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Research Article

The Solvability and Optimal Controls for Some Fractional Impulsive Equations of Order $1 < \alpha < 2$

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We study the existence of solutions and optimal controls for some fractional impulsive equations of order $1 < \alpha < 2$. By means of Gronwall's inequality and Leray-Schauder's fixed point theorem, the sufficient condition for the existence of solutions and optimal controls is presented. Finally, an example is given to illustrate our main results.

1. Introduction

In this paper, we study some fractional evolution equation with finite impulsive:

$$\begin{gathered} {}^{c}D_t^{\alpha} x(t) = Ax(t) + f(t, x(t)) + B(t)u(t), \\ t \in J = [0, b], \ t \neq t_k, \\ \Delta x(t_k) = I_k(x(t_k^-)), \qquad \Delta x'(t_k) = I_k^*(x(t_k^-)), \\ k = 1, 2, 3, \ldots, m, \\ x(0) = x_0, \qquad x'(0) = x_1, \end{gathered} \tag{1}$$

where ${}^{c}D_t^{\alpha}$ is the standard Caputo fractional derivative of order $\alpha, b > 0$, $1 < \alpha < 2$, and $A : D(A) \subset X \to X$ is a sectorial operator of type (M, θ, α, μ) defined on a complex Banach space X. Let $f : J \times X \to X$ be a given function satisfying some assumptions that will be specified later. The function $I_k : X \to X$ is continuous and $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_m = T$; $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, where $x(t_k^+)$ and $x(t_k^-)$ denote the right and the left limits of $x(t)$ at $t = t_k (k = 1, 2, \ldots, n)$, and respectively, $\Delta x'(t_k)$ has the similar meaning for $x'(t_k)$. The control function u is given in a suitable admissible control set U_{ad}; B is a linear operator from a separable reflexive Banach space Y into X. The associated cost functions to be minimized over the family of admissible state control pairs (x, u) are given by

$$\mathcal{J}(x, u) = \int_J \mathscr{L}(t, x(t), u(t))\, dt. \tag{2}$$

For the last decades, fractional differential equations have been receiving intensive attention because they provide an excellent tool for the description of memory and hereditary properties of various materials and processes, such as physics, mechanics, chemistry, and engineering. For more details on fractional calculus theory, one can see the monographs of Miller and Ross [1], Podlubny [2], and Kilbas et al. [3] and the references therein.

Recently, impulsive differential equations have been proved to be valuable tools in the modelling of many phenomena in various fields of engineering, physics, and economics. The reason for the interest in the study of them is that the impulsive differential systems can be used to model processes which are subjected to abrupt changes and which cannot be described by the classical differential problem. For example, Liu and Li [4] utilized the well-known fixed point theorems to investigate the existence and uniqueness of solutions for the nonlinear impulsive fractional differential equations. Shu and Wang [5] studied the existence of mild solutions for fractional differential equations with nonlocal conditions of order $1 < \alpha < 2$:

$$D_t^{\alpha} u(t) = Au(t) + f(t, u(t)) + \int_0^t q(t-s)\, g(s, u(s))\, ds, \quad t \in J = [0, T],$$
$$u(0) + m(u) = u_0, \qquad u'(0) + n(u) = u_1 \in X, \tag{3}$$

where D_t^{α} is Caputo's fractional derivative of order $1 < \alpha < 2$ and A is a sectorial operator of type (M, θ, α, μ).

In [6], Dabas and Chauhan researched the existence and uniqueness of mild solution which is expressed by Mittag-Leffler functions for an impulsive neutral fractional integro-differential equation with infinite delay:

$$\begin{aligned} {}^cD_t^{\alpha}\left[x(t) + g(t, x_t)\right] &= A\left[x(t) + g(t, x_t)\right] + J_t^{1-\alpha} f(t, x_t, Bx(t)), \\ &\qquad t \in I = [0, T],\ t \neq t_k, \\ \Delta x(t_k) &= I_k(x(t_k)), \quad k = 1, 2, \ldots, m, \\ x_0 &= \phi \in \mathfrak{B}_h, \end{aligned} \tag{4}$$

where ${}^cD^{\alpha}$ denotes the Caputo fractional derivative of order $0 < \alpha \le 1$. Bazhlekova [7], Li and Peng [8] were concerned with the controllability of nonlocal fractional differential systems of order $1 < \alpha \le 2$ in Banach spaces. Wang et al. [9] discussed the new concept of solutions and existence results for impulsive fractional evolution equations.

To the best of our knowledge, the system (1) is still untreated in the literature and it is the motivation for the present work. The rest of this paper is organized as follows. In Section 2, some notations and preparations are given. In Section 3, mainly some results of (1) are obtained. At last, an example is given to demonstrate our results.

2. Preliminaries

In this section, we will give some definitions and preliminaries which will be used in the paper.

Firstly, we will define $PC(J, X)$ and $PC^1(J, X)$. The norm of the space X will be defined by $\|\cdot\|_X$; let $C(J, X)$ denote the Banach space of all X-value continuous functions from $J = [0, T]$ into X, the norm $\|\cdot\|_c = \sup\{\|\cdot\|_X\}$. Let another Banach space $PC(J, X) = \{x : J \to X, x \in C((t_k, t_{k+1}], X)$, $k = 0, 1, 2, \ldots, n$, there exist $x(t_k^-)$, $x(t_k^+)$, $k = 1, 2, \ldots, n$, and $x(t_k^-) = x(t_k)\}$; the norm $\|x\|_{PC} = \max\{\sup \|x(t+0)\|, \sup \|x(t-0)\|\}$. $PC^1(J, X) = \{x : J \to X, x \in C^1((t_k, t_{k+1}], X)$, $k = 0, 1, 2, \ldots, n$, there exist $x'(t_k^+)$, $x'(t_k^-)$, $k = 1, 2, \ldots, n$, and $x'(t_k^-) = x'(t_k)\}$, the norm $\|x\|_{PC^1} = \sup\{\|x(t)\|_{PC}, \|x'(t)\|_{PC} : t \in J\}$. Obviously $PC(J, X)$ and $PC^1(J, X)$ are Banach spaces.

We denote by $L^p(J, R)$ the Banach space of all Lebesgue measurable functions from J to R with $\|f\|_{L^p(J,R)} = (\int_J |f(t)|^p dt)^{1/p}$.

Let us recall some known definitions of fractional calculus; for more details, see [1–3, 10].

Let $\alpha, \beta > 0$; then $n-1 < \alpha < n, n-1 < \beta < n$, and f is a suitable function.

Definition 1 (Riemann-Liouville fractional integral and derivative operators). The integral operator I_a^{α} is defined on $L_1[a, b]$ by

$$I_a^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt, \quad (a \le x \le b). \tag{5}$$

The derivative operator is defined as $D_a^{\alpha} f(x) = D_a^n (I_a^{n-\alpha}) f(x)$, where $D_a^n = d^n/dt^n$ and

$$I_a^{\alpha} I_a^{\beta} f(x) = I_a^{\alpha+\beta} f(x). \tag{6}$$

Definition 2. Caputo's fractional derivative of $f(x)$ of order α is defined as

$${}^cD_a^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^{(n)}(t)\, dt. \tag{7}$$

If $a = 0$, we can write the Caputo derivative of the function $f(t) \in C^n[0, \infty), f : [0, \infty) \to R$ via the above Riemann-Liouville fractional derivative as

$${}^cD_0^{\alpha} f(x) = {}^LD^{\alpha}\left[f(x) - \sum_{k=0}^{n-1} \frac{x^k}{k!} f^{(k)}(0)\right]. \tag{8}$$

Definition 3 (see [11]). Let $A : D(A) \subset X \to X$ be a sectorial operator of type (M, θ, α, μ) if there exists $0 < \theta < \pi/2$, $M > 0$, $\mu \in R$, such that the α-resolvent of A exists outside the sector:

$$\begin{gathered} \mu + S_{\theta} = \{\mu + \lambda : \lambda \in C, |\mathrm{Arg}(-\lambda)| < \theta\}, \\ \left\|(\lambda - A)^{-1}\right\| \le \frac{M}{\lambda - \mu}, \quad \lambda \notin S_{\theta}. \end{gathered} \tag{9}$$

Theorem 4. *According to Lemma 2.6 in [4], one can get that if $u(t) \in PC^2(J, X)$, then*

$$I^q\, {}^cD^q u(t) = \begin{cases} u(t) - tu'(0) - u(0), & t \in [0, t_1], \\ u(t) - \sum\limits_{i=1}^{k} I_i(u(t_i)) \\ \quad - \sum\limits_{i=1}^{k} (t - t_i) I_i^*(u(t_i)) - tu'(0) - u(0), & t \in (t_k, t_{k+1}]. \end{cases} \tag{10}$$

Proof. If $t \in [0, t_1]$, then

$$\begin{aligned} I^q\, {}^cD^q u(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \frac{1}{\Gamma(2-q)} \int_0^s (s-\tau)^{1-q} u''(\tau)\, d\tau\, ds \\ &= \frac{1}{\Gamma(q)\Gamma(2-q)} \int_0^t u''(\tau) \int_{\tau}^t (t-s)^{q-1} (s-\tau)^{1-q} ds\, d\tau \\ &= \int_0^t u''(\tau)(t-\tau)\, d\tau = u(t) - tu'(0) - u(0). \end{aligned} \tag{11}$$

If $t \in (t_k, t_{k+1}]$, $k \geq 1$, then

$$
\begin{aligned}
&I^{q\,c}D^q u(t) \\
&= \frac{1}{\Gamma(q)\Gamma(2-q)} \int_0^t (t-s)^{q-1} \int_0^s (s-\tau)^{1-q} u''(\tau)\, d\tau\, ds \\
&= \frac{1}{\Gamma(q)\Gamma(2-q)} \int_0^{t_1} (t-s)^{q-1} \int_0^s (s-\tau)^{1-q} u''(\tau)\, d\tau\, ds \\
&\quad + \sum_{i=1}^{k-1} \frac{1}{\Gamma(q)\Gamma(2-q)} \int_{t_i}^{t_{i+1}} (t-s)^{q-1} \\
&\quad \times \int_0^s (s-\tau)^{1-q} u''(\tau)\, d\tau\, ds \\
&\quad + \frac{1}{\Gamma(q)\Gamma(2-q)} \int_{t_k}^t (t-s)^{q-1} \int_0^s (s-\tau)^{1-q} u''(\tau)\, d\tau\, ds \\
&= \frac{1}{\Gamma(q)\Gamma(2-q)} \int_0^{t_1} (t-s)^{q-1} \int_0^s (s-\tau)^{1-q} u''(\tau)\, d\tau\, ds \\
&\quad + \sum_{i=1}^{k-1} \frac{1}{\Gamma(q)\Gamma(2-q)} \int_{t_i}^{t_{i+1}} (t-s)^{q-1} \\
&\quad \times \left[\sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} (s-\tau)^{1-q} u''(\tau)\, d\tau + \int_{t_i}^s (s-\tau)^{1-q} u''(\tau)\, d\tau \right] ds \\
&\quad + \frac{1}{\Gamma(q)\Gamma(2-q)} \int_{t_k}^t (t-s)^{q-1} \\
&\quad \times \left[\sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} (s-\tau)^{1-q} u''(\tau)\, d\tau + \int_{t_k}^s (s-\tau)^{1-q} u''(\tau)\, d\tau \right] ds \\
&= \frac{1}{\Gamma(q)\Gamma(2-q)} \int_0^{t_1} u''(\tau)\, d\tau \int_\tau^{t_1} (t-s)^{q-1} (s-\tau)^{1-q} ds \\
&\quad + \sum_{i=1}^{k-1} \sum_{j=1}^{i-1} \frac{1}{\Gamma(q)\Gamma(2-q)} \int_{t_j}^{t_{j+1}} u''(\tau)\, d\tau \\
&\quad \times \int_{t_i}^{t_{i+1}} (t-s)^{q-1} (s-\tau)^{1-q} ds \\
&\quad + \sum_{i=1}^{k-1} \frac{1}{\Gamma(q)\Gamma(2-q)} \int_{t_i}^{t_{i+1}} u''(\tau)\, d\tau \\
&\quad \times \int_\tau^{t_{i+1}} (t-s)^{q-1} (s-\tau)^{1-q} ds \\
&\quad + \sum_{j=0}^{k-1} \frac{1}{\Gamma(q)\Gamma(2-q)} \int_{t_j}^{t_{j+1}} u''(\tau)\, d\tau \\
&\quad \times \int_{t_k}^t (t-s)^{q-1} (s-\tau)^{1-q} ds \\
&\quad + \frac{1}{\Gamma(q)\Gamma(2-q)} \int_{t_k}^t u''(\tau)\, d\tau \int_\tau^t (t-s)^{q-1} (s-\tau)^{1-q} ds \\
&= u(t) - \sum_{i=1}^k I_i(u(t_i)) - \sum_{i=1}^k (t-t_i) I_i^*(u(t_i)) \\
&\quad - tu'(0) - u(0),
\end{aligned} \tag{12}
$$

where

$$
\begin{aligned}
&\int_\tau^t (t-s)^{q-1} (s-\tau)^{1-q} ds \\
&\quad = (t-\tau) \int_0^1 (1-z)^{q-1} z^{1-q} dz \\
&\quad = (t-\tau) B(2-q, q) = (t-\tau) \frac{\Gamma(2-q)\Gamma(q)}{\Gamma(2-q+q)} \\
&\quad = (t-\tau) \frac{\Gamma(2-q)\Gamma(q)}{\Gamma(2)} = (t-\tau)\Gamma(2-q)\Gamma(q);
\end{aligned} \tag{13}
$$

with the help of the substitution $s = z(t-\tau) + \tau$, the proof is completed. □

Lemma 5 (see [12]). *Let $x \in PC(J, X)$ satisfy the following inequality:*

$$
\|x(t)\| \leq c_1 + c_2 \int_0^t (t-s)^{\beta-1} \|x(s)\| ds + \sum_{0<t_k<t} h_k \left\| x(t_k^-) \right\|, \tag{14}
$$

where c_1, c_2, and $h_k \geq 0$ are constants. Then

$$
\begin{aligned}
\|x(t)\| \leq c_1 \left(1 + H^* E_\beta \left(c_2 \Gamma(\beta) t^\beta\right)\right)^k E_\beta \left(c_2 \Gamma(\beta) t^\beta\right) \\
\text{for } t \in [t_k, t_{k+1}],
\end{aligned} \tag{15}
$$

where $H^* = \max\{h_k : k = 1, 2, \ldots, m\}$.

Theorem 6 (Hölder's inequality). *Assume that $p > 0, q > 0$, and $1/p + 1/q = 1$; if $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, then $f \cdot g \in L^1(\Omega)$ and $\|fg\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}$.*

Theorem 7 (Arzela-Ascoli theorem). *If a sequence (f_n) in $C(X)$ is bounded and equicontinuous, then it has a uniformly convergent subsequence.*

Remark 8. A subset F of $C(X)$ is compact if and only if it is closed, bounded, and equicontinuous.

Theorem 9 (Leray-Schauder's fixed point theorem). *If C is a closed bounded and convex subset of Banach space X and $F : C \to C$ is completely continuous, then F has a fixed point in C.*

3. Existence and Uniqueness of Mild Solution

In this section, we will investigate the existence and uniqueness for impulsive fractional differential equations with the help of Schauder's fixed point theorem and someone else.

Firstly, we will make the following assumptions.

$H(1)$ The function $f : J \times X \to X$ satisfies the following.

(i) f is measurable for all $t \in J$.

(ii) There exists a constant $L_f > 0$ such that $\|f(t,x) - f(t,y)\| \le L_f\|x-y\|$, for all $x, y \in X$.

(iii) There exist a real function $\phi(t) \in L^{1/\gamma}(J, R^+)$, $\gamma \in (0,\alpha)$, and a constant $\theta > 0$, such that $\|f(t,x)\| \le \phi(t) + \theta\|x\|$, for a.e. $t \in J$ and all $x \in X$.

$H(2)$ $I_k, I_k^* : X \to X$ $(k = 1, 2, \ldots, m)$ satisfies the following.

(i) I_k and I_k^* are continuous and map a bounded set to a bounded set.

(ii) There exist constants $h_k > 0, h_k^* > 0$ $(k = 1, 2, \ldots, m)$ such that

$$\begin{aligned} \left\|I_k(x) - I_k(y)\right\| &\le h_k\left\|x - y\right\|, \quad x, y \in X, \\ \left\|I_k^*(x) - I_k^*(y)\right\| &\le h_k^*\left\|x - y\right\|, \quad x, y \in X. \end{aligned} \tag{16}$$

Specially, if $y = 0$,

$$\begin{aligned} \left\|I_k(x) - I_k(0)\right\| &\le h_k\|x\|, \quad x \in X. \\ \left\|I_k^*(x) - I_k^*(0)\right\| &\le h_k^*\|x\|, \quad x \in X. \end{aligned} \tag{17}$$

We can make $\|I(0)\| = \sup\{\|I_k(0)\|,\ k = 1, 2, \ldots, m\}$, $\|I^*(0)\| = \sup\{I_k^*(0),\ k = 1, 2, \ldots, m\}$.

$H(3)$ Operator $B \in L^\infty(J, L(Y, X))$ and bounded, so there exists $M_B > 0$, $\|B\| \le M_B$.

$H(4)$ The multivalued maps $U : J \to P_f(Y)$ (where $P_f(Y)$ is a class of nonempty closed and convex subsets of Y) are measurable and $U(\cdot) \subseteq \Omega$, where Ω is a bounded set of Y.

Set the admissible control set:

$$U_{\text{ad}} = S_U^p = \left\{u \in L^2(J, \Omega) : u(t) \in U(t) \text{ a.e.}\right\}. \tag{18}$$

Then, $U_{\text{ad}} \neq \emptyset$ (see Proposition 2.1.7 and Lemma 2.3.2 of [13]). And it is obvious that $Bu \in L^2(J, X)$ for all $u \in U_{\text{ad}}$.

According to Definitions 1–2 and Theorem 4, without loss of generality, let $t \in (t_k, t_{k+1}]$ and $1 \le k \le m-1$, by comparson with the fractional differential equations given in [4, 5, 8, 9, 12]; we will define the concept of mild solution for problem (1) as follows.

Definition 10. A function $x \in PC^1(J, X)$ is said to be a solution (mild solution) of the problem (1) such that

$$\begin{aligned} x(t) &= S_\alpha(t)x_0 + Q_\alpha(t)x_1 \\ &\quad + \sum_{i=1}^{k} S_\alpha(t - t_i) I_i\left(x\left(t_i^-\right)\right) + \sum_{i=1}^{k} Q_\alpha(t - t_i) I_i^*\left(x\left(t_i^-\right)\right) \\ &\quad + \int_0^t T_\alpha(t - \tau)\left(f(\tau, x(\tau)) + B(\tau)u(\tau)\right) d\tau, \end{aligned} \tag{19}$$

where

$$\begin{aligned} S_\alpha(t) &= \frac{1}{2\pi i}\int_c e^{\lambda t}\lambda^{\alpha-1}R\left(\lambda^\alpha, A\right) d\lambda, \\ Q_\alpha(t) &= \frac{1}{2\pi i}\int_c e^{\lambda t}\lambda^{\alpha-2}R\left(\lambda^\alpha, A\right) d\lambda, \\ T_\alpha(t) &= \frac{1}{2\pi i}\int_c e^{\lambda t}R\left(\lambda^\alpha, A\right) d\lambda, \end{aligned} \tag{20}$$

with c being a suitable path such that $\lambda^\alpha \notin \mu + S_\theta$ for $\lambda \in c$. For more details, one can see [5].

Lemma 11 (see [5]). *For any fixed $t \ge 0$, $S_\alpha(t)$, $Q_\alpha(t)$, and $T_\alpha(t)$ are compacted and bounded operators; that is, for any $t \ge 0$,*

$$\left\|S_\alpha(t)\right\| \le M, \qquad \left\|Q_\alpha(t)\right\| \le M, \qquad \left\|T_\alpha(t)\right\| \le M. \tag{21}$$

Theorem 12. *If the assumptions $H(1)$, $H(2)$, $H(3)$, and $H(4)$ are satisfied and Lemma 5 and (1) is mildly solvable on [0,b], then there exists a constant $\omega > 0$ such that $\|x(t)\| \le \omega$, for all $t \in J$.*

Proof. If (1) can be solvable on [0,b], we may suppose $x(t)$ is the mild solution of it, so $x(t)$ must satisfy (19).

From Theorem 6, we also get that

$$\begin{aligned} \|x(t)\| &\le \left\|S_\alpha(t)x_0\right\| + \left\|Q_\alpha(t)x_1\right\| \\ &\quad + \left\|\sum_{i=1}^{k} S_\alpha(t - t_k) I_k\left(x\left(t_k^-\right)\right)\right\| \\ &\quad + \left\|\sum_{i=1}^{k} Q_\alpha(t - t_k) I_k^*\left(x\left(t_k^-\right)\right)\right\| \\ &\quad + \int_0^t \left\|T_\alpha(t - \tau)\left(f(\tau, x(\tau)) + B(\tau)u(\tau)\right)\right\| d\tau \\ &\le M\left(\left\|x_0\right\| + \left\|x_1\right\|\right) + M\sum_{i=1}^{m} h_k\left\|x\left(t_k^-\right)\right\| + Mm\|I(0)\| \\ &\quad + M\sum_{i=1}^{m} h_k^*\left\|x\left(t_k^-\right)\right\| + Mm\left\|I^*(0)\right\| \\ &\quad + M\int_0^t \left[\phi(s) + \theta\|x(\tau)\|\right] d\tau + MM_B\|u\|_{L^2}b^2 \\ &\le M\left(\left\|x_0\right\| + \left\|x_1\right\|\right) + M\sum_{i=1}^{m}\left(h_k + h_k^*\right)\left\|x\left(t_k^-\right)\right\| \\ &\quad + Mm\left(\|I(0)\| + \left\|I^*(0)\right\|\right) + Mb^{1-\gamma}\|\phi\|_{L^{1/\gamma}} \\ &\quad + M\theta\int_0^t \|x(\tau)\| d\tau + MM_B\|u\|_{L^2}b^2. \end{aligned} \tag{22}$$

Let $\rho = M(\|x_0\| + \|x_1\|) + Mm(\|I(0)\| + \|I^*(0)\|) + Mb^{1-\gamma}\|\phi\|_{L^{1/\gamma}} + MM_B\|u\|_{L^2}b^2$; then

$$\|x(t)\| \le \rho + M\sum_{k=1}^{m}(h_k + h_k^*)\|x(t_k^-)\| + M\theta\int_0^t \|x(\tau)\|\,d\tau, \tag{23}$$

so it follows from Lemma 5 that

$$\|x(t)\| \le \rho(1 + H^* E_1(M\theta b))^k E_1(M\theta b) = \omega, \tag{24}$$

where

$$H^* = \max\{M(h_k + h_k^*) : k = 1, 2, \ldots, m\}; \tag{25}$$

the proof is completed. □

Theorem 13. *Assume that the hypotheses $H(1)$, $H(2)$, and $H(3)$ are satisfied Theorem 12; then the problem (1) has a unique mild solution on J provided that*

$$\left(\sum_{i=1}^{m}(h_i + h_i^*) + \theta b\right)M < 1. \tag{26}$$

Proof. Transform problem (1) into a fixed point theorem. Consider the operator $F : PC^1(J, X) \to PC^1(J, X)$ defined by

$$\begin{aligned}(Fx)(t) &= S_\alpha(t)x_0 + Q_\alpha(t)x_1 + \sum_{i=1}^{k}S_\alpha(t-t_i)I_i(x(t_i^-))\\ &\quad + \sum_{i=1}^{k}Q_\alpha(t-t_i)I_i^*(x(t_i^-))\\ &\quad + \int_0^t T_\alpha(t-\tau)(f(\tau, x(\tau)) + B(\tau)u(\tau))\,d\tau.\end{aligned} \tag{27}$$

Clearly, the problem of finding mild solutions of (1) is reduced to finding the fixed points of the F. The proof is based on Theorem 9. Now we prove that the operators F satisfy all the conditions of Theorem 9.

Firstly, choose

$$\begin{aligned}&M\left[(\|x_0\| + \|x_1\|) + m(\|I(0)\| + \|I^*(0)\|) + b^{1-\gamma}\|\phi\|_{L^{1/\gamma}} + M_B\|u\|_{L^2}b^2\right]\\ &\quad\times\left(1 - M\sum_{k=1}^{n}(h_k + h_k^*) - Mb\theta\right)^{-1} \le r\end{aligned} \tag{28}$$

and consider the bounded set $B_r = \{x \in PC^1 : \| x \| \le r\}$. Next, we divide the proof into four steps.

Step 1. We prove that $FB_r \subseteq B_r$:

$$\begin{aligned}\|(Fx)(t)\| &\le \|S_\alpha(t)x_0\| + \|Q_\alpha(t)x_1\|\\ &\quad + \left\|\sum_{i=1}^{k}S_\alpha(t-t_k)I_k(x(t_k^-))\right\|\\ &\quad + \left\|\sum_{i=1}^{k}Q_\alpha(t-t_k)I_k^*(x(t_k^-))\right\|\\ &\quad + \int_0^t \|T_\alpha(t-\tau)(f(\tau, x(\tau)) + B(\tau)u(\tau))\|\,d\tau\\ &\le M(\|x_0\| + \|x_1\|) + M\sum_{k=1}^{m}h_k\|x(t_k^-)\|\\ &\quad + Mm\|I(0)\| + M\sum_{k=1}^{m}h_k^*\|x(t_k^-)\| + Mn\|I^*(0)\|\\ &\quad + M\int_0^t [\phi(\tau) + \theta\|x(\tau)\|]\,d\tau + MM_B\|u\|_{L^2}b^2\\ &\le M(\|x_0\| + \|x_1\|) + M\sum_{k=1}^{m}(h_k + h_k^*)\|x(t_k^-)\|\\ &\quad + Mm(\|I(0)\| + \|I^*(0)\|) + Mb^{1-\gamma}\|\phi\|_{L^{1/\gamma}}\\ &\quad + M\theta\int_0^t \|x(\tau)\|\,d\tau + MM_B\|u\|_{L^2}b^2\\ &\le M(\|x_0\| + \|x_1\|) + Mb^{1-\gamma}\|\phi\|_{L^{1/\gamma}}\\ &\quad + Mm(\|I(0)\| + \|I^*(0)\|) + MM_B\|u\|_{L^2}b^2\\ &\quad + \left(M\sum_{i=1}^{m}(h_i + h_i^*) + Mb\theta\right)r \le r.\end{aligned} \tag{29}$$

Hence, we can make $FB_r \subseteq B_r$. So F is a contraction mapping.

Step 2. We show that F is continuous.

Let $\{x_n\}$ be a sequence such that $x_n \to x$ in $PC^1(J, X)$ as $n \to \infty$. Then for each $t \in J$, we obtain

$$\begin{aligned}&\|(Fx_n)(t) - (Fx)(t)\|\\ &\quad\le \left\|\sum_{i=1}^{k}S_\alpha(t-t_i)[I_i(x_n(t_i^-)) - I_i(x(t_i^-))]\right\|\\ &\qquad + \left\|\sum_{i=1}^{k}Q_\alpha(t-t_i)[I_i^*(x_n(t_i^-)) - I_i^*(x(t_i^-))]\right\|\\ &\qquad + \int_0^t \|T_\alpha(t-\tau)[f(\tau, x_n(\tau)) - f(\tau, x(\tau))]\|\,d\tau\\ &\quad\le M\sum_{i=1}^{m}h_i\|x_n - x\| + M\sum_{i=1}^{m}h_i^*\|x_n - x\|\\ &\qquad + ML_f\int_0^t \|x_n(\tau) - x(\tau)\|\,d\tau\\ &\quad\le \left[M\sum_{i=1}^{m}h_i + M\sum_{i=1}^{m}h_i^* + ML_f b\right]\|x_n - x\|.\end{aligned} \tag{30}$$

As $x_n \to x$, it is easy to see that

$$\|Fx_n - Fx\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{31}$$

Step 3. F is equicontinuous on B_r.

Let $0 \le \tau_1 < \tau_2 \le b$; then for each $x \in B_r$, we obtain

$$\begin{aligned}
&\|(Fx)(\tau_2) - (Fx)(\tau_1)\| \\
&\quad \le \|[S_\alpha(\tau_2) - S_\alpha(\tau_1)]x_0\| + \|[Q_\alpha(\tau_2) - Q_\alpha(\tau_1)]x_1\| \\
&\qquad + \left\| \left[\sum_{i=1}^{k} S_\alpha(\tau_2 - t_k) - \sum_{i=1}^{k} S_\alpha(\tau_1 - t_k) \right] I_k(x(t_k^-)) \right\| \\
&\qquad + \left\| \left[\sum_{i=1}^{k} Q_\alpha(\tau_2 - t_k) - \sum_{i=1}^{k} Q_\alpha(\tau_1 - t_k) \right] I_k^*(x(t_k^-)) \right\| \\
&\qquad + \left\| \int_0^{\tau_2} T_\alpha(\tau_2 - \tau) f(\tau, x(\tau))\, d\tau \right. \\
&\qquad\qquad \left. - \int_0^{\tau_1} T_\alpha(\tau_1 - \tau) f(\tau, x(\tau))\, d\tau \right\| \\
&\qquad + \left\| \int_0^{\tau_2} T_\alpha(\tau_2 - \tau) B(\tau) u(\tau)\, d\tau \right. \\
&\qquad\qquad \left. - \int_0^{\tau_1} T_\alpha(\tau_1 - \tau) B(\tau) u(\tau)\, d\tau \right\| \\
&\quad \le \|S_\alpha(\tau_2) - S_\alpha(\tau_1)\| \|x_0\| + \|Q_\alpha(\tau_2) - Q_\alpha(\tau_1)\| \|x_1\| \\
&\qquad + \sum_{i=1}^{k} \|S_\alpha(\tau_2 - t_k) - S_\alpha(\tau_1 - t_k)\| \\
&\qquad\qquad \times (h_k \|x(t_k^-)\| + \|I_k(0)\|) \\
&\qquad + \sum_{i=1}^{k} \|Q_\alpha(\tau_2 - t_k) - Q_\alpha(\tau_1 - t_k)\| \\
&\qquad\qquad \times (h_k^* \|x(t_k^-)\| + \|I_k^*(0)\|) \\
&\qquad + \left\| \int_0^{\tau_1} (T_\alpha(\tau_2 - \tau) - T_\alpha(\tau_1 - \tau)) f(\tau, x(\tau))\, d\tau \right\| \\
&\qquad + \left\| \int_{\tau_1}^{\tau_2} T_\alpha(\tau_2 - \tau) f(\tau, x(\tau))\, d\tau \right\| \quad \text{denoted by } Q_1 \\
&\qquad + \left\| \int_0^{\tau_1} (T_\alpha(\tau_2 - \tau) - T_\alpha(\tau_1 - \tau)) B(\tau) u(\tau)\, d\tau \right\| \\
&\qquad + \left\| \int_{\tau_1}^{\tau_2} T_\alpha(\tau_2 - \tau) B(\tau) u(\tau)\, d\tau \right\| \quad \text{denoted by } Q_2.
\end{aligned} \tag{32}$$

Let

$$\begin{aligned}
\Lambda = {} & \|S_\alpha(\tau_2) - S_\alpha(\tau_1)\| \|x_0\| + \|Q_\alpha(\tau_2) - Q_\alpha(\tau_1)\| \|x_1\| \\
& + \sum_{i=1}^{k} \|S_\alpha(\tau_2 - t_k) - S_\alpha(\tau_1 - t_k)\| \\
& \qquad \times (h_k \|x(t_k^-)\| + \|I_k(0)\|) \\
& + \sum_{i=1}^{k} \|Q_\alpha(\tau_2 - t_k) - Q_\alpha(\tau_1 - t_k)\| \\
& \qquad \times (h_k^* \|x(t_k^-)\| + \|I_k^*(0)\|) \\
& + \left\| \int_0^{\tau_1} (T_\alpha(\tau_2 - \tau) - T_\alpha(\tau_1 - \tau)) f(\tau, x(\tau))\, d\tau \right\| \\
& + \left\| \int_0^{\tau_1} (T_\alpha(\tau_2 - \tau) - T_\alpha(\tau_1 - \tau)) B(\tau) u(\tau)\, d\tau \right\|.
\end{aligned} \tag{33}$$

By Lemma 11, we have

$$\lim_{\tau_2 \to \tau_1} \Lambda = 0. \tag{34}$$

By assumption $H(2)$, we obtain

$$\begin{aligned}
Q_1 &\le M\left(\|\phi\|_{L^{1/\gamma}} + \theta b^\gamma r\right)(\tau_2 - \tau_1)^{1-\gamma}, \\
Q_2 &\le M M_B \|u\|_{L^2} (\tau_2 - \tau_1)^2.
\end{aligned} \tag{35}$$

Combining the estimations for Λ, Q_1, and Q_2, let $\tau_2 \to \tau_1$; we know that $\|(Fx)(\tau_2) - (Fx)(\tau_1)\| \to 0$, which implies that F is equicontinuous.

Step 4. Now we show that F is compact.

Let $t \in J$ be fixed; we show that the set $\Pi(t) = \{(Fx)(t) : x \in B_r\}$ is relatively compact in X. From Step 1 and (24), we know that

$$\|Fx(t)\| \le \rho(1 + H^* E_1(M\theta b))^k E_1(M\theta b) = \omega < \infty. \tag{36}$$

Then the set $\Pi(t) = \{(Fx)(t) : x \in B_r\}$ is uniformly bounded. From Step 3 and Arzela -Ascoli theorem, we know that the set $\Pi(t) = \{(Fx)(t) : x \in B_r\}$ is relatively compact in X.

As a result, by the conclusion of Theorem 9, we obtain that F has a fixed point x on B_r; therefore system (1) has a unique mild solution on J. The proof is completed. □

4. Optimal Control Results

In the following, we will consider the Lagrange problem (P).

Find a control pair $(x^0, u^0) \in PC(J, X) \times U_{ad}$ such that

$$\mathcal{J}(x^0, u^0) \le J(x^u, u), \quad \forall (x, u) \in PC(J, X) \times U_{ad}, \tag{37}$$

where

$$\mathcal{J}(x^u, u) := \int_0^b \mathcal{L}(t, x^u(t), u(t))\, dt, \tag{38}$$

and x^u denotes the mild solution of system (1) corresponding to the control $u \in U_{ad}$.

For the existence of solution for problem (P), we will introduce the following assumption.

$H(5)$ The function $\mathcal{L} : J \times X \times Y \to R \cup \{\infty\}$ satisfies the following.

(i) The function $\mathcal{L} : J \times X \times Y \to R \cup \{\infty\}$ is Borel measurable;

(ii) $\mathcal{L}(t, \cdot, \cdot)$ is sequentially lower semicontinuous on $X \times Y$ for almost all $t \in J$.

(iii) $\mathscr{L}(t, x, \cdot)$ is convex on Y for each $x \in X$ and almost all $t \in J$.

(iv) There exist constants $c \geq 0, d > 0$ and φ is nonnegative, and $\varphi \in L^1(J, R)$ such that

$$\mathscr{L}(t, x, u) \geq \varphi(t) + c\|x\|_X + d\|u\|_Y^p. \tag{39}$$

Next, we can give the following result on existence of optimal controls for problem (P).

Theorem 14. *Let the assumptions of Theorem 13 and $H(5)$ hold. Suppose that B is a strongly continuous operator. Then Lagrange problem (P) admits at least one optimal pair; that is, there exists an admissible control pair $(x^0, u^0) \in PC(J, X) \times U_{ad}$ such that*

$$\begin{gathered}\mathcal{J}\left(x^0, u^0\right) = \int_0^b \mathscr{L}\left(t, x^0(t), u^0(t)\right) dt \leq \mathcal{J}\left(x^u, u\right), \\ \forall \left(x^u, u\right) \in PC(J, X) \times U_{ad}.\end{gathered} \tag{40}$$

Proof. If $\inf\{\mathcal{J}(x^u, u) : (x^u, u) \in PC(J, X) \times U_{\text{ad}}\} = +\infty$, there is nothing to prove.

Without loss of generality, we assume that $\inf\{J(x^u, u) : (x^u, u) \in PC(J, X) \times U_{\text{ad}}\} = \rho < +\infty$. Using $H(5)$, we have $\rho > -\infty$. By definition of infimum, there exists a minimizing sequence feasible pair $\{(x^n, u^n)\} \subset \mathscr{P}_{\text{ad}} \equiv \{(x, u) : x$ is a mild solution of system (1) corresponding to $u \in U_{\text{ad}}\}$, such that $J(x^n, u^n) \to \rho$ as $m \to +\infty$. Since $\{u^n\} \subseteq U_{\text{ad}}, m = 1, 2, \ldots, \{u^n\}$ is a bounded subset of the separable reflexive Banach space $L^p(J, Y)$; there exists a subsequence, relabeled as $\{u^n\}$, and $u^0 \in L^p(J, Y)$ such that

$$u^n \xrightarrow{w} u^0 \quad \text{in } L^p(J, Y). \tag{41}$$

Since U_{ad} is closed and convex, due to Mazur lemma, $u_0 \in U_{\text{ad}}$. Let $\{x^n\}$ denote the sequence of solutions of the system (1) corresponding to $\{u^m\}$; x^0 is the mild solution of the system (1) corresponding to u^0. x^n and x^0 satisfy the following integral equation, respectively:

$$\begin{aligned}x^n(t) = {} & S_\alpha(t) x_0 + Q_\alpha(t) x_1 + \sum_{i=1}^{k} S_\alpha(t - t_i) I_i(x^n(t_i^-)) \\ & + \sum_{i=1}^{k} Q_\alpha(t - t_i) I_i^*(x^n(t_i^-)) \\ & + \int_0^t T_\alpha(t - \tau)\left(f\left(\tau, x^n(\tau)\right) + B(\tau) u^n(\tau)\right) d\tau, \\ x^0(t) = {} & S_\alpha(t) x_0 + Q_\alpha(t) x_1 + \sum_{i=1}^{k} S_\alpha(t - t_i) I_i\left(x^0(t_i^-)\right) \\ & + \sum_{i=1}^{k} Q_\alpha(t - t_i) I_i^*\left(x^0(t_i^-)\right) \\ & + \int_0^t T_\alpha(t - \tau)\left(f\left(\tau, x^0(\tau)\right) + B(\tau) u^0(\tau)\right) d\tau.\end{aligned} \tag{42}$$

It follows the boundedness of $\{u^n\}, \{u^0\}$, and Theorem 12; one can check that there exists a positive number ω such that $\| x^n \| \leq \omega, \| x^0 \| \leq \omega$.

For $t \in J$, we obtain

$$\begin{aligned}& \left\|x^n(t) - x^0(t)\right\| \\ & = \underbrace{\left\|\sum_{i=1}^{k} S_\alpha(t - t_i)\left[I_i(x^n(t_i^-)) - I_i\left(x^0(t_i^-)\right)\right]\right\|}_{\text{denoted by } \eta_1(t)} \\ & + \underbrace{\left\|\sum_{i=1}^{k} Q_\alpha(t - t_i)\left[I_i^*(x^n(t_i^-)) - I_i^*\left(x^0(t_i^-)\right)\right]\right\|}_{\text{denoted by } \eta_2(t)} \\ & + \underbrace{\int_0^t \left\|T_\alpha(t - \tau)\left[f(s, x^n(\tau)) - f\left(\tau, x^0(\tau)\right)\right]\right\| d\tau}_{\text{denoted by } \eta_3(t)} \\ & + \underbrace{\int_0^t \left\|T_\alpha(t - \tau)\left[B(\tau) u^n(\tau) - B(\tau) u^0(\tau)\right]\right\| d\tau.}_{\text{denoted by } \eta_4(t).}\end{aligned} \tag{43}$$

By $H(3)$(ii), we have

$$\begin{aligned}\eta_1(t) & = \left\|\sum_{i=1}^{k} S_\alpha(t - t_i)\left[I_i(x^n(t_i^-)) - I_i\left(x^0(t_i^-)\right)\right]\right\| \\ & \leq M \sum_{i=1}^{m} h_i \left\|x^n - x^0\right\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \\ \eta_2(t) & = \left\|\sum_{i=1}^{k} Q_\alpha(t - t_i)\left[I_i^*(x^n(t_i^-)) - I_i^*\left(x^0(t_i^-)\right)\right]\right\| \\ & \leq M \sum_{i=1}^{m} h_i^* \left\|x^n - x^0\right\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{44}$$

Using Lemma 11 and by $H(1)$(ii), one can obtain

$$\begin{aligned}\eta_3(t) & = \int_0^t \left\|T_\alpha(t - \tau)\left[f(s, x^n(\tau)) - f\left(\tau, x^0(\tau)\right)\right]\right\| d\tau \\ & \leq ML_f \int_0^t \left\|x^n(s) - x^0(s)\right\| ds.\end{aligned} \tag{45}$$

Similarly, one has

$$\begin{aligned}\eta_4(t) & = \int_0^t \left\|T_\alpha(t - \tau)\left[B(\tau) u^n(\tau) - B(\tau) u^0(\tau)\right]\right\| d\tau \\ & \leq Mb^2 \left(\int_0^t \left\|B(\tau) u^n(\tau) - B(\tau) u^0(s)\right\|^{1/2} d\tau\right)^2 \\ & \leq Mb^2 \left\|Bu^n - Bu^0\right\|_{L^2(J,Y)}.\end{aligned} \tag{46}$$

Since B is strongly continuous, we have

$$\left\|Bu^n - Bu^0\right\|_{L^2(J,Y)} \xrightarrow{s} \quad \text{as } n \longrightarrow \infty, \tag{47}$$

which implies

$$\eta_4(t) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{48}$$

Thus

$$\begin{aligned} &\left\|x^n(t) - x^0(t)\right\| \\ &\leq \eta_1(t) + \eta_3(t) + \eta_4(t) + ML_f \int_0^t \left\|x^n(s) - x^0(s)\right\| ds, \end{aligned} \tag{49}$$

and by virtue of singular version Gronwall inequality (see Remark 3.2, in [12]), we obtain

$$\left\|x^n(t) - x^0(t)\right\| \leq [\eta_1(t) + \eta_3(t) + \eta_4(t)] E_1\left(ML_f b\right). \tag{50}$$

This yields that

$$x^n \xrightarrow{s} x^0 \quad \text{in } PC(J, X) \text{ as } n \longrightarrow \infty. \tag{51}$$

Note that $H(5)$ implies that all of the assumptions of Balder (see Theorem 2.1, in [11]) are satisfied. Hence, from Balder's theorem, we can conclude that $(x, u) \rightarrow \int_0^b \mathscr{L}(t, x(t), u(t))dt$ is sequentially lower semicontinuous in the strong topology of $L^1(J, X)$. Since $L^p(J, Y) \subset L^1(J, Y)$, $\mathscr{J}$ is weakly lower semicontinuous on $L^p(J, Y)$, and since, by $H(5)$(iv), $\mathscr{J} > -\infty$, $\mathscr{J}$ attains its infimum at $u_0 \in U_{\text{ad}}$; that is,

$$\begin{aligned} \rho &= \lim_{n\to\infty} \int_0^b \mathscr{L}(t, x^n(t), u^m(t)) dt \\ &\geq \int_0^b \mathscr{L}\left(t, x^0(t), u^0(t)\right) dt = J\left(x^0, u^0\right) \geq \rho. \end{aligned} \tag{52}$$

The proof is completed. □

5. An Example

We can consider the following initial-boundary value problem of fractional impulsive parabolic system:

$$\begin{aligned} \frac{\partial^{3/2}}{\partial t^{3/2}} x(t, y) &= \frac{\partial^2}{\partial y^2} x(t, y) + e^{-t} \\ &\quad + \frac{1}{10} \sin x(t, y) + \int_0^1 q(y, \tau) u(\tau, t) d\tau, \\ &\qquad t \in J' = [0, 1] \setminus \left\{\frac{1}{2}\right\}, \ y \in [0, \pi], \\ \Delta x\left(\frac{1}{2}, y\right) &= \frac{|x(y)|}{6 + |x(y)|}, \\ \Delta x'\left(\frac{1}{2}, y\right) &= \frac{|x(y)|}{20 + |x(y)|}, \quad y \in [0, \pi], \\ x(t, 0) &= x(t, \pi) = 0, \quad t \in J = [0, 1], \\ x(0, y) &= x_0(y), \qquad x'(0, y) = x_1(y), \quad y \in [0, \pi]. \end{aligned} \tag{53}$$

Take $X = Y = L^2[0, \pi]$ and the operator $A : D(A) \subset X \rightarrow X$ is defined by

$$A\omega = \omega'', \tag{54}$$

where the domain $D(A)$ is given by

$$\begin{aligned} &\{\omega \in X : \omega, \omega' \text{ are absolutely continuous,} \\ &\quad \omega'' \in X, \ \omega(0) = \omega(\pi) = 0\}. \end{aligned} \tag{55}$$

Then A can be written as

$$A\omega = \sum_{n=1}^{\infty} n^2 (\omega, \omega_n) \omega_n, \quad \omega \in D(A), \tag{56}$$

where $\omega_n(x) = \sqrt{2/\pi} \sin nx (n = 1, 2, \ldots)$ is an orthonormal basis of X. It is well known that A is the infinitesimal generator of a compact semigroup $T(t)(t > 0)$ in X given by

$$\begin{aligned} T(t)x &= \sum_{n=1}^{\infty} \exp^{-n^2 t} (x, x_n) x_n, \quad x \in X, \\ \|T(t)\| &\leq e^{-1} < 1. \end{aligned} \tag{57}$$

From Theorems 3.3 and 3.4 of [5], we can easily get $M = 3$,

$$\begin{aligned} f(t, x(t, y)) &= e^{-t} + \frac{1}{10} \sin x(t, y), \\ I_k(x(t, y)) &= \frac{|x(y)|}{6 + |x(y)|}, \qquad I_k^*\left(x\left(t_k^-\right)\right) = \frac{|x(y)|}{20 + |x(y)|}, \\ Bu(y) &= \int_0^1 q(y, \tau) u(\tau, t) d\tau. \end{aligned} \tag{58}$$

Denote $x(t, y) = x(t)(y)$; then it is easy to see that

$$\begin{aligned} \|f(t, x(t))\| &\leq e^{-t} + \frac{1}{10} \|x(t)\|, \\ \|I_k(x(t))\| &\leq \frac{\|x(t)\|}{6}, \qquad \left\|I_k^*\left(x\left(t_k^-\right)\right)\right\| \leq \frac{\|x(t)\|}{20}, \\ \|f(t, x(t)) - f(t, y(t))\| &\leq \frac{1}{10} \|x - y\|, \\ \|I_k(x(t)) - I_k(y(t))\| &\leq \frac{1}{6} \|x - y\|, \\ \|I_k^*(x(t)) - I_k^*(y(t))\| &\leq \frac{1}{20} \|x - y\|. \end{aligned} \tag{59}$$

Moreover,

$$\left(\sum_{i=1}^{n} h_k + \theta b\right) M = \left(\frac{1}{6} + \frac{1}{20} + \frac{1}{10} \times 1\right) \times 3 = \frac{19}{20} < 1. \tag{60}$$

Hence, all the conditions of Theorem 13 are satisfied, system (53) has a unique mild solution.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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