*Research Article*

# Existence and Exponential Stability of Solutions for Stochastic Cellular Neural Networks with Piecewise Constant Argument

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By using the concept of differential equations with piecewise constant argument of generalized type, a model of stochastic cellular neural networks with piecewise constant argument is developed. Sufficient conditions are obtained for the existence and uniqueness of the equilibrium point for the addressed neural networks. $p$th moment exponential stability is investigated by means of Lyapunov functional, stochastic analysis, and inequality technique. The results in this paper improve and generalize some of the previous ones. An example with numerical simulations is given to illustrate our results.

## 1. Introduction

Since Chua and Yang introduced cellular neural networks (CNNs) [1, 2] in 1988, delayed cellular neural networks (DCNNs) were proposed by Chua and Roska [3] in 1990; the dynamics of CNNs and DCNNs have received great attention due to their wide applications in classification of patterns, associative memories, optimization problems, and so forth. As is well known, such applications depend on the existence of an equilibrium point and its stability. Up to now, many results on stability of delayed neural networks have been developed [3–25].

In real nervous systems, there are many stochastic perturbations that affect the stability of neural networks. The results in [26] suggested that one neural network could be stabilized or destabilized by certain stochastic inputs. It implies that the stability analysis of stochastic neural networks has primary significance in the design and applications of neural networks, such as [13–26], but only few works have been done on the $p$th moment exponentially stable for stochastic cellular neural networks [22–25].

The theory of differential equations with piecewise constant argument was initiated in Cooke and Wiener [27] and Shah and Wiener [28]. These equations represent a hybrid of continuous and discrete dynamical systems and combine the properties of both the differential and difference equations. It is well known that the reduction of differential equations with piecewise constant argument to discrete equations has been the main and possibly a unique way of stability analysis for these equations [27–32]. Particularly, one cannot investigate the problem of stability completely, as only elements of a countable set are allowed to be discussed for initial moments. By introducing arbitrary piecewise constant functions as arguments, the concept of differential equations with a piecewise constant argument has been generalized in [12, 33–38], where an integral representation formula was proposed as another approach to meet the challenges discussed above.

To the best of our knowledge, the equations with piecewise constant arguments were not considered as models of neural networks, except possibly in [12, 36–38]. In [12], the authors assume CNNs may "memorize" values of the phase variable at certain moments of time to utilize the values during middle process till the next moment. Thus, they arrive at differential equations with a piecewise constant argument. Obviously, the distances between the "memorized" moments may be very variative. Consequently, the concept of generalized type of piecewise constant argument is fruitful. But these systems are deterministic; the dynamical behavior of stochastic neural networks with piecewise constant arguments has never been tackled.

Motivated by the discussion above, our paper attempts to fill the gap by considering stochastic cellular neural networks

with piecewise constant arguments. In this paper, criteria on the $p$th moment exponential stability of equilibrium point can be derived by constructing a suitable Lyapunov functional; the results obtained in this paper generalize and improve some of the existing results in [12, 38].

The remainder of this paper is organized as follows. In Section 2, we introduce some notations and assumptions. In Section 3, a sufficient condition for the existence and uniqueness of the solution is obtained. In Section 4, we establish our main results on $p$th moment exponentially stable. A numerical example is given in Section 5 to demonstrate the theoretical results of this paper. Finally, some conclusions are given in Section 6.

## 2. Preliminaries

In this paper, let $\mathbb{R}^+ = [0, \infty)$, $\mathscr{L}^p(\mathbb{R}^+, \mathbb{R}^n)$ be the family of $\mathbb{R}^n$-valued $\mathscr{F}_t$-adapted process $\{f(t)\}_{t \geq 0}$ such that, for every $T > 0$, $\int_0^T |f(t)|^p dt < \infty$, $\mathscr{M}^p([t_0, \infty), \mathbb{R}^n)$ be the family of $\mathbb{R}^n$-valued $\mathscr{F}_t$-adapted process $\{f(t)\}_{t \geq t_0}$ such that, for every $T > t_0$, $E \int_{t_0}^T |f(t)|^p dt < \infty$. $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\mathbb{R}^n$ denotes the $n$-dimensional real space, $n \in \mathbb{N}$. We fix real-valued sequences $\theta_i$ such that $0 = \theta_0 < \theta_1 < \cdots < \theta_i < \theta_{i+1} < \cdots$ with $\theta_i \to \infty$ as $i \to \infty$.

We study stochastic cellular neural networks with piecewise constant arguments described by the differential equations:

$$
dx_i(t) = \left[ -a_i x_i(t) + \sum_{j=1}^n b_{ij} f_j(x_j(t)) \right.
$$
$$
\left. + \sum_{j=1}^n c_{ij} g_j(x_j(\beta(t))) + I_i \right] dt
$$
$$
+ \sum_{l=1}^n \sigma_{il}(t, x_i(t), x_i(\beta(t))) dw_l(t),
$$
$$
i \in \Lambda = \{1, 2, \ldots, n\}.
$$

If $t \in [\theta_k, \theta_{k+1})$, $\beta(t) = \theta_k$, $k \in \mathbb{N}$. $n$ corresponds to the number of units in a neural network, $x_i(t)$ stands for the potential (or voltage) of cell $i$ at time $t$, $f_j(\cdot)$, $g_j(\cdot)$ are activation functions, $a_i > 0$ denotes the rate with which cell $i$ resets its potential to the resting state when isolated from the other cells and inputs, and $b_{ij}$ and $c_{ij}$ denote the strengths of connectivity between cells $i$ and $j$ at times $t$ and $\beta(t)$, respectively. $I_i$ denotes the external bias on the $i$th unit. Moreover, $w(t) = (w_1(t), \ldots, w_n(t))^T$ is $n$-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathscr{F}, P)$ with a natural filtration $\{\mathscr{F}_t\}_{t \geq 0}$ generated by $\{w(s) : 0 \leq s \leq t\}$, where we associate $\Omega$ with the canonical space generated by $w(t)$, and denote by $\mathscr{F}$ the associated $\sigma$-algebra generated by $w(t)$ with the probability measure $P$. Let $L^p(\Omega, \mathbb{R}^n)$ be the family of $\mathbb{R}^n$-valued random variables $\xi$ with $E|\xi|^p < \infty$, $\sigma(t, x, y) = (\sigma_{il}(t, x_i, y_i))_{n \times n} \in \mathbb{R}^{n \times n}$ and let $\sigma_i(t, x_i, y_i)$ be the $i$th row vector of $\sigma(t, x, y)$.

Throughout this paper, the following standard hypotheses are needed.

(H1) Functions $f_j$, $g_j$ are Lipschitz-continuous on $\mathbb{R}$ with Lipschitz constants $L_j^f > 0$, $L_j^g > 0$, respectively. That is,

$$
\left| f_j(u) - f_j(v) \right| \leq L_j^f |u - v|,
$$
$$
\left| g_j(u) - g_j(v) \right| \leq L_j^g |u - v|,
$$

for all $u, v \in \mathbb{R}, i \in \Lambda$.

(H2) There exists a positive number $\theta$ such that $\theta_{i+1} - \theta_i \leq \theta$, $i \in \mathbb{N}$.

(H3) Assume that $f(0) = 0$, $g(0) = 0$, $\sigma(t, 0, 0) = 0$.

(H4) There exist nonnegative constants $e_i$, $l_i$ such that

$$
\left[ \sigma_i(t, u', v') - \sigma_i(t, u, v) \right] \left[ \sigma_i(t, u', v') - \sigma_i(t, u, v) \right]^T
$$
$$
\leq e_i \left| u' - u \right|^2 + l_i \left| v' - v \right|^2,
$$

for all $u, v, u', v' \in \mathbb{R}, i \in \Lambda$.

In the following, for further study, we give the following definitions and lemmas.

$\|x\|$ denotes a vector norm defined by

$$
\|x\| = \left( \sum_{i=1}^n |x_i(t)|^p \right)^{1/p}.
$$

*Definition 1* (see [26]). The equilibrium point $x^* = (x_1^*, x_2^*, \ldots, x_n^*)^T$ of system (1) is said to be the $p$th moment exponentially stable if there exist $\lambda > 0$ and $M \geq 1$ such that $\forall t \geq t_0$,

$$
E\|x(t) - x^*\|^p \leq M\|x_0 - x^*\|^p e^{-\lambda(t - t_0)},
$$

where $x(t)$ is a solution of system (1) with initial value $x(t_0) = x_0 \in \mathbb{R}^n$.

In such a case,

$$
\limsup_{t \to \infty} \frac{1}{t} \ln E\left( \|x(t) - x^*\|^p \right) \leq -\lambda.
$$

The right hand of (6) is commonly known as the $p$th moment Lyapunov exponent of this solution.

When $p = 2$, it is usually said to be exponentially stable in mean square.

**Lemma 2** (see [26], (Burkholder-Davis-Gundy inequality)). *Let $g \in \mathscr{L}^2(\mathbb{R}^+, \mathbb{R}^n)$. Define, for $t \geq 0$,*

$$
x(t) = \int_0^t g(s) dw(s), \qquad A(t) = \int_0^t |g(t)|^2 ds. \quad (7)
$$

*Then, for every $p > 0$, there exist universal positive constants $C_p$ (depending only on $p$), such that*

$$
E\left( \sup_{0 \leq s \leq t} |x(s)|^p \right) \leq C_p E|A(t)|^{p/2}, \quad (8)
$$

*for all $t \geq 0$.*

*In particular, one may take*

$$\text{if } 0 < p < 2,\ C_p = (32/p)^{p/2},$$

$$\text{if } p = 2,\ C_p = 4,$$

$$\text{if } p > 2, C_p = [p^{p+1}/2(p-1)^{p-1}]^{p/2}.$$

**Lemma 3** (see [39]). *If $a_i$ ($i = 1, 2, \ldots, p$) denote $p$ nonnegative real numbers, then*

$$a_1 a_2 \cdots a_p \le \frac{a_1^p + a_2^p + \cdots + a_p^p}{p}, \tag{9}$$

*where $p \ge 1$ denotes an integer.*
    *A particular form of (9) is, namely,*

$$a_1^{p-1} a_2 \le \frac{(p-1) a_1^p}{p} + \frac{a_2^p}{p}, \quad \text{for } p = 1, 2, 3, \ldots. \tag{10}$$

**Lemma 4** (see [39]). *Assuming that there exists constant $a_k \ge 0$, $k = 1, 2, \ldots, n$, if $p > 1$, then the following inequality holds:*

$$\left( \sum_{k=1}^{n} a_k \right)^p \le n^{p-1} \sum_{k=1}^{n} a_k^p. \tag{11}$$

## 3. Existence and Uniqueness of Solutions

In this section, we will study the existence and uniqueness of the equilibrium point of neural networks (1).

**Theorem 5.** *Assume that (H1)–(H4) are fulfilled. Then, for every $(t_0, x^0) \in \mathbb{R}^+ \times \mathbb{R}^n$, there exists a unique solution $x(t) = x(t, t_0, x^0)$, $t \ge t_0$ of (1) such that $x(t_0) = x^0$ and $x(t) \in \mathcal{M}^p([t_0, \infty); \mathbb{R}^n)$; $x_i(t)$ is a solution of the following integral equation:*

$$x_i(t) = x_i^0 + \int_{t_0}^{t} \left[ -a_i x_i(s) + \sum_{j=1}^{n} b_{ij} f_j \left( x_j(s) \right) \right.$$

$$\left. + \sum_{j=1}^{n} c_{ij} g_j \left( x_j(\beta(s)) \right) + I_i \right] ds \tag{12}$$

$$+ \int_{t_0}^{t} \sum_{l=1}^{n} \sigma_{il} \left( s, x_i(s), x_i(\beta(s)) \right) dw_l(s).$$

*Proof. Existence.* Fix $t_0 \in \mathbb{R}^+$; then there exists $k \in \mathbb{N}$, such that $\theta_k \le t_0 < \theta_{k+1}$. Without loss of generality, we assume that $\theta_k \le t_0 \le t \le \theta_{k+1}$. We will prove that, for every $(t_0, x^0) \in [\theta_k, \theta_{k+1}) \times \mathbb{R}^n$, there exists a solution $x(t) = x(t, t_0, x^0)$ of (1) such that $x(t_0) = x^0$, $t \ge t_0$.

For each $m = 1, 2, \ldots$, set $x^0(t) = x^0$ and define, by the Picard iterations,

$$x_i^m(t) = x_i^0 + \int_{t_0}^{t} \left[ -a_i x_i^{m-1}(s) + \sum_{j=1}^{n} b_{ij} f_j \left( x_j^{m-1}(s) \right) \right.$$

$$\left. + \sum_{j=1}^{n} c_{ij} g_j \left( x_j^{m-1}(\beta(s)) \right) + I_i \right] ds \tag{13}$$

$$+ \int_{t_0}^{t} \sum_{l=1}^{n} \sigma_{il} \left( s, x_i^{m-1}(s), x_i^{m-1}(\beta(s)) \right) dw_l(s).$$

Obviously, $x^0(\cdot) \in \mathcal{M}^p([\theta_k, \theta_{k+1}]; \mathbb{R}^n)$. Moreover, for $m \ge 1$, it is easy to see by induction that $x^m(\cdot) \in \mathcal{M}^p([\theta_k, \theta_{k+1}]; \mathbb{R}^n)$. For simpleness, we let

$$\zeta = \max_{1 \le i \le n} \left( a_i + \sum_{j=1}^{n} \left( \left| b_{ji} \right| L_i^f + \left| c_{ji} \right| L_i^g \right) \right),$$

$$e = \max_{1 \le i \le n} (e_i), \qquad l = \max_{1 \le i \le n} (l_i). \tag{14}$$

Then

$$E \left( \sup_{t_0 \le s \le t} \left\| x^m(s) \right\|^p \right)$$

$$= E \left[ \sup_{t_0 \le s \le t} \sum_{i=1}^{n} \left| x_i^0 + \int_{t_0}^{t} \left[ -a_i x_i^{m-1}(s) + \sum_{j=1}^{n} b_{ij} f_j \left( x_j^{m-1}(s) \right) \right. \right. \right.$$

$$\left. + \sum_{j=1}^{n} c_{ij} g_j \left( x_j^{m-1}(\beta(s)) \right) + I_i \right] ds$$

$$+ \int_{t_0}^{t} \sum_{l=1}^{n} \sigma_{il} \left( s, x_i^{m-1}(s), \right.$$

$$\left. \left. \left. x_i^{m-1}(\beta(s)) \right) dw_l(s) \right|^p \right]$$

$$\le 3^{p-1} E \left\| x^0 \right\|^p + (3\theta)^{p-1}$$

$$\times \sum_{i=1}^{n} E \int_{t_0}^{t} \left( \left( a_i + \sum_{j=1}^{n} \left| b_{ij} \right| L_i^f \right) \left| x_i^{m-1}(s) \right| \right.$$

$$\left. + \sum_{j=1}^{n} \left| c_{ji} \right| L_i^g \left| x_i^{m-1}(\beta(s)) \right| + I_i \right)^p ds$$

$$+ 3^{p-1} C_p E$$

$$\times \sum_{i=1}^{n} \left( \int_{t_0}^{t} \left( e_i \left| x_i^{m-1}(s) \right|^2 + l_i \left| x_i^{m-1}(\beta(s)) \right|^2 \right) \right)^{p/2}$$

$$\le C_1 + C_2 \int_{t_0}^{t} E \left( \sup_{t_0 \le r \le s} \left\| x^{m-1}(r) \right\|^p \right) ds, \tag{15}$$

where

$$C_1 = 3^{p-1}E\|x^0\|^p + (9\theta)^{p-1}\theta\sum_{i=1}^{n}I_i^p,$$

$$C_2 = 2\zeta^p(9\theta)^{p-1} + 3^{p-1}C_p(2\theta)^{p/2-1}\left(e^{p/2} + l^{p/2}\right). \tag{16}$$

For any $K \geq 1$,

$$\max_{1\leq m\leq K} E\left(\sup_{t_0\leq s\leq t}\|x^m(s)\|^p\right)$$

$$\leq C_1 + C_2\int_{t_0}^{t}\max_{1\leq m\leq K}E\left(\sup_{t_0\leq r\leq s}\|x^{m-1}(r)\|^p\right)ds$$

$$\leq C_1 + C_2\theta E\|x^0\|^p \tag{17}$$

$$+ C_2\int_{t_0}^{t}\max_{1\leq m\leq K}E\left(\sup_{t_0\leq r\leq s}\|x^m(r)\|^p\right)ds$$

$$= C_3 + C_2\int_{t_0}^{t}\max_{1\leq m\leq K}E\left(\sup_{t_0\leq r\leq s}\|x^m(r)\|^p\right)ds,$$

where $C_3 = C_1 + C_2\theta E\|x^0\|^p$.

The Gronwall inequality implies

$$\max_{1\leq m\leq K}E\left(\sup_{t_0\leq s\leq t}\|x^m(s)\|^p\right) \leq C_3 e^{C_2(t-t_0)} \leq C_3 e^{C_2\theta}. \tag{18}$$

Since $K$ is arbitrary, for $\theta_k \leq t_0 \leq t \leq \theta_{k+1}$, we must have

$$E\left(\sup_{t_0\leq s\leq t}\|x^m(s)\|^p\right) \leq C_3 e^{C_2\theta}. \tag{19}$$

Therefore $\{x^m(t)\} \in \mathcal{M}^p([\theta_k, \theta_{k+1}]; \mathbb{R}^n)$.

Now we claim that, for all $m \geq 0$,

$$E\left(\sup_{\theta_k\leq t_0\leq s<t}\|x^{m+1}(s) - x^m(s)\|^p\right) \leq \frac{C[M(t-\theta_k)]^m}{m!}, \tag{20}$$

where $M = 2\zeta^p(4\theta)^{p-1} + 2^{p-1}C_p(2\theta)^{p/2-1}(e^{p/2} + l^{p/2})$ and $C$ will be defined below.

Firstly, we compute

$$E\left(\sup_{t_0\leq s<t}\|x^1(s) - x^0(s)\|^p\right)$$

$$\leq 2^{p-1}E\sum_{i=1}^{n}\left(\int_{t_0}^{t}\left[-a_i x_i^0 + \sum_{j=1}^{n}b_{ij}f_j\left(x_j^0\right)\right.\right.$$

$$\left.\left. + \sum_{j=1}^{n}c_{ij}g_j\left(x_j^0\right) + I_i\right]ds\right)^p$$

$$+ 2^{p-1}E\sum_{i=1}^{n}\left(\sup_{t_0\leq s<t}\left|\int_{t_0}^{s}\sum_{l=1}^{n}\sigma_{il}\left(r, x_i^0, x_i^0\right)dw_l(r)\right|^p\right)$$

$$\leq (2\theta)^{p-1}E\sum_{i=1}^{n}\int_{t_0}^{t}\left[-a_i x_i^0 + \sum_{j=1}^{n}b_{ij}f_j\left(x_j^0\right)\right.$$

$$\left. + \sum_{j=1}^{n}c_{ij}g_j\left(x_j^0\right) + I_i\right]^p ds$$

$$+ 2^{p-1}C_p E\sum_{i=1}^{n}\left(\int_{t_0}^{t}\left(e_i\left|x_i^0\right|^2 + l_i\left|x_i^0\right|^2\right)ds\right)^{p/2} \tag{21}$$

$$\leq (2\theta)^{p-1}E\sum_{i=1}^{n}\int_{t_0}^{t}\left[\left(a_i + \sum_{j=1}^{n}\left|b_{ji}\right|L_i^f\right.\right.$$

$$\left.\left. + \sum_{j=1}^{n}\left|c_{ji}\right|L_i^g\right)\left|x_i^0\right| + I_i\right]^p ds$$

$$+ 2^{p-1}C_p((e+l)\theta)^{p/2}E\|x^0\|^p$$

$$\leq (4\theta)^{p-1}\left(\zeta^p\theta E\|x^0\|^p + \theta\sum_{i=1}^{n}I_i^p\right)$$

$$+ 2^{p-1}C_p((e+l)\theta)^{p/2}E\|x^0\|^p$$

$$= \left[(4\theta)^{p-1}\zeta^p\theta + 2^{p-1}C_p((e+l)\theta)^{p/2}\right]E\|x^0\|^p$$

$$+ (4\theta)^{p-1}\theta\sum_{i=1}^{n}I_i^p := C,$$

where Lemma 4 is used in the first inequality and Lemma 2 and Hölder inequality are used in the second inequality. So (20) holds for $m = 0$.

Next, assume (20) holds for $m \geq 0$; then

$$E\left(\sup_{\theta_k \leq t_0 \leq s \leq t} \left\| x^{m+2}(s) - x^{m+1}(s) \right\|^p \right)$$

$$\leq (2\theta)^{p-1} E \sum_{i=1}^{n} \int_{t_0}^{t} \left[ a_i \left| x_i^{m+1}(s) - x_i^m(s) \right| \right.$$

$$+ \sum_{j=1}^{n} \left| b_{ij} \right| L_j^f \left| x_j^{m+1}(s) - x_j^m(s) \right|$$

$$+ \sum_{j=1}^{n} \left| c_{ij} \right| L_j^g \left| x_j^{m+1}(\beta(s)) \right.$$

$$\left. \left. - x_j^m(\beta(s)) \right| \right]^p ds$$

$$+ 2^{p-1} E$$

$$\times \sum_{i=1}^{n} \left( \sup_{\theta_k \leq t_0 \leq s \leq t} \left| \int_{\theta_k}^{s} \sum_{l=1}^{n} \left( \sigma_{il} \left( r, x_i^{m+1}(r), x_i^{m+1}(\beta(r)) \right) \right. \right. \right.$$

$$- \sigma_{il} \left( r, x_i^m(r), \right.$$

$$\left. \left. \left. x_i^m(\beta(r)) \right) \right) dw_l(r) \right|^p \right)$$

$$\leq (4\theta)^{p-1} E \sum_{i=1}^{n} \int_{t_0}^{t} \left( \zeta^p \left| x_i^{m+1}(s) - x_i^m(s) \right|^p \right.$$

$$\left. + \zeta^p \left| x_i^{m+1}(\beta(s)) - x_i^m(\beta(s)) \right|^p \right) ds$$

$$+ 2 C_p (8\theta)^{p/2-1} E \sum_{i=1}^{n} \int_{t_0}^{t} \left( e_i^{p/2} \left| x_i^{m+1}(s) - x_i^m(s) \right|^p \right) ds$$

$$+ 2 C_p (8\theta)^{p/2-1} E$$

$$\times \sum_{i=1}^{n} \int_{t_0}^{t} \left( l_i^{p/2} \left| x_i^{m+1}(\beta(s)) - x_i^m(\beta(s)) \right|^p \right) ds$$

$$\leq \left[ 2\zeta^p (4(t - \theta_k))^{p-1} + 2 C_p (8\theta)^{p/2-1} \left( e^{p/2} + l^{p/2} \right) \right]$$

$$\cdot \int_{\theta_k}^{t} E\left( \sup_{\theta_k \leq r \leq s} \left\| x^{m+1}(r) - x^m(r) \right\|^p \right) ds$$

$$\leq M \int_{\theta_k}^{t} \frac{C\left[ M(s - \theta_k) \right]^m}{m!} ds \leq \frac{C\left[ M(t - \theta_k) \right]^{m+1}}{(m+1)!},$$

$$(22)$$

where $M = 2\zeta^p (4\theta)^{p-1} + 2^{p-1} C_p (2\theta)^{p/2-1} (e^{p/2} + l^{p/2})$.

That is, (20) holds for $m + 1$. Hence, by induction, (20) holds for all $m \geq 0$.

One can see from (20) that, for every $t$, $\{x^m(t)\}$ is a Cauchy sequence in $L^p$. Hence we have $x^m(t) \to x(t)$ as $m \to \infty$ in $L^p$. Letting $m \to \infty$ in (19) gives

$$E\left( \sup_{t_0 \leq s \leq t} \| x(s) \|^p \right) \leq C_3 e^{C_2 \theta}. \tag{23}$$

Therefore, $x(t) \in \mathcal{M}^p([\theta_k, \theta_{k+1}]; \mathbb{R}^n)$.

It remains to show that $x(t)$ is a solution of system (1) satisfying $(t_0, x^0) \in [\theta_k, \theta_{k+1}) \times \mathbb{R}^n$. Note that

$$E \sum_{i=1}^{n} \left| \int_{t_0}^{t} \left[ -a_i x_i(s) + \sum_{j=1}^{n} b_{ij} f_j\left( x_j(s) \right) \right. \right.$$

$$\left. + \sum_{j=1}^{n} c_{ij} g_j\left( x_j(\beta(s)) \right) + I_i \right] ds$$

$$- \int_{t_0}^{t} \left[ -a_i x_i^{m-1}(s) + \sum_{j=1}^{n} b_{ij} f_j\left( x_j^{m-1}(s) \right) \right.$$

$$\left. \left. + \sum_{j=1}^{n} c_{ij} g_j\left( x_j^{m-1}(\beta(s)) \right) + I_i \right] ds \right|^p \tag{24}$$

$$\leq 2\theta^{p-1} \zeta^p \int_{t_0}^{t} E \sup_{\theta_k \leq r \leq s} \left\| x(r) - x^{m-1}(r) \right\|^p ds \longrightarrow 0,$$

$$m \longrightarrow \infty,$$

$$E \sum_{i=1}^{n} \left| \int_{t_0}^{t} \sum_{l=1}^{n} \sigma_{il}\left( s, x_i(s), x_i(\beta(s)) \right) dw_l(s) \right.$$

$$\left. - \int_{t_0}^{t} \sum_{l=1}^{n} \sigma_{il}\left( s, x_i^{m-1}(s), x_i^{m-1}(\beta(s)) \right) dw_l(s) \right|^p$$

$$\longrightarrow 0, \quad m \longrightarrow \infty.$$

Hence, we can let $m \to \infty$ in (13), and (12) is derived. Again, using the same argument, we can continue $x(t)$ from $t = \theta_{k+1}$ to $t = \theta_{k+2}$. Hence, the mathematical induction completes the proof.

*Uniqueness.* Let $x(t)$ and $\overline{x}(t)$ be the two solutions, $i \in \Lambda$. By noting

$$x_i(t) - \overline{x}_i(t)$$

$$= \int_{t_0}^{t} \left[ -a_i\left( x_i(s) - \overline{x}_i(s) \right) \right.$$

$$+ \sum_{j=1}^{n} b_{ij}\left( f_j\left( x_j(s) \right) - f_j\left( \overline{x}_j(s) \right) \right)$$

$$\left. + \sum_{j=1}^{n} c_{ij}\left( g_j\left( x_j(\beta(s)) \right) - g_j\left( \overline{x}_j(\beta(s)) \right) \right) \right] ds$$

$$+ \int_{t_0}^{t} \sum_{l=1}^{n} \left( \sigma_{il} \left( s, x_i \left( s \right), x_i \left( \beta \left( s \right) \right) \right) \right.$$

$$\left. - \sigma_{il} \left( s, \overline{x}_i \left( s \right), \overline{x}_i \left( \beta \left( s \right) \right) \right) \right) dw_l \left( s \right), \tag{25}$$

we can easily show that

$$E \left( \sup_{\theta_k \leq t_0 \leq s \leq t} \| x \left( s \right) - \overline{x} \left( s \right) \|^p \right)$$

$$\leq M \int_{\theta_k}^{t} E \left( \sup_{\theta_k \leq r \leq s} \| x(r) - \overline{x}(r) \|^p \right) ds. \tag{26}$$

The Gronwall inequality then yields that

$$E \left( \sup_{\theta_k \leq t \leq \theta_{k+1}} \| x \left( t \right) - \overline{x} \left( t \right) \|^p \right) = 0. \tag{27}$$

This implies that $x(t) = \overline{x}(t)$ for $\theta_k \leq t_0 \leq t \leq \theta_{k+1}$. The uniqueness has been proved for $t \in [\theta_k, \theta_{k+1}]$. For every $T > t_0$, we can prove the uniqueness by mathematical induction on $[t_0, T]$. As the assumptions of the existence-and-uniqueness theorem hold on every finite subinterval $[t_0, T]$ of $[t_0, \infty)$, then (1) has a unique solution $x(t)$ on the entire interval $[t_0, \infty)$ (see [26]). Hence, the theorem is proved. □

From the proof of Theorem 5, we easily obtain the following theorem.

**Theorem 6.** *Assume that (H1)–(H3) are fulfilled. Then, for every $(t_0, x^0) \in \mathbb{R}^+ \times \mathbb{R}^n$, there exists a unique solution $x(t) = x(t, t_0, x^0)$ of the following system such that $x(t_0) = x^0$, $t \geq t_0$:*

$$\dot{x}_i \left( t \right) = -a_i x_i \left( t \right) + \sum_{j=1}^{n} b_{ij} f_j \left( x_j \left( t \right) \right)$$

$$+ \sum_{j=1}^{n} c_{ij} g_j \left( x_j \left( \beta \left( t \right) \right) \right) + I_i, \quad i \in \Lambda. \tag{28}$$

*Remark 7.* When the system (1) neglects stochastic perturbations, it becomes the system (28) which is studied in [12, 38]. In order to obtain the existence and uniqueness, more restrictions on coefficients are needed in [12, 38], which might greatly restrict the application domain of the neural networks. Clearly, our results in this paper contain the results given in [12, 38], and the conditions are less restrictive compared with [12, 38].

## 4. $p$th Moment Exponential Stability

In this section, we will establish some sufficient conditions ensuring the $p$th moment exponential stability of the equilibrium point $x^*$ of system (1).

Let $PC^{1,2}([\theta_k, \theta_{k+1}] \times \mathbb{R}^n; \mathbb{R}^+)$ denote the family of all nonnegative functions $V(t, x)$ on $[\theta_k, \theta_{k+1}] \times \mathbb{R}^n$ which are continuous once differentiable in $t$ and twice differentiable in

$x$. If $V(t, x) \in PC^{1,2}([\theta_k, \theta_{k+1}] \times \mathbb{R}^n; \mathbb{R}^+)$, define an operator $\mathscr{L}V$ associated with (1) as

$$\mathscr{L}V \left( t, x \right) = V_t \left( t, x \right)$$

$$+ \sum_{i=1}^{n} V_{x_i} \left( t, x \right) \left[ -a_i x_i \left( t \right) + \sum_{j=1}^{n} b_{ij} f_j \left( x_j \left( t \right) \right) \right.$$

$$\left. + \sum_{j=1}^{n} c_{ij} g_j \left( x_j \left( \beta \left( t \right) \right) \right) + I_i \right]$$

$$+ \frac{1}{2} \text{trace} \left[ \sigma^T V_{xx} \left( t, x \right) \sigma \right], \tag{29}$$

where

$$V_t \left( t, x \right) = \frac{\partial V \left( t, x \right)}{\partial t},$$

$$V_{x_i} \left( t, x \right) = \frac{\partial V \left( t, x \right)}{\partial x_i}, \tag{30}$$

$$V_{xx} \left( t, x \right) = \left( \frac{\partial V \left( t, x \right)}{\partial x_i \partial x_j} \right)_{n \times n}.$$

Let $y_i(t) = x_i(t) - x^*$; then (1) can be written by

$$dy_i \left( t \right) = \left[ -a_i y_i \left( t \right) + \sum_{j=1}^{n} b_{ij} \tilde{f}_j \left( y_j \left( t \right) \right) \right.$$

$$\left. + \sum_{j=1}^{n} c_{ij} \tilde{g}_j \left( y_j \left( \beta \left( t \right) \right) \right) \right] dt \tag{31}$$

$$+ \sum_{l=1}^{n} \tilde{\sigma}_{il} \left( t, y_i \left( t \right), y_i \left( \beta \left( t \right) \right) \right) dw_l \left( t \right),$$

where

$$\tilde{f}_j \left( y_j \left( t \right) \right) = f_j \left( y_j \left( t \right) + x_j^* \right) - f_j \left( x_j^* \right),$$

$$\tilde{g}_j \left( y_j \left( \beta \left( t \right) \right) \right) = g_j \left( y_j \left( \beta \left( t \right) \right) + x_j^* \right) - g_j \left( x_j^* \right),$$

$$\tilde{\sigma}_{ij} \left( t, y_i \left( t \right), y_i \left( \beta \left( t \right) \right) \right) \tag{32}$$

$$= \sigma_{ij} \left( t, y_i \left( t \right) + x_i^*, y_i \left( \beta \left( t \right) \right) + x_i^* \right)$$

$$- \sigma_{ij} \left( t, x_i^*, x_i^* \right).$$

It is clear that the stability of the zero solution of (31) is equivalent to that of the equilibrium $x^*$ of (1). Therefore, we restrict our discussion to the stability of the zero solution of (31).

For simplicity of notation, we denote

$$m_1 = \max_{1 \le i \le n} \left[ a_i \left( \frac{a_i}{n} \right)^{p-1} + \sum_{j=1}^{n} \left( |b_{ji}| L_i^f \right)^p \right],$$

$$m_2 = \max_{1 \le i \le n} \left[ \sum_{j=1}^{n} \left( |c_{ji}| L_i^g \right)^p \right], \qquad (33)$$

$$\lambda_1 = m_2 \theta (3n\theta)^{p-1} + \theta l C_p (2\theta l)^{p/2-1},$$

$$\lambda_2 = 3^{p-1} \left( (3n\theta)^{p-1} m_1 + C_p (2\theta)^{p/2-1} e^{p/2} \right).$$

In order to obtain our results, the following assumption and Lemmas are needed:

(H5) $3^{p-1}(\lambda_1 + \lambda_2(1 + \lambda_1)e^{\lambda_2 \theta}\theta) < 1$.

**Lemma 8.** *Let $y(t) = (y_1(t), y_2(t), \dots, y_n(t))^T$ be a solution of (31) and let (H1)–(H5) be satisfied. Then the following inequality*

$$E\|y(\beta(t))\|^p \le \mu E\|y(t)\|^p \qquad (34)$$

*holds for all $t \in \mathbb{R}^+$, where*

$$\mu = 3^{p-1} \left( 1 - 3^{p-1} \left( \lambda_1 + \lambda_2 (1 + \lambda_1) e^{\lambda_2 \theta}\theta \right) \right)^{-1}. \qquad (35)$$

*Proof.* Fix $t \in \mathbb{R}^+$; there exists $k \in \mathbb{N}$, such that $t \in [\theta_k, \theta_{k+1})$. Then from Lemma 4, it follows that

$$E\|y(t)\|^p$$

$$= E \left( \sum_{i=1}^{n} |y_i(t)|^p \right)$$

$$= \sum_{i=1}^{n} E \left| y_i(\theta_k) + \int_{\theta_k}^{t} \left( -a_i y_i(s) + \sum_{j=1}^{n} b_{ij} \tilde{f}_j (y_j(s)) \right. \right.$$

$$\left. + \sum_{j=1}^{n} c_{ij} \tilde{g}_j (y_j(\beta(s))) \right) ds$$

$$\left. + \int_{\theta_k}^{t} \sum_{l=1}^{n} \tilde{\sigma}_{il}(s, y_i(s), y_i(\beta(s))) dw_l(s) \right|^p$$

$$\le 3^{p-1} \sum_{i=1}^{n} E \left[ |y_i(\theta_k)|^p \right.$$

$$+ \left| \int_{\theta_k}^{t} \sum_{l=1}^{n} \tilde{\sigma}_{il}(s, y_i(s), y_i(\beta(s))) dw_l(s) \right|^p$$

$$+ \left| \int_{\theta_k}^{t} \left( -a_i y_i(s) + \sum_{j=1}^{n} b_{ij} \tilde{f}_j (y_j(s)) \right. \right.$$

$$\left. \left. + \sum_{j=1}^{n} c_{ij} \tilde{g}_j (y_j(\beta(s))) \right) ds \right|^p \right]$$

$$= 3^{p-1} \left( E\|y(\theta_k)\|^p + I_1 + I_2 \right). \qquad (36)$$

Now we compute $I_1$ and $I_2$:

$$I_1 = \sum_{i=1}^{n} E \left| \int_{\theta_k}^{t} \left( -a_i y_i(s) + \sum_{j=1}^{n} b_{ij} \tilde{f}_j (y_j(s)) \right. \right.$$

$$\left. \left. + \sum_{j=1}^{n} c_{ij} \tilde{g}_j (y_j(\beta(s))) \right) ds \right|^p$$

$$\le \sum_{i=1}^{n} E \left( \int_{\theta_k}^{t} a_i |y_i(s)| ds \right.$$

$$+ \sum_{j=1}^{n} |b_{ij}| L_j^f \int_{\theta_k}^{t} |y_j(s)| ds$$

$$\left. + \sum_{j=1}^{n} |c_{ij}| L_j^g \int_{\theta_k}^{t} |y_j(\beta(s))| ds \right)^p$$

$$\le 3^{p-1} \sum_{i=1}^{n} E \left[ \left( \int_{\theta_k}^{t} a_i |y_i(s)| ds \right)^p \right.$$

$$+ \left( \sum_{j=1}^{n} |b_{ij}| L_j^f \int_{\theta_k}^{t} |y_j(s)| ds \right)^p$$

$$\left. + \left( \sum_{j=1}^{n} |c_{ij}| L_j^g \int_{\theta_k}^{t} |y_j(\beta(s))| ds \right)^p \right]$$

$$= 3^{p-1} \sum_{i=1}^{n} [I_{11} + I_{12} + I_{13}]. \qquad (37)$$

The Hölder inequality yields

$$I_{11} = E\left(\int_{\theta_k}^{t} a_i \left|y_i(s)\right| ds\right)^p \le a_i^p \theta^{p-1} E \int_{\theta_k}^{t} \left|y_i(s)\right|^p ds, \tag{38}$$

$$\begin{aligned} I_{12} &= E\left(\sum_{j=1}^{n} \left|b_{ij}\right| L_j^f \int_{\theta_k}^{t} \left|y_j(s)\right| ds\right)^p \\ &\le n^{p-1} E\left[\sum_{j=1}^{n} \left(\left|b_{ij}\right| L_j^f\right)^p \left(\int_{\theta_k}^{t} \left|y_j(s)\right| ds\right)^p\right] \\ &\le n^{p-1}\theta^{p-1} \sum_{j=1}^{n} \left(\left|b_{ij}\right| L_j^f\right)^p E \int_{\theta_k}^{t} \left|y_j(s)\right|^p ds, \end{aligned} \tag{39}$$

$$\begin{aligned} I_{13} &= E\left(\sum_{j=1}^{n} \left|c_{ij}\right| L_j^g \int_{\theta_k}^{t} \left|y_j(\beta(s))\right| ds\right)^p \\ &\le n^{p-1} \sum_{j=1}^{n} \left(\left|c_{ij}\right| L_j^g \theta\right)^p E \left|y_j(\theta_k)\right|^p. \end{aligned} \tag{40}$$

Substituting (38)–(40) into (37) yields that

$$I_1 \le (3n\theta)^{p-1}\left(m_1 E \int_{\theta_k}^{t} \|y(s)\|^p ds + \theta m_2 E\|y(\theta_k)\|^p\right), \tag{41}$$

where $m_1 = \max_{1\le i\le n}[a_i(a_i/n)^{p-1} + \sum_{j=1}^{n}(|b_{ji}|L_i^f)^p]$, $m_2 = \max_{1\le i\le n}[\sum_{j=1}^{n}(|c_{ji}|L_i^g)^p]$.

On the other hand, Lemma 2 and (H3) yield

$$\begin{aligned} I_2 &= E\left(\sum_{i=1}^{n} \int_{\theta_k}^{t} \sum_{l=1}^{n} \tilde{\sigma}_{il}\left(s, y_i(s), y_i(\theta_k)\right) dw_l(s)\,|^p\right) \\ &\le \sum_{i=1}^{n} C_p E\left(\left|\int_{\theta_k}^{t} \sum_{l=1}^{n} \tilde{\sigma}_{il}^2\left(s, y_i(s), y_i(\theta_k)\right) ds\right|^{p/2}\right) \\ &\le C_p \sum_{i=1}^{n} E\left[\int_{\theta_k}^{t} e_i \left|y_i(s)\right|^2 ds + l_i\theta\left|y_i(\theta_k)\right|^2\right]^{p/2} \\ &\le C_p 2^{p/2-1}\sum_{i=1}^{n} E\left[e_i^{p/2}\theta^{p/2-1}\int_{\theta_k}^{t} \left|y_i(s)\right|^p ds \right. \\ &\qquad\qquad \left. + (l_i\theta)^{p/2}\left|y_i(\theta_k)\right|^p\right] \\ &\le C_p(2\theta)^{p/2-1}\left[e^{p/2}\int_{\theta_k}^{t} E\|y(s)\|^p ds \right. \\ &\qquad\qquad \left. + \theta l^{p/2} E\|y(\theta_k)\|^p\right]. \end{aligned} \tag{42}$$

Substituting (41)-(42) into (36), we have

$$\begin{aligned} E\|y(t)\|^p &\le 3^{p-1}E\|y(\theta_k)\|^p + 3^{p-1}(3n\theta)^{p-1} \\ &\quad \times\left(m_1 E\int_{\theta_k}^{t}\|y(s)\|^p ds + \theta m_2 E\|y(\theta_k)\|^p\right) \\ &\quad + 3^{p-1}C_p(2\theta)^{p/2-1} \\ &\quad \times\left[e^{p/2}\int_{\theta_k}^{t} E\|y(s)\|^p ds + \theta l^{p/2}E\|y(\theta_k)\|^p\right] \\ &= 3^{p-1}(1+\lambda_1)E\|y(\theta_k)\|^p + \lambda_2\int_{\theta_k}^{t} E\|y(s)\|^p ds, \end{aligned} \tag{43}$$

where $\lambda_1 = m_2\theta(3n\theta)^{p-1} + \theta l C_p(2\theta l)^{p/2-1}$, $\lambda_2 = 3^{p-1}((3n\theta)^{p-1}m_1 + C_p(2\theta)^{p/2-1}e^{p/2})$.

By Gronwall inequality

$$E\|y(t)\|^p \le 3^{p-1}(1+\lambda_1)e^{\lambda_2\theta}E\|y(\theta_k)\|^p. \tag{44}$$

Furthermore, for $t \in [\theta_k, \theta_{k+1})$, we have

$$\begin{aligned} E\|y(\theta_k)\|^p &\le 3^{p-1}\left(E\|y(t)\|^p + \lambda_1 E\|y(\theta_k)\|^p\right) \\ &\quad + \lambda_2\int_{\theta_k}^{t} E\|y(s)\|^p ds \\ &\le 3^{p-1}E\|y(t)\|^p + 3^{p-1}\lambda_1 E\|y(\theta_k)\|^p \\ &\quad + \lambda_2 3^{p-1}(1+\lambda_1)e^{\lambda_2\theta}E\|y(\theta_k)\|^p\theta. \end{aligned} \tag{45}$$

Thus, it follows from condition (H5) that

$$E\|y(\beta(t))\|^p \le \mu E\|y(t)\|^p. \tag{46}$$

Therefore, (34) holds for $t \in [\theta_k, \theta_{k+1})$; the extension of (34) for all $t \in \mathbb{R}^+$ is obvious. This proves the theorem.  □

**Lemma 9** (Mao [26]). *Assume that there is a function* $V(t,x) \in PC^{1,2}([t_0,\infty)\times\mathbb{R}^n; \mathbb{R}^+)$ *and positive constants* $c_1$, $c_2$, *and* $c_3$, *such that*

$$c_1\|x\|^p \le V(x,t) \le c_2\|x\|^p, \tag{47}$$

$$\mathscr{L}V \le -c_3 V(t,x) \tag{48}$$

*for all* $(t,x) \in [t_0,\infty)\times\mathbb{R}^n$. *Then*

$$E\|x(t;t_0,x^0)\|^p \le \frac{c_2}{c_1}e^{-c_3(t-t_0)}\|x^0\|^p, \quad t \ge t_0 \tag{49}$$

*for all* $x^0 \in \mathbb{R}^n$.

For convenience, we adopt the following notation:

$$
\begin{aligned}
k_1 = \min_{1 \le i \le n} \Bigg\{ & pa_i - (p-1)\sum_{j=1}^{n}\left(L_j^f\left|b_{ij}\right| + L_j^g\left|c_{ij}\right|\right) \\
& - \sum_{j=1}^{n}\left|b_{ji}\right|L_i^f - \frac{p(p-1)}{2}e_i \\
& - \frac{(p-1)(p-2)}{2}l_i \Bigg\} > 0,
\end{aligned}
$$

$$
k_2 = \max_{1 \le i \le n}\left\{ L_i^g \sum_{j=1}^{n}\left|c_{ji}\right| + (p-1)l_i \right\}. \tag{50}
$$

**Theorem 10.** *Assume that (H1)–(H5) hold and, furthermore, that the following inequality is satisfied:*

$$
k_1 > k_2\mu. \tag{51}
$$

*Then system (31) is the pth moment exponentially stable.*

*Proof.* We define a Lyapunov function

$$
V(t, y(t)) = \sum_{i=1}^{n}\left|y_i(t)\right|^p = \|y(t)\|^p. \tag{52}
$$

Obviously, (47) is satisfied with $c_1 = c_2 = 1$. Now we prove (48) holds.

For $t \ne \theta_i$, $i \in \mathbb{N}$, the operator $\mathscr{L}V$ with respect to (12) is given by

$$
\begin{aligned}
\mathscr{L}V = & \, p\sum_{i=1}^{n}\left|y_i(t)\right|^{p-1}\operatorname{sgn}(y_i(t)) \\
& \times \Bigg[ -a_i y_i(t) + \sum_{j=1}^{n}b_{ij}\tilde{f}_j\left(y_j(t)\right) \\
& \quad + \sum_{j=1}^{n}c_{ij}\tilde{g}_j\left(y_j(\beta(t))\right) \Bigg] \\
& + \frac{p(p-1)}{2}\sum_{i=1}^{n}\left|y_i(t)\right|^{p-2} \\
& \times \sum_{l=1}^{m}\tilde{\sigma}_{il}^2\left(t, y_i(t), y_i(\beta(t))\right)
\end{aligned}
$$

$$
\begin{aligned}
\le & -p\sum_{i=1}^{n}a_i\left|y_i(t)\right|^p \\
& + p\sum_{i=1}^{n}\sum_{j=1}^{n}\left|b_{ij}\right|L_j^f\left|y_i(t)\right|^{p-1}\left|y_j(t)\right| \\
& + p\sum_{i=1}^{n}\sum_{j=1}^{n}\left|c_{ij}\right|L_j^g\left|y_i(t)\right|^{p-1}\left|y_j(\beta(t))\right| \\
& + \frac{p(p-1)}{2}\sum_{i=1}^{n}\left|y_i(t)\right|^{p-2} \\
& \times \left(e_i\left|y_i(t)\right|^2 + l_i\left|y_i(\beta(t))\right|^2\right) \\
\le & -p\sum_{i=1}^{n}a_i\left|y_i(t)\right|^p \\
& + \sum_{i=1}^{n}\sum_{j=1}^{n}\left|b_{ij}\right|L_j^f\left((p-1)\left|y_i(t)\right|^p + \left|y_j(t)\right|^p\right) \\
& + \sum_{i=1}^{n}\sum_{j=1}^{n}\left|c_{ij}\right|L_j^g\left((p-1)\left|y_i(t)\right|^p + \left|y_j(\beta(t))\right|^p\right) \\
& + \frac{p(p-1)}{2}\sum_{i=1}^{n}e_i\left|y_i(t)\right|^p + \frac{(p-1)}{2} \\
& \times \sum_{i=1}^{n}l_i\left((p-2)\left|y_i(t)\right|^p + 2\left|y_i(\beta(t))\right|^p\right) \\
= & -\sum_{i=1}^{n}\Bigg[ pa_i - (p-1)\sum_{j=1}^{n}\left(L_j^f\left|b_{ij}\right| + L_j^g\left|c_{ij}\right|\right) \\
& \qquad - \sum_{j=1}^{n}\left|b_{ji}\right|L_i^f - \frac{p(p-1)}{2}e_i \Bigg]\left|y_i(t)\right|^p \\
& - \sum_{i=1}^{n}\frac{(p-1)(p-2)}{2}l_i\left|y_i(t)\right|^p \\
& + \sum_{i=1}^{n}\Bigg[ L_i^g\sum_{j=1}^{n}\left|c_{ij}\right| + (p-1)l_i \Bigg]\left|y_i(\beta(t))\right|^p \\
\le & -k_1\|y(t)\|^p + k_2\|y(\beta(t))\|^p.
\end{aligned} \tag{53}
$$

By using Lemma 8, we obtain

$$
\mathscr{L}V \le -(k_1 - k_2\mu)\|y(t)\|^p. \tag{54}
$$

This, together with (51), we get that (48) is true. Now, define $\gamma$ for convenience as follows:

$$
\gamma = k_1 - k_2\mu > 0. \tag{55}
$$

By Lemma 9, the equilibrium of (31) is the $p$th moment exponentially stable and the $p$th moment Lyapunov exponent should not be greater than $-\gamma$. □

**Theorem 11** (see [12]). *Suppose that (H1)–(H3) and (H5) hold true. Assume, furthermore, that the following inequality is satisfied:*

$$\min_{1 \le i \le n} \left\{ a_i - \sum_{j=1}^{n} \left| b_{ji} \right| L_i^f \right\} > \max_{1 \le i \le n} \left\{ \sum_{j=1}^{n} \left| c_{ji} \right| L_i^g \right\} \mu, \quad (56)$$

*where*

$$\mu = \left( 1 - \left( \lambda_1 + \lambda_2 \left( 1 + \lambda_1 \right) e^{\lambda_2 \theta} \theta \right) \right)^{-1},$$

$$\lambda_1 = \max_{1 \le i \le n} \left\{ \sum_{j=1}^{n} \left| c_{ji} \right| L_i^g \right\}, \quad (57)$$

$$\lambda_2 = \max_{1 \le i \le n} \left\{ a_i + \sum_{j=1}^{n} \left| b_{ji} \right| L_i^f \right\};$$

*then system* (28) *is globally exponentially stable.*

*Proof.* Let $\sigma_{il}(t, x_i(t), x_i(\beta(t))) = 0$, $i, l \in \Lambda$; the conclusion is straightforward. □

*Remark 12.* Theorem 10 generalizes the work of Akhmet et al. [12] and the conditions in the theorem are easy to verify.

## 5. Illustrative Example

In the following, we will give an example to illustrate our results.

*Example 1.* Consider the following model:

$$dx_i(t) = \left[ -a_i x_i(t) + \sum_{j=1}^{2} b_{ij} f_j \left( x_j(t) \right) \right.$$

$$\left. + \sum_{j=1}^{2} c_{ij} g_j \left( x_j(\beta(t)) \right) + I_i \right] dt \quad (58)$$

$$+ \sigma_i \left( t, x_i(t), x_i(\beta(t)) \right) dw(t), \quad i = 1, 2.$$

Suppose the activation function is described by $f_i(x) = \tanh(x)$, $g_i(x) = (1/2)[|x + 1| - |x - 1|]$.

Let $\theta_k = k$, when $t \in [\theta_k, \theta_{k+1})$, $\beta(t) = \theta_k$. $a_1 = 1.3 \times 10^{-2}$, $a_2 = 1.2 \times 10^{-2}$, $b_{11} = 2 \times 10^{-3}$, $b_{21} = 3 \times 10^{-3}$, $b_{12} = 10^{-3}$, $i = 1, 2$. $c_{ij} = 2 \times 10^{-3}$, $i, j = 1, 2$. $\sigma_1 = 2 \times 10^{-2} x_1(t) + 10^{-2} x_1(\beta(t))$, $\sigma_2 = 2 \times 10^{-2} x_2(t) + 10^{-2} x_2(\beta(t))$.

It is easy to check that $\theta = 1$, $L_i^f = L_i^g = 1$, $e_i = e = 4 \times 10^{-4}$, $l_i = l = 10^{-4}$, $n = 2$.

When $p = 2$, $C_p = 4$, we compute $m_1 = 9.75 \times 10^{-5}$, $m_2 = 8 \times 10^{-6}$, $\lambda_1 = 4.48 \times 10^{-4}$, $\lambda_2 = 6.555 \times 10^{-3}$, $\mu = 3.0648$, $k_1 = 1.36 \times 10^{-2}$, and $k_2 = 4.1 \times 10^{-3}$.

It is obvious that $k_1 = 1.36 \times 10^{-2} > k_2 \mu = 1.31 \times 10^{-2}$. Thus all conditions of Theorem 10 in this paper are satisfied; the equilibrium solution of (58) is exponentially stable in mean square.

*Remark 13.* When $t \in [\theta_k, \theta_{k+1})$, let $\beta(t) = t - \tau_j(t) = \theta_k$, $j = \{1, 2, \ldots\}$; then system (58) can be viewed as the system (2) in [22]. We can compute $\rho[C^{-1}(MM_1K + MM_2K + NN_1 + NN_2)] = 4$, where $\rho[C^{-1}(MM_1K + MM_2K + NN_1 + NN_2)]$ was defined in [22] and the condition $\rho[C^{-1}(MM_1K + MM_2K + NN_1 + NN_2)] \le 1$ is not satisfied. Hence results in [22] are useless to judge the exponential stability of system (58).

## 6. Conclusion

This is the first time that stochastic cellular neural networks with piecewise constant argument of generalized type are considered. Sufficient conditions on existence, uniqueness, and the $p$th moment exponential stability of the equilibrium point are derived by applying stochastic analysis techniques and some known inequalities. The obtained results could be useful in the design and applications of cellular neural networks. Furthermore, the method given in this paper may be extended to study more complex systems, such as stochastic neural networks with piecewise constant argument and impulsive perturbations.

## Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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