0$ independent of n, such that $\|u_n\| \geqslant \alpha$.*

Proof. As in [19], let $J_n(v) = \int_\Omega Q_n|v|^q dx$ and

$$s_n := \inf_{J_n(v)>0} \frac{\|v\|^p}{|J_n(v)|^{p/q}} = \inf_{J_n(v)>0} \frac{\int_\Omega |\nabla v|^p + V|v|^p dx}{\left(\int_\Omega Q_n|v|^q dx\right)^{p/q}}. \tag{9}$$

Suppose that (v_k) is a minimizing sequence for s_n, normalized by $J_n(v_k) = 1$; then $\|v_k\|$ is bounded. Hence, $v_n \rightharpoonup v$ in E and $v_k(x) \to v(x)$ a.e. in Ω (by choosing a subsequence). Note that $Q_n < 0$ on $|x| > 1$ for n large. The Rellich-Kondrachov Theorem and Fatous's Lemma say that

$$\begin{aligned} s_n &= \lim_{k\to\infty} \|v_k\|^p \\ &= \lim_{k\to\infty} \frac{\|v_k\|^p}{\left(\int_{|x|<1} Q_n|v_k|^q dx + \int_{|x|>1} Q_n|v_k|^q dx\right)^{p/q}} \\ &\geqslant \frac{\|v\|^p}{J_n(v)^{p/q}} \geqslant s_n. \end{aligned} \tag{10}$$

Thus v is a minimizer.

And then, the lagrange multiple rule implies that $u_n = c_n v_n$ is a solution to (1) for some appropriate constant $c_n > 0$. Moreover, since v_n may be replaced by $|v_n|$, $v_n \geqslant 0$ (and hence $u_n \geqslant 0$). To show that $u_n > 0$, we note that u_n satisfies

$$\begin{aligned} &-\operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right) + \left(V(x)u_n^{p-2} + Q_n^-(x)u_n(x)^{q-2}\right)v \\ &\quad = Q_n^+(x)u_n(x)^{q-1} \geqslant 0, \end{aligned} \tag{11}$$

where $Q_n^\pm := \max\{\pm Q_n(x), 0\}$. Since $V(x)u_n^{p-2} + Q_n^-(x)u_n(x)^{q-2} \geqslant 0$, it follows from the strong maximum principle (see [20, 21]) that $u_n > 0$.

If $u_n \neq 0$ is a solution to (1), then, via multiplying the equation by u_n, integrating by parts, and using the Sobolev inequality, one deduces that

$$\|u_n\|^p = \int_\Omega Q_n|u_n|^q dx \leqslant c_1|u_n|_q^q \leqslant c_2\|u_n\|^q; \tag{12}$$

hence, $\|u_n\| \geqslant \alpha$ for some $\alpha > 0$ and all large n. □

The next step is to consider the property of the nontrivial solution $\{u_n\}$ to (1) and $w_n := u_n/\|u_n\|$.

Lemma 2. *Consider*

$$\|u_n\| \longrightarrow \infty \quad \text{as } n \longrightarrow \infty. \tag{13}$$

Proof. We present an abridged version of the proof highlighting the main differences to that in [19]. It will be proved by contradiction. Assume $u_n \rightharpoonup u$ in E and $u_n \to u$ in $L^q_{\text{loc}}(\Omega)$ after passing to a subsequence. Multiplying (1) (with $u = u_n$) by u_n, integrating by parts, and recalling that $Q_n < 0$ for each $\epsilon > 0$ and $n \geqslant N_\epsilon$, it holds that

$$\begin{aligned} \limsup_{n\to\infty}\|u_n\|^p &= \limsup_{n\to\infty} \int_\Omega Q_n|u_n|^q dx \\ &\leqslant \limsup_{n\to\infty} \int_{|x|<\epsilon} Q_n|u_n|^q dx \leqslant c\int_{|x|<\epsilon} |u|^q dx. \end{aligned} \tag{14}$$

□

If $\epsilon \to 0$, $u_n \to 0$ in E. It is a contradiction to $\|u_n\| \geqslant \alpha > 0$ given in Theorem 1.

Lemma 3. *Consider*

$$w_n \rightharpoonup 0 \text{ in } E \quad \text{as } n \longrightarrow \infty \tag{15}$$

Proof. We prove it by contradiction as well. We may assume that $w_n \rightharpoonup w(\not\equiv 0)$ in E. Multiplying (1) (with $u = u_n$) by $u_n/\|u_n\|^p$ yields that

$$1 = \|w_n\|^p = \|u_n\|^{q-p} \int_\Omega Q_n |w_n|^q dx. \tag{16}$$

Due to Lemma 2 with $q > p$, $\int_\Omega Q_n |w_n|^q \to 0$.

On the other hand, we have for $0 < \epsilon < \epsilon_1$

$$\begin{aligned} 0 &= \lim_{n\to\infty} \int_\Omega Q_n |w_n|^q dx \\ &= \lim_{n\to\infty} \left(\int_{|x|<\epsilon} Q_n |w_n|^q dx + \int_{|x|>\epsilon} Q_n |w_n|^q dx \right) \\ &\leqslant \lim_{n\to\infty} \left(\int_{|x|<\epsilon} Q_n |w_n|^q dx + \int_{|x|>\epsilon_1} Q_n |w_n|^q dx \right) \\ &\leqslant c \int_{|x|<\epsilon} |w_n|^q dx - \delta_{\epsilon_1} \int_{|x|>\epsilon_1} |w_n|^q dx. \end{aligned} \tag{17}$$

We may choose small ϵ_1 such that the second integral on the right-hand side above is positive as $w \not\equiv 0$. Then we get the contradiction as $\epsilon \to 0$.

In the sequel, we study concentration of $\{u_n\}$ as $n \to \infty$. Let $\epsilon > 0$ be given and $\chi \in C^\infty(\Omega, [0,1])$ be such that $\chi(x) = 0$ for $x \in B_{\epsilon/2}(0)$ and $\chi(x) = 1$ for $x \notin B_\epsilon(0)$.

Multiplying (1) (with $u = u_n$) by χu_n we obtain

$$\int_\Omega \left(|\nabla u_n|^{p-2} \nabla u_n \cdot \nabla (\chi u_n) + \chi V u_n^p \right) dx = \int_\Omega \chi Q_n |u_n|^q dx, \tag{18}$$

namely,

$$\begin{aligned} &\int_\Omega \chi \left(|\nabla u_n|^p + V u_n^p \right) dx - \int_\Omega \chi Q_n |u_n|^q dx \\ &\quad = - \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla \chi \cdot u_n dx. \end{aligned} \tag{19}$$

Given $\epsilon > 0$, we have $Q_n \leqslant -\delta_\epsilon$ on $\operatorname{supp}\chi$, provided that n is large enough. Hence for all such n,

$$\begin{aligned} 0 &\leqslant \int_{\Omega\setminus B_\epsilon(0)} \left(|\nabla u_n|^p + V u_n^p \right) dx + \delta_\epsilon \int_{\Omega\setminus B_\epsilon(0)} |u_n|^q dx \\ &\leqslant \int_\Omega \chi \left(|\nabla u_n|^p + V u_n^p \right) dx - \int_\Omega \chi Q_n |u_n|^q dx \\ &= - \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla \chi \cdot u_n dx \\ &\leqslant d_\epsilon \int_{B_\epsilon(0)\setminus B_{\epsilon/2}(0)} |u_n| \, |\nabla u_n|^{p-1} dx, \end{aligned} \tag{20}$$

where d_ϵ is a constant independent of n. Since $w_n = u_n/\|u_n\| \to 0$ in $L^p_{\text{loc}}(\Omega)$ according to Lemma 3, it follows from Hölder inequality that

$$\int_{B_\epsilon(0)\setminus B_{\epsilon/2}(0)} |w_n| \, |\nabla w_n|^{p-1} dx \longrightarrow 0. \tag{21}$$

So (20) implies

$$\int_{\Omega\setminus B_\epsilon(0)} \left(|\nabla w_n|^p + V w_n^p \right) dx + \delta_\epsilon \|u_n\|^{q-p} \int_{\Omega\setminus B_\epsilon(0)} |w_n|^q dx = 0. \tag{22}$$

□

Theorem 4. *Suppose that Q_n satisfies the assumption $(*)$ and $q \in (p, p^*)$. Let u_n be a nontrivial solution to (1) and put $w_n = u_n/\|u_n\|$. Then for every $\epsilon > 0$ they hold that*

$$\lim_{n\to\infty} \int_{\Omega\setminus B_\epsilon(0)} \left(|\nabla w_n|^p + V w_n^p \right) dx = 0, \tag{23}$$

$$\lim_{n\to\infty} \|u_n\|^{q-p} \int_{\Omega\setminus B_\epsilon(0)} |w_n|^q dx = 0. \tag{24}$$

Moreover,

$$\begin{aligned} &\lim_{n\to\infty} \frac{\int_{\Omega\setminus B_\epsilon(0)} \left(|\nabla w_n|^p + V w_n^p \right) dx}{\int_\Omega \left(|\nabla w_n|^p + V w_n^p \right) dx} = 0, \\ &\lim_{n\to\infty} \frac{\int_{\Omega\setminus B_\epsilon(0)} |w_n|^q dx}{\int_\Omega |w_n|^q dx} = 0. \end{aligned} \tag{25}$$

Proof. (23) and (24) can be easily obtained by (22). Note that

$$\int_\Omega \left(|\nabla w_n|^p + V w_n^p \right) dx = \|w_n\|^p = 1. \tag{26}$$

From (23), one concludes that

$$\begin{aligned} &\lim_{n\to\infty} \frac{\int_{\Omega\setminus B_\epsilon(0)} \left(|\nabla u_n|^p + V u_n^p \right) dx}{\int_\Omega \left(|\nabla u_n|^p + V u_n^p \right) dx} \\ &\quad = \lim_{n\to\infty} \frac{\int_{\Omega\setminus B_\epsilon(0)} \left(|\nabla w_n|^p + V w_n^p \right) dx}{\int_\Omega \left(|\nabla w_n|^p + V w_n^p \right) dx} = 0. \end{aligned} \tag{27}$$

According to (16), we get

$$c\|u_n\|^{q-p} \int_\Omega |w_n|^q dx \geqslant \|u_n\|^{q-p} \int_\Omega Q_n |w_n|^q dx = \|w_n\|^p = 1. \tag{28}$$

This and (24) imply

$$\lim_{n\to\infty} \frac{\int_{\Omega\setminus B_\epsilon(0)} |w_n|^q dx}{\int_\Omega |w_n|^q dx} = \lim_{n\to\infty} \frac{\|u_n\|^{q-p} \int_{\Omega\setminus B_\epsilon(0)} |w_n|^q dx}{\|u_n\|^{q-p} \int_\Omega |w_n|^q dx} = 0. \tag{29}$$

□

3. Concentration in the L^s-Norm

The next is to consider the concentration in other norms.

Theorem 5. *Let u_n denote a nontrivial solution to* (1) *for each $n \in \mathbb{N}$. Suppose that the assumption* $(*)$ *holds and there exists $R, \lambda > 0$ such that $V \geqslant \lambda$ whenever $x \in \Omega \setminus B_R(0)$, and there exists $\epsilon > 0$ such that $\overline{B_\epsilon(0)} \subset \Omega$; then one can get that*

(a) *$\exists C$, for all $s \in [1, \infty]$, $n \in \mathbb{N}$, $|u_n|_{s,\Omega\setminus B_\epsilon(0)} \leqslant C$;*

(b) *if $\delta = \delta_\epsilon > 0$ in $(*)$ can be chosen independently of ϵ (> 0), then $\lim_{n\to\infty}|u_n|_{s,\Omega\setminus B_\epsilon(0)} = 0$, for every $s \in [1, \infty]$;*

(c) *for all $s\,(\geqslant 1) \in (N(q-p)/p, \infty]$, one has $\lim_{n\to\infty} |u_n|_s = \infty$ and*

$$\lim_{n\to\infty} \frac{|u_n|_{s,\Omega\setminus B_\epsilon(0)}}{|u_n|_s} = 0; \tag{30}$$

(d) *if $N(q-p)/p \geqslant 1$, then for $s = N(q-p)/p$ it holds that*

$$\liminf_{n\to\infty} |u_n|_s > 0. \tag{31}$$

If the hypotheses in (b) are satisfied, then (30) *also holds for this s.*

Proof. There is clearly a positive classical solution w to the equation

$$\begin{gathered} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = -\delta_{\epsilon/2}|u|^{q-2}u, \quad x \in \mathbb{R}^n \setminus \overline{B_{\epsilon/2}(0)} \\ \lim_{|x|\to\epsilon/2} w(x) = \infty, \qquad \lim_{|x|\to\infty} w(x) = 0. \end{gathered} \tag{32}$$

In fact, by [22, 23], the radial solution $u_p(x) = u_p(|x|)$ satisfies the ordinary differential equation

$$\begin{gathered} \left(r^{n-1}|u'|^{p-2}u'\right)' = -\delta_{\epsilon/2} r^{n-1} u^q \\ u(r) = \infty \quad \text{as} \quad r \longrightarrow \epsilon/2, \\ u(r) \longrightarrow 0 \quad \text{as} \quad r \longrightarrow \infty. \end{gathered} \tag{33}$$

Set $z_n = w - u_n$ and

$$\varphi_n(x) := (q-1)\int_0^1 |sw(x) + (1-s)u_n(x)|^{q-2} ds \geqslant 0,$$

$$\phi_n(x) := (p-1)\int_0^1 |sw(x) + (1-s)u_n(x)|^{p-2} ds \geqslant 0,$$

$$\begin{aligned} &\varphi_n(x) z_n \\ &\quad = (q-1)\int_0^1 |sw(x) + (1-s)u_n(x)|^{q-2}(w - u_n)\,ds \\ &\quad = \int_0^1 \frac{d}{ds}\left(|sw + (1-s)u_n|^{p-2}(sw + (1-s)u_n)\right) ds \\ &\quad = w^{q-1} - |u_n|^{q-2}u_n, \\ &\phi_n(x) z_n \\ &\quad = (q-1)\int_0^1 |sw(x) + (1-s)u_n(x)|^{p-2}(w - u_n)\,ds \\ &\quad = \int_0^1 \frac{d}{ds}\left(|sw + (1-s)u_n|^{p-2}(sw + (1-s)u_n)\right) ds \\ &\quad = w^{p-1} - |u_n|^{p-2}u_n \end{aligned} \tag{34}$$

and hence from $(*)$

$$\begin{aligned} &-\operatorname{div}\left(|\nabla w|^{p-2}\nabla w\right) - \left(-\operatorname{div}\left(|\nabla u_n|^{p-2}\nabla u_n\right)\right) \\ &+ \left(V\phi_n(x) - Q_n\psi_n\right) z_n \\ &\quad = -\operatorname{div}|\nabla w|^{p-2} + V|w|^{p-2}w - Q_n w^{q-1} \\ &\qquad - \left[-\operatorname{div}|\nabla u|^{p-2}\nabla u + V|\nabla u|^{p-2}\nabla u - Q_n|u_n|^{q-2}u_n\right] \\ &\quad = -\operatorname{div}|\nabla w|^{p-2} + V|w|^{p-2}w - Q_n w^{q-1} \\ &\quad \geqslant -\operatorname{div}|\nabla w|^{p-2} + \delta_{\epsilon/2} w^{q-1} = 0. \end{aligned} \tag{35}$$

Note that $V\phi_n(x) - Q_n\varphi_n \geqslant 0$ in $\Omega \setminus \overline{B_{\epsilon/2}(0)}$ when $n \geqslant N_{\epsilon/2}$. Due to the continuity of u_n and the fact that $w_n(x) \to \infty$ as $x \to \partial B_{\epsilon/2}(0)$, there is $r \in (\epsilon/2, \epsilon)$ such that $z_n \geqslant 0$ on $\partial B_r(0)$. Moreover, $z_n \geqslant 0$ on $\partial\Omega$. If Ω is bounded, the maximum principle says that $z_n \geqslant 0$ in $\Omega \setminus B_r(0)$ (see [20, 21]). If Ω is unbounded, by virtue of $w(x)$ tending to 0 as $|x| \to \infty$ by construction, thus for any $\gamma > 0$, we may pick $\widetilde{R} > 0$ such that $z_n \geqslant -\gamma$ in $\Omega \setminus B_{\widetilde{R}}(0)$. Moreover, applying regularity theory to $u_n \in W_0^{1,p}(\Omega)$, we can get $u_n(x) \to 0$ as $|x| \to \infty$. Now the same maximum principle is applied on $\Omega \cap (B_{\widetilde{R}} \setminus \overline{B_r(0)})$, which implies that $z_n \geqslant -\gamma$ in all of $\Omega \setminus B_r(0)$. Letting $\gamma \to 0$, we obtain $z_n > 0$ again. By analogy we obtain $u_n \geqslant -w$ (take $z_n := w + u_n$); hence

$$|z_n| \geqslant w \qquad \text{in } \Omega \setminus B_\epsilon(0), \ \forall n \geqslant N_{\epsilon/2}. \tag{36}$$

Hence (a) follows from above arguments with the fact that w is continuous in $\Omega \setminus B_\epsilon(0)$.

Next, the hypotheses in (b) imply that there is $\delta > 0$ such that $Q_n \leqslant -\delta$ on $\Omega \setminus B_{1/n}(0)$ for each n large enough. Let w_n be a positive solution to

$$\begin{gathered} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = -\delta|u|^{q-2}u, \quad x \in \mathbb{R}^n \setminus B_{1/n}(0) \\ \lim_{|x|\to 1/n} w_n(x) = +\infty, \qquad \lim_{|x|\to\infty} w_n(x) = 0. \end{gathered} \tag{37}$$

Then the sequence w_n is monotone decreasing, by using the maximum principle to $w_n \geqslant w_{n+1}$ on $\partial B_{1/n}(0)$ for every $n \in \mathbb{N}$. Therefore, w_n converges locally and uniformly to a nonnegative solution w to (37) on $\mathbb{R}^n \setminus \{0\}$. It follows from our hypotheses on N and p that w is an entire solution to (37) by applying the argument as in [24]. And then, due to [25], $w \equiv 0$. For another, the function w_n dominates the solution u_n on $\overline{\Omega} \setminus B_r(0)$ for some $r \in (\epsilon/2, \epsilon)$, as seen in the proof of (a). Thus, u_n also converges to 0 locally and uniformly in $\Omega \setminus B_r(0)$; that is, $\lim_{n\to\infty} |u_n|_{s,\Omega\setminus B_r(0)} = 0$.

For (c), we first consider the case $s\,(\geqslant 1) \in (N(q-p)/p, q]$. By interpolation inequality, we have the following estimate for solution u_n:

$$\begin{aligned} \|u_n\|^p &= \int_\Omega Q_n |u_n|^q dx \leqslant c_1 |u_n|_q^q \leqslant c_1 |u_n|_s^{q\theta} |u_n|_{p^*}^{q(1-\theta)} \\ &\leqslant c_2 |u_n|_s^{q\theta} \|u_n\|^{q(1-\theta)}. \end{aligned} \tag{38}$$

Here c_1, c_2 are independent of n, and θ satisfies that

$$\frac{1}{q} = \frac{\theta}{s} + \frac{1-\theta}{p^*}. \tag{39}$$

According to Lemma 2, it suffices to impose that $q(1-\theta) < p$ or equivalent $s > N(q-p)/p$. This and (a) prove the case $s \in (N(q-p)/p, q]$. And then, (38) and (a) yield $|u_n|_{q,B_\epsilon(0)} \to \infty$; hence $|u_n|_{s,B_\epsilon(0)} \to \infty$ for every $s \in (q, \infty]$ as $n \to \infty$. Using (a) again we get (30).

Note that (38) implies (30) for $s = N(q-p)/p$, so case (d) is easily followed. □

4. Concentration at Several Points

Now we assume that the function Q_n is positive in a neighbourhood of two distinct points $x_1, x_2 \in \Omega$ (indeed, the following argument is also valid for any finite number of points in Ω). More precisely, we assume.

$(**)$ $Q_n > 0$ in a neighbourhood of $\{x_1\} \cup \{x_2\}$, and there exists a constant C such that $|Q_n|_{L^\infty(\Omega)} \leqslant C$ for all n. Moreover, for each $\epsilon > 0$ there exist constants $\delta_\epsilon > 0$ and N_ϵ such that $Q_n \leqslant -\delta_\epsilon$ whenever $x \notin B_\epsilon(x_1) \cup B_\epsilon(x_2)$ and $n \geqslant N_\epsilon$.

As in Section 2, we put $J_n(u) = \int_\Omega Q_n |u|^q dx$:

$$s_n := \inf_{J_n(u)} \frac{\|u\|^p}{J_n(u)^{p/q}} \equiv \inf_{J_n(u)} \frac{\int_\Omega (|\nabla u|^p + V|u|^p)\, dx}{\left(\int_\Omega Q_n |u|^q dx\right)^{p/q}}. \tag{40}$$

Theorem 6. *Suppose Q_n satisfies $(**)$ and $q \in (p, p^*)$, and u_n is a ground state solution to (1). Then, for n large, u_n concentrates at x_1 or x_2. More precisely, for each $\epsilon > 0$ we have by passing to a subsequence*

$$\begin{aligned} &\lim_{n\to\infty} \frac{\int_{\Omega\setminus B_\epsilon(x_j)} \left(|\nabla u|^p + V u_n^p\right) dx}{\int_\Omega \left(|\nabla u|^p + V u_n^p\right) dx} = 0, \\ &\lim_{n\to\infty} \frac{\int_{\Omega\setminus B_\epsilon(x_j)} Q_n u_n^p dx}{\int_\Omega Q_n u_n^p dx} = 0 \end{aligned} \tag{41}$$

for $j = 1$ or 2 (but not for $j = 1$ and 2).

Remark 7. Note that, in view of the obvious modification of Theorem 4, the limits in (41) are 0 if $\Omega \setminus B_\epsilon(x_j)$ is replaced by $\Omega \setminus B_\epsilon(x_1) \cup B_\epsilon(x_2)$. So if $j = 1$ in (41), then concentration occurs at x_1 and if $j = 2$, it occurs at x_2.

Proof. As in [19], we may assume that $J_n(u_n) = \int_\Omega Q_n |u_n|^p dx = 1$ by renormalizing (u_n may not be a solution to (1), but we still have $s_n := \|u_n\|^p / J_n(u_n)^{p/q}$). Let $\xi_j \in C_0^\infty(\Omega, [0,1])$ be a function such that $\xi_j = 1$ on $B_{\epsilon/2}(x_j)$ and $\xi_j = 0$ on $\Omega \setminus B_\epsilon(x_j)$, $j = 1, 2$, where ϵ is so small that $\overline{B_\epsilon(x_j)} \subset \Omega$ and $\overline{B_\epsilon(x_1)} \cap \overline{B_\epsilon(x_2)} = \emptyset$. Set $v_n := \xi_1 u_n$, $w_n := \xi_2 u_n$, and $z_n := u_n - v_n - w_n$. Since $\operatorname{supp} z_n \subset \Omega \setminus (B_{\epsilon/2}(x_1) \cup B_{\epsilon/2}(x_2))$ and the conclusion of Theorem 4 remains valid after a modification, we have

$$\begin{aligned} \|u_n\|^p &= \int_\Omega \left(|\nabla u_n|^p + V u_n^p\right) dx \\ &= \left(\int_\Omega \left(|\nabla v_n|^p + V u_n^p\right) dx \right. \\ &\quad \left. + \int_\Omega \left(|\nabla w_n|^p + V w_n^p\right) dx \right) (1 + o(1)) \\ &= \left(\|v_n\|^p + \|w_n\|^p\right)(1 + o(1)), \\ 1 = J_n(u_n) &= \int_\Omega Q_n |u_n|^q dx \\ &= \int_\Omega Q_n |v_n|^q dx + \int_\Omega Q_n |w_n|^q dx + o(1) \\ &= J_n(v_n) + J_n(w_n) + o(1). \end{aligned} \tag{42}$$

First, we assume that $\limsup_{n\to\infty} J_n(v_n) \geqslant 0$ and $\limsup_{n\to\infty} J_n(w_n) \geqslant 0$. By passing to a subsequence, we may assume that $J_n(v_n) \to c_0 \in [0,1]$ and $J_n(w_n) \to 1 - c_0 \in [0,1]$. If $c_0 \in (0,1)$, recalling that $q > p$, we get a contradiction from the following inequality:

$$\begin{aligned} s_n &= \frac{\|u_n\|^p}{J_n(u_n)^{p/q}} = \frac{\left(\|v_n\|^p + \|w_n\|^p\right)(1 + o(1))}{\left(J_n(v_n) + J_n(w_n) + o(1)\right)^{p/q}} \\ &> \frac{\|v_n\|^p + \|w_n\|^p}{J_n(v_n)^{p/q} + J_n(w_n)^{p/q}} \\ &\geqslant \min\left\{ \frac{\|v_n\|^p}{J_n(v_n)^{p/q}}, \frac{\|w_n\|^p}{J_n(w_n)^{p/q}} \right\} \geqslant s_n. \end{aligned} \tag{43}$$

So $c_0 = 0$ or 1. If $c_0 = 1$ (say), then the second limit in (41) is 0 for $j = 1$ because $\operatorname{supp} v_n \subset B_\epsilon(x_1)$. The first limit is 0 as well, since $\|w_n\|^p / \|v_n\|^p$ is otherwise bounded away from 0 for large n, and we obtain a contradiction again from

$$s_n = \frac{\left(\|v_n\|^p + \|w_n\|^p\right)(1 + o(1))}{\left(J_n(v_n) + J_n(w_n) + o(1)\right)^{p/q}} > \frac{\|v_n\|^p}{J_n(v_n)^{p/q}} \geqslant s_n. \tag{44}$$

Finally, suppose $\limsup_{n\to\infty} J_n(w_n) < 0$ (the case $\limsup_{n\to\infty} J_n(v_n) < 0$ is of course analogous); it passes to a subsequence $J_n(w_n) \leqslant -\eta$ for some $\eta > 0$ when n is large enough. Then a contradiction (44) holds for such n because $J_n(v_n) > J_n(v_n) + J_n(w_n) + o(1)$. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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