

Research Article

Asymptotic Stability of Caputo Type Fractional Neutral Dynamical Systems with Multiple Discrete Delays

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We discuss the delay-independent asymptotic stability of Caputo type fractional-order neutral differential systems with multiple discrete delays. Based on the algebraic approach and matrix theory, the sufficient conditions are derived to ensure the asymptotic stability for all time-delay parameters. By applying the stability criteria, one can avoid solving the roots of transcendental equations. The results obtained are computationally flexible and convenient. Moreover, an example is provided to illustrate the effectiveness and applicability of the proposed theoretical results.

1. Introduction

Fractional calculus is regarded as a generalization of the classical integer-order calculus to arbitrary order. Since the fractional-order derivative has nonlocal property and weakly singular kernels, it provides an excellent tool for the description of memory and hereditary properties of dynamical processes. Recently, it has gained increasing interests from researchers in various areas and has become one of the central subjects [1–12]. For more details on fractional calculus theory, one can see the monographs of Miller and Ross [1], Podlubny [2], Kilbas et al. [3], and Diethelm [4]. Lakshmikantham et al. [5] and Baleanu et al. [6] have elaborated the theory of fractional-order dynamics systems.

Stability is an important performance metric for dynamic systems. Meanwhile, time delay has an important effect on the stability and performance of dynamic systems. In the past few decades, numerous results on the stability problem of integer-order delay differential systems have been obtained (see [13–20] and references therein). Recently, there are some results on the stability of fractional-order differential systems [21–33]. For example, Matignon [23, 24]

discussed the asymptotic stability of linear fractional-order autonomous systems. In terms of comparison principle [25, 26] and Lyapunov direct method [27], Li et al. [25, 27] obtained the Mittag-Leffler stability theorems of fractional-order systems. Linear matrix inequality (LMI) method [28] and variational Lyapunov method [29] were also used to investigate the stability of fractional-order systems. Wang et al. [30] investigated Hyers-Ulam-Rassias stability of a certain fractional differential equation by means of the fixed point theorem. Moreover, Rivero et al. [31], Li and Zhang [32], and Choi et al. [33] summarized and reviewed the developments and advances in stability of fractional-order dynamical systems, respectively.

It is worth pointing out that the notable contributions have been made to the stability of fractional-order delay differential systems [34–44]. Many methods have been applied to discuss various types of stability problems for fractional-order delay dynamical systems, such as Gronwall integral inequality method [34–36], final-value theorem of Laplace transform [37], Lyapunov functional method [38, 39], analytical and numerical methods [40], fixed point theorems [41–43], and semigroup theory [42].

In this paper, motivated by the aforementioned works, we are devoted to discussing the delay-independent asymptotic stability for linear Caputo fractional neutral differential difference system with multiple discrete delays as follows:

$${}^C D^\alpha \left[x(t) - \sum_{i=1}^m C_i x(t - \tau_i) \right] = Ax(t) + \sum_{i=1}^m B_i x(t - \tau_i),$$

$$t \geq 0,$$

$$x(t) = \varphi(t), \quad t \in [-\tau, 0], \tag{1}$$

where ${}^C D^\alpha x(t)$ denotes an α order Caputo fractional derivative of $x(t)$, $0 < \alpha < 1$, A, B_i, C_i are $n \times n$ constant matrices, τ_i ($i = 1, 2, \dots, m$) are real constants with $0 < \tau_1 \leq \tau_2 \leq \dots \leq \tau_m = \tau$, the initial function $\varphi \in C^1([-\tau, 0], \mathbb{R}^n)$, and $C^1([-\tau, 0], \mathbb{R}^n)$ denotes space of continuously differentiable functions mapping the interval $[-\tau, 0]$ into \mathbb{R}^n .

Compared to integer-order differential systems, the research on the stability of fractional dynamical systems is still at the stage of exploiting and developing. Different from the methods in [34–43], we apply the algebraic approach and matrix theory to establish the delay-independent asymptotic stability criteria for system (1), which do not contain information on delays. We establish the sufficient conditions which ensure that all the roots of characteristic equation lie in open left-half complex plane and are uniformly bounded away from the imaginary axis. At the same time, by applying these stability criteria, one can avoid solving the roots of the transcendental equations. The results obtained are computationally flexible and efficient.

The rest of this paper is organized as follows. In Section 2, we present some definitions, notations, and lemmas related to the main results. In Section 3, the sufficient conditions of the delay-independent asymptotic stability for system (1) are derived based on the algebraic approach and matrix theory. In Section 4, an example is provided to illustrate the effectiveness and applicability of the proposed criteria. Finally, some concluding remarks are drawn in Section 5.

2. Preliminaries

In this section, we present some definitions of fractional calculus (see [1–4]), notations, and lemmas used in the paper.

For the sake of convenience, some notations are introduced firstly. Throughout this paper, $\det(A)$ represents the determinant of matrix A , $\sigma(A)$ denotes the spectrum of matrix A , $\rho[A]$ represents the spectral radius of matrix A , and $\arg(\sigma(A))$ stands for the principal argument of $\sigma(A)$ defined on $(-\pi, \pi]$.

- (a) Riemann-Liouville’s fractional integral of order $q > 0$ for a function $f : \mathbb{R}^+ \rightarrow \mathbb{R}^n$ is given by

$$D^{-q} f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) ds, \quad t > 0, \tag{2}$$

where $\Gamma(\cdot)$ is Euler’s gamma function.

- (b) Riemann-Liouville’s fractional derivative of order q for a function $f : \mathbb{R}^+ \rightarrow \mathbb{R}^n$ is given by

$${}^{RL} D^q f(t) = \frac{1}{\Gamma(m-q)} \frac{d^m}{dt^m} \int_0^t (t-s)^{m-q-1} f(s) ds, \tag{3}$$

where $0 \leq m-1 \leq q < m$, $m \in \mathbb{Z}^+$.

- (c) Caputo’s fractional derivative of order q for a function $f : \mathbb{R}^+ \rightarrow \mathbb{R}^n$ is defined as

$${}^C D^q f(t) = \frac{1}{\Gamma(m-q)} \int_0^t (t-s)^{m-q-1} f^{(m)}(s) ds, \tag{4}$$

where $0 \leq m-1 \leq q < m$, $m \in \mathbb{Z}^+$. Here, ${}^C D^q$ is still written as D^q .

- (d) The Laplace transform of a function $f(t)$ is defined as

$$F(s) = \mathcal{L}[f(t)] = \int_0^{+\infty} e^{-st} f(t) dt, \quad s \in \mathbb{C}, \tag{5}$$

where \mathbb{C} denotes the complex plane and $f(t)$ is n -dimensional vector-valued function. For $m-1 \leq q < m$, it follows from [1–4] that

$$\mathcal{L}[D^q f(t)] = s^q \mathcal{L}[f(t)] - \sum_{k=0}^{m-1} s^{q-k-1} f^{(k)}(0). \tag{6}$$

The following Mittag-Leffler function plays an important role in the study on fractional-order differential systems, which is considered as a natural generalization of the exponential function.

- (e) The Mittag-Leffler function in two parameters is defined as

$$E_{q,\beta}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(kq + \beta)}, \quad q > 0, \beta > 0, z \in \mathbb{C}. \tag{7}$$

In particular, for $\beta = 1$, the Mittag-Leffler function in one parameter is given by

$$E_q(z) := E_{q,1}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(1+kq)}, \quad q > 0, z \in \mathbb{C}. \tag{8}$$

Applying the method of steps [34], we obtain the following lemma which generalizes well-known results of integer-order delay differential systems [13] to fractional-order neutral differential systems.

Lemma 1. *For system (1), there exists a unique continuous solution on $[0, +\infty)$.*

Proof. For system (1), on the interval $[-\tau, 0]$, $x(t) = \varphi(t)$. Thus, when $t \in [0, \tau_1]$, system (1) is given by

$$D^\alpha x(t) = Ax(t) + \sum_{i=1}^m B_i \varphi(t - \tau_i) + D^\alpha \left[\sum_{i=1}^m C_i \varphi(t - \tau_i) \right]. \tag{9}$$

Since $\sum_{i=1}^m B_i \varphi(t - \tau_i) + D^\alpha [\sum_{i=1}^m C_i \varphi(t - \tau_i)]$ is continuous on $[0, \tau_1]$, from [3], we know that there is a unique continuous solution for system (1) on $[-\tau, \tau_1]$, which is denoted as $x_1(t)$, $t \in [-\tau, \tau_1]$. Furthermore, $x_1(t)$ can be expressed by the following form:

$$x_1(t) = \begin{cases} \varphi(t), & t \in [-\tau, 0], \\ E_\alpha (At^\alpha) \varphi(0) + \int_0^t (t-\theta)^{\alpha-1} E_{\alpha,\alpha} [A(t-\theta)^\alpha] \times \left\{ \sum_{i=1}^m B_i \varphi(\theta - \tau_i) + D^\alpha \left[\sum_{i=1}^m C_i \varphi(\theta - \tau_i) \right] \right\} d\theta, & t \in [0, \tau_1]. \end{cases} \tag{10}$$

For $t \in [\tau_1, 2\tau_1]$, system (1) is given by

$$D^\alpha x(t) = Ax(t) + \sum_{i=1}^m B_i x_1(t - \tau_i) + D^\alpha \left[\sum_{i=1}^m C_i x_1(t - \tau_i) \right]. \tag{11}$$

Similarly, since $\sum_{i=1}^m B_i x_1(t - \tau_i) + D^\alpha [\sum_{i=1}^m C_i x_1(t - \tau_i)]$ is continuous on $[-\tau, 2\tau_1]$, we obtain that $x_2(t)$ is a unique continuous solution of system (1) on $[-\tau, 2\tau_1]$ and

$$x_2(t) = \begin{cases} x_1(t), & t \in [-\tau, \tau_1], \\ E_\alpha (At^\alpha) \varphi(0) + \int_0^t (t-\theta)^{\alpha-1} E_{\alpha,\alpha} [A(t-\theta)^\alpha] \times \left\{ \sum_{i=1}^m B_i x_1(\theta - \tau_i) + D^\alpha \left[\sum_{i=1}^m C_i x_1(\theta - \tau_i) \right] \right\} d\theta, & t \in [\tau_1, 2\tau_1]. \end{cases} \tag{12}$$

Assume that system (1) has a unique solution $x_k(t)$ on $[(k-1)\tau_1, k\tau_1]$. For $t \in [k\tau_1, (k+1)\tau_1]$, system (1) is given by

$$D^\alpha x(t) = Ax(t) + \sum_{i=1}^m B_i x_k(t - \tau_i) + D^\alpha \left[\sum_{i=1}^m C_i x_k(t - \tau_i) \right]. \tag{13}$$

Similarly, since $\sum_{i=1}^m B_i x_k(t - \tau_i) + D^\alpha [\sum_{i=1}^m C_i x_k(t - \tau_i)]$ is continuous on $[k\tau_1, (k+1)\tau_1]$, we obtain that $x_{k+1}(t)$ is a unique continuous solution of system (1) on $[k\tau_1, (k+1)\tau_1]$ and

$$x_{k+1}(t) = \begin{cases} x_k(t), & t \in [-\tau, k\tau_1], \\ E_\alpha (At^\alpha) \varphi(0) + \int_0^t (t-\theta)^{\alpha-1} E_{\alpha,\alpha} [A(t-\theta)^\alpha] \times \left\{ \sum_{i=1}^m B_i x_k(\theta - \tau_i) + D^\alpha \left[\sum_{i=1}^m C_i x_k(\theta - \tau_i) \right] \right\} d\theta, & t \in [k\tau_1, (k+1)\tau_1]. \end{cases} \tag{14}$$

According to the mathematical induction, we know that system (1) has a unique continuous solution on $[0, k\tau_1]$, $k = 1, 2, \dots$

Now, for any $T > 0$, we assert that system (1) has a unique continuous solution on $[0, T]$. In fact, three cases are discussed as follows.

Case 1. When $(k+1)\tau_1 = T$, we know that the assertion is true.

Case 2. When $0 < T - (k+1)\tau_1 < \tau_1$, we only need to prove that system (1) has a unique continuous solution on $[\tau_1, T]$. For $t \in [\tau_1, T]$, we denote $t_1 = t - \tau_1 \in [0, T - \tau_1] \subset [0, (k+1)\tau_1]$, and we can use the similar proof to obtain the conclusion.

Case 3. When $T - (k+1)\tau_1 > \tau_1$, we can repeat the above process until the condition of Case 2 is satisfied.

Note that T is an arbitrary positive real number; then, we know that system (1) has a unique continuous solution on $[0, +\infty)$. Therefore, the proof is completed. \square

Remark 2. Lemma 1 ensures the existence and uniqueness of solution for system (1) on $[0, +\infty)$. Evidently, Caputo's fractional derivative of a constant is equal to zero; then, $x(t) \equiv 0$ is the zero solution of system (1).

Definition 3. The zero solution $x(t) \equiv 0$ of system (1) is called delay-independent globally asymptotically stable if, for any initial function $\varphi(\cdot) \in \mathbf{C}^1([-\tau, 0], \mathbb{R}^n)$, its analytic solution $x(t)$ satisfies $\lim_{t \rightarrow +\infty} x(t) = 0$ for all the time delays $0 < \tau_1 \leq \tau_2 \leq \dots \leq \tau_m$.

Next, we discuss the characteristic equation and delay-independent globally asymptotic stability of system (1).

From [1-4], the Laplace transform of Caputo fractional-order derivative $D^\alpha f(t)$ is given as follows:

$$\mathcal{E} [D^\alpha f(t)] = s^\alpha \mathcal{E} [f(t)] - s^{\alpha-1} f(0), \quad 0 < \alpha < 1. \tag{15}$$

Applying the Laplace transform on both sides of system (1) yields

$$s^\alpha \mathcal{E} \left[x(t) - \sum_{i=1}^m C_i x(t - \tau_i) \right] - s^{\alpha-1} \varphi(0) + s^{\alpha-1} \sum_{i=1}^m C_i \varphi(-\tau_i) = A \mathcal{E} [x(t)] + \sum_{i=1}^m B_i \mathcal{E} [x(t - \tau_i)]. \tag{16}$$

Note that

$$\mathcal{E} [x(t - \tau_i)] = e^{-s\tau_i} \mathcal{E} [x(t)] + e^{-s\tau_i} \int_{-\tau_i}^0 e^{-st} \varphi(t) dt, \tag{17}$$

$i = 1, 2, \dots, m;$

thus, we obtain

$$\begin{aligned} \Delta(s, \tau_i) \mathcal{E}[x(t)] &= s^{\alpha-1} \varphi(0) - s^{\alpha-1} \sum_{i=1}^m C_i \varphi(-\tau_i) \\ &\quad + \sum_{i=1}^m B_i e^{-s\tau_i} \int_{-\tau_i}^0 e^{-st} \varphi(t) dt \\ &\quad + s^\alpha \sum_{i=1}^m C_i e^{-s\tau_i} \int_{-\tau_i}^0 e^{-st} \varphi(t) dt, \end{aligned} \tag{18}$$

where

$$\Delta(s, \tau_i) = \left[s^\alpha I - A - \sum_{i=1}^m B_i e^{-s\tau_i} - s^\alpha \sum_{i=1}^m C_i e^{-s\tau_i} \right] \tag{19}$$

is the characteristic matrix of system (1). Multiplying s on both sides of (18) yields

$$\begin{aligned} \Delta(s, \tau_i) \{s\mathcal{E}[x(t)]\} &= s \left\{ s^{\alpha-1} \varphi(0) - s^{\alpha-1} \sum_{i=1}^m C_i \varphi(-\tau_i) \right. \\ &\quad + \sum_{i=1}^m B_i e^{-s\tau_i} \int_{-\tau_i}^0 e^{-st} \varphi(t) dt \\ &\quad \left. + s^\alpha \sum_{i=1}^m C_i e^{-s\tau_i} \int_{-\tau_i}^0 e^{-st} \varphi(t) dt \right\}. \end{aligned} \tag{20}$$

By means of the final-value theorem of Laplace transform [45] and Definition 3, if all the roots of characteristic equation $\det[\Delta(s, \tau_i)] = 0$ lie in open left-half complex plane and are uniformly bounded away from the imaginary axis, then the zero solution of system (1) is delay-independent globally asymptotically stable.

Therefore, we immediately have the following conclusion.

Lemma 4. *If all the roots of characteristic equation*

$$\det \left[s^\alpha I - A - \sum_{i=1}^m B_i e^{-s\tau_i} - s^\alpha \sum_{i=1}^m C_i e^{-s\tau_i} \right] = 0 \tag{21}$$

lie in open left-half complex plane and are uniformly bounded away from the imaginary axis, then the zero solution of system (1) is delay-independent globally asymptotically stable.

Remark 5. As we know, when $\alpha = 1$ and $\tau_1 = \tau_2 = \dots = \tau_m = 0$, the characteristic equation

$$\det \left[\lambda I - A - \sum_{i=1}^m (B_i + \lambda C_i) \right] = 0 \tag{22}$$

is an algebraic equation of λ , and (22) only has n roots distributed in the complex plane. However, the characteristic equation $\det[\Delta(s, \tau_i)] = 0$ has countably infinite roots with $\alpha = 1$ and some $\tau_i > 0$ (see [13]). For $0 < \alpha < 1$ and $\tau_i > 0$ ($i = 1, 2, \dots, m$), it is very difficult to solve the roots of the transcendental equation (21) in practice. Based on these considerations, we are devoted to establishing the algebraic stability criteria of system (1) in the next section.

3. Stability Criteria for System (1)

In this section, we derive the sufficient conditions of delay-independent globally asymptotic stability for system (1). Applying the algebraic method, we investigate the distribution of roots for equation $\det[\Delta(s, \tau_i)] = 0$ in any neighborhood of the infinity and find a positive number $\delta > 0$ such that any characteristic root s satisfies $\Re e(s) < -\delta < 0$, where $\Re e(s)$ represents the real part of the complex number s .

Theorem 6. *The zero solution $x(t) \equiv 0$ of system (1) is delay-independent globally asymptotically stable if the following conditions are satisfied:*

$$\begin{aligned} (H_1) \quad & |\arg(\sigma(A))| > \frac{\alpha\pi}{2}, \\ (H_2) \quad & \rho \left[\sum_{i=1}^m C_i \right] < 1, \\ (H_3) \quad & \sup_{\Re e(s) \geq 0} \rho \left[(s^\alpha I - A)^{-1} \sum_{i=1}^m (B_i + s^\alpha C_i) \right] < 1. \end{aligned} \tag{23}$$

Proof. Note that $|\arg(\sigma(A))| > \alpha\pi/2$; then, all the roots of equation $\det(\lambda I - A) = 0$ satisfy $|\arg(\lambda)| > \alpha\pi/2$. Let $s^\alpha = \lambda$; then, all the roots of equation $\det(s^\alpha I - A) = 0$ satisfy $|\arg(s)| > \pi/2$; that is, $\Re e(s) < 0$. Therefore, matrix A is invertible and $(s^\alpha I - A)^{-1}$ is well defined when $|\arg(\sigma(A))| > \alpha\pi/2$ and $\Re e(s) \geq 0$.

For $\Re e(s) \geq 0$, it follows from the characteristic polynomial of system (1) that

$$\begin{aligned} \det \Delta(s, \tau_i) &= \det \left[s^\alpha I - A - \sum_{i=1}^m B_i e^{-s\tau_i} - s^\alpha \sum_{i=1}^m C_i e^{-s\tau_i} \right] \\ &= \det [s^\alpha I - A] \det \left[I - (s^\alpha I - A)^{-1} \right. \\ &\quad \left. \times \sum_{i=1}^m (B_i + s^\alpha C_i) e^{-s\tau_i} \right] \\ &= \det [s^\alpha I - A] \\ &\quad \times \prod_{j=1}^n \left(1 - \lambda_j \left[(s^\alpha I - A)^{-1} \sum_{i=1}^m (B_i + s^\alpha C_i) e^{-s\tau_i} \right] \right). \end{aligned} \tag{24}$$

Then, we have

$$\begin{aligned} & \left| \lambda_j \left[(s^\alpha I - A)^{-1} \sum_{i=1}^m (B_i + s^\alpha C_i) e^{-s\tau_i} \right] \right| \\ & \leq \rho \left[(s^\alpha I - A)^{-1} \sum_{i=1}^m (B_i + s^\alpha C_i) \right] |e^{-s\tau}| < 1, \end{aligned} \tag{25}$$

$j = 1, 2, \dots, n.$

Combining (24) and (25), we have $\det \Delta(s, \tau_i) \neq 0$ for $\Re e(s) \geq 0$ and $\tau_i > 0, i = 1, 2, \dots, m$; that is, if

conditions (H_1) and (H_3) are satisfied, then the characteristic equation (21) implies that $\Re e(s) < 0$.

Suppose that there exists a sequence of roots $\{s_n\}$ of the characteristic equation (21) whose real parts are not uniformly bounded away from zero; that is, $\Re e(s_n) < 0$ and $\Re e(s_n) \rightarrow 0$ as $n \rightarrow +\infty$. Note that any eigenvalue $\lambda_j[(s^\alpha I - A)^{-1} \sum_{i=1}^m (B_i + s^\alpha C_i)]$ is a continuous function of s for $\Re e(s) \geq 0$; then, it follows from (H_2) that

$$\rho \left[(s^\alpha I - A)^{-1} \sum_{i=1}^m (B_i + s^\alpha C_i) \right] \rightarrow \rho \left[\sum_{i=1}^m C_i \right] < 1, \tag{26}$$

$$s \rightarrow +\infty.$$

Hence, $|\lambda_j[(s^\alpha I - A)^{-1} \sum_{i=1}^m (B_i + s^\alpha C_i)]|$ reach the maximum value for $\Re e(s) \geq 0$. From condition (H_3) , there exists a positive constant ε such that

$$\begin{aligned} & \sup_{\Re e(s) \geq 0} \rho \left[(s^\alpha I - A)^{-1} \sum_{i=1}^m (B_i + s^\alpha C_i) \right] \\ &= \sup_{\Re e(s) \geq 0} \max_{1 \leq j \leq n} \left| \lambda_j \left[(s^\alpha I - A)^{-1} \sum_{i=1}^m (B_i + s^\alpha C_i) \right] \right| = 1 - \varepsilon. \end{aligned} \tag{27}$$

Then, equality (27) implies that

$$\sup_{\Re e(\zeta) = 0} \rho \left[(\zeta^\alpha I - A)^{-1} \sum_{i=1}^m (B_i + \zeta^\alpha C_i) \right] \leq 1 - \varepsilon. \tag{28}$$

When the positive integer n is large enough, there exist a positive constant ε^* ($0 < \varepsilon^* < \varepsilon$) and a characteristic root s_n such that $|\Re e(s_n)|$ is sufficiently small, $\Re e(s_n) < 0$ and

$$\begin{aligned} & \left| \max_{1 \leq j \leq n} \lambda_j \left[(s_n^\alpha I - A)^{-1} \sum_{i=1}^m (B_i + s_n^\alpha C_i) \right] \right. \\ & \left. - \sup_{\Re e(\zeta) = 0} \rho \left[(\zeta^\alpha I - A)^{-1} \sum_{i=1}^m (B_i + \zeta^\alpha C_i) \right] \right| < \varepsilon^*. \end{aligned} \tag{29}$$

For $\Re e(\zeta) = 0$, from (28) and (29), we have

$$\begin{aligned} & \left| \lambda_j \left[(s_n^\alpha I - A)^{-1} \sum_{i=1}^m (B_i + s_n^\alpha C_i) \right] \right| \\ & \leq \sup_{\Re e(\zeta) = 0} \rho \left[(\zeta^\alpha I - A)^{-1} \sum_{i=1}^m (B_i + \zeta^\alpha C_i) \right] + \varepsilon^* \tag{30} \\ & \leq 1 - \varepsilon + \varepsilon^* < 1, \quad j = 1, 2, \dots, n. \end{aligned}$$

Choosing n large enough yields

$$\left| \lambda_j \left[(s_n^\alpha I - A)^{-1} \sum_{i=1}^m (B_i + s_n^\alpha C_i) e^{-s_n \tau_i} \right] \right| < 1, \quad j = 1, 2, \dots, n. \tag{31}$$

Therefore, for $\Re e(s_n) < 0$ and $\Re e(s_n) \rightarrow 0$ as $n \rightarrow +\infty$, one can obtain

$$\begin{aligned} \det \Delta(s_n, \tau_i) = \det \left[s_n^\alpha I - A - \sum_{i=1}^m B_i e^{-s_n \tau_i} \right. \\ \left. - s_n^\alpha \sum_{i=1}^m C_i e^{-s_n \tau_i} \right] \neq 0, \end{aligned} \tag{32}$$

which contradicts the assumption that $\{s_n\}$ is a sequence of roots of the characteristic equation (21). In view of Lemma 4, the proof is completed. \square

Theorem 7. *The zero solution $x(t) \equiv 0$ of system (1) is delay-independent globally asymptotically stable if the following conditions are satisfied:*

$$(H_1) \quad |\arg(\sigma(A))| > \frac{\alpha\pi}{2}, \quad 0 < \alpha < 1,$$

$$(H_4) \quad \rho \left[\sum_{i=1}^m \xi_i C_i \right] < 1, \quad \xi_i \in \mathbb{C}, \quad |\xi_i| \leq 1, \quad i = 1, 2, \dots, m,$$

$$(H_5) \quad \rho \left[(s^\alpha I - A)^{-1} \sum_{i=1}^m (B_i + s^\alpha C_i) \xi_i \right] < 1,$$

$$\Re e(s) = 0, \quad \xi_i \in \mathbb{C}, \quad |\xi_i| = 1, \quad i = 1, 2, \dots, m. \tag{33}$$

Proof. According to condition (H_1) and the proof of Theorem 6, we know that matrix A is invertible and $(s^\alpha I - A)^{-1}$ is well defined when $|\arg(\sigma(A))| > \alpha\pi/2$ and $\Re e(s) = 0$.

On the one hand, any eigenvalue $\lambda[(s^\alpha I - A)^{-1} (\sum_{i=1}^m \xi_i B_i + s^\alpha \sum_{i=1}^m \xi_i C_i)]$ is an algebraic function of $(\xi_1, \xi_2, \dots, \xi_m)$ with $|\xi_i| < 1$ and is continuous with $|\xi_i| \leq 1, i = 1, 2, \dots, m$. An application of the maximum modulus principle yields that

$$\begin{aligned} & \rho \left[(s^\alpha I - A)^{-1} \left(\sum_{i=1}^m \xi_i B_i + s^\alpha \sum_{i=1}^m \xi_i C_i \right) \right] < 1, \tag{34} \\ & \Re e(s) = 0, \quad \xi_i \in \mathbb{C}, \quad |\xi_i| = 1, \end{aligned}$$

is equivalent to

$$\begin{aligned} & \rho \left[(s^\alpha I - A)^{-1} \left(\sum_{i=1}^m \xi_i B_i + s^\alpha \sum_{i=1}^m \xi_i C_i \right) \right] < 1, \tag{35} \\ & \Re e(s) = 0, \quad \xi_i \in \mathbb{C}, \quad |\xi_i| \leq 1. \end{aligned}$$

In view of the expression of $(s^\alpha I - A)^{-1} (\sum_{i=1}^m \xi_i B_i + s^\alpha \sum_{i=1}^m \xi_i C_i)$, as $\Re e(s) = 0$ and $s \rightarrow \infty$, one can obtain

$$\begin{aligned} & \rho \left[(s^\alpha I - A)^{-1} \left(\sum_{i=1}^m \xi_i B_i + s^\alpha \sum_{i=1}^m \xi_i C_i \right) \right] \rightarrow \rho \left[\sum_{i=1}^m \xi_i C_i \right] < 1, \\ & \forall \xi_i \in \mathbb{C}, \quad |\xi_i| \leq 1. \end{aligned} \tag{36}$$

It follows from (36) that

$$\sup_{\Re(s)=0} \rho \left[(s^\alpha I - A)^{-1} \left(\sum_{i=1}^m \xi_i B_i + s^\alpha \sum_{i=1}^m \xi_i C_i \right) \right] \leq 1, \tag{37}$$

$$\forall \xi_i \in \mathbb{C}, \quad |\xi_i| \leq 1.$$

Applying the maximum modulus principle on the unbounded region $\{s|s \in \mathbb{C}, \Re(s) \geq 0\}$, then (35) implies that

$$\rho \left[(s^\alpha I - A)^{-1} \left(\sum_{i=1}^m \xi_i B_i + s^\alpha \sum_{i=1}^m \xi_i C_i \right) \right] < 1, \tag{38}$$

$$\Re(s) \geq 0, \quad \xi_i \in \mathbb{C}, \quad |\xi_i| \leq 1.$$

On the other hand, suppose that there exists some $s_0 \in \{s|s \in \mathbb{C}, \Re(s) \geq 0\}$ such that

$$\det \left[s_0^\alpha I - A - \sum_{i=1}^m B_i e^{-s_0 \tau_i} - s_0^\alpha \sum_{i=1}^m C_i e^{-s_0 \tau_i} \right] = 0, \tag{39}$$

$$s_0 \in \{s|s \in \mathbb{C}, \Re(s) \geq 0\}.$$

Choose $\xi_i^0 = e^{-s_0 \tau_i}$ such that $|\xi_i^0| \leq 1, i = 1, 2, \dots, m$. It follows from (39) that

$$1 \in \sigma \left[(s_0^\alpha I - A)^{-1} \left(\sum_{i=1}^m B_i e^{-s_0 \tau_i} + s_0^\alpha \sum_{i=1}^m C_i e^{-s_0 \tau_i} \right) \right], \tag{40}$$

$$s_0 \in \{s|s \in \mathbb{C}, \Re(s) \geq 0\},$$

which contradicts inequality (38). Therefore, if conditions (H_1) , (H_4) , and (H_5) are satisfied, then the characteristic equation (21) implies that $\Re(s) < 0$.

The rest of the proof is similar to that of Theorem 6; then, the conclusion holds. \square

According to the proof of Theorem 7, we have the result as follows.

Corollary 8. *The zero solution $x(t) \equiv 0$ of system (1) is delay-independent globally asymptotically stable if the following conditions are satisfied:*

$$(H_1) \quad |\arg(\sigma(A))| > \frac{\alpha\pi}{2}, \quad 0 < \alpha < 1,$$

$$(H_4) \quad \rho \left[\sum_{i=1}^m \xi_i C_i \right] < 1, \quad \xi_i \in \mathbb{C}, \quad |\xi_i| \leq 1, \quad i = 1, 2, \dots, m,$$

$$(H_6) \quad \rho \left[(s^\alpha I - A)^{-1} \sum_{i=1}^m (B_i + s^\alpha C_i) \xi_i \right] < 1,$$

$$\Re(s) \geq 0, \quad \xi_i \in \mathbb{C}, \quad |\xi_i| \leq 1, \quad i = 1, 2, \dots, m. \tag{41}$$

Assume that $|\arg(\sigma(A))| > \alpha\pi/2, 0 < \alpha < 1$. Define the following matrices:

$$L = (I - A)^{-1} \sum_{i=1}^m (B_i + C_i),$$

$$M = (I - A)^{-1} \sum_{i=1}^m (B_i - C_i), \tag{42}$$

$$N = (I - A)^{-1} (I + A).$$

Let

$$s^\alpha = \frac{1 - z}{1 + z}, \quad \Re(s) \geq 0, \quad 0 < \alpha < 1; \tag{43}$$

then, we have $|z| \leq 1$.

Theorem 9. *The zero solution $x(t) \equiv 0$ of system (1) is delay-independent globally asymptotically stable if the following conditions are satisfied:*

$$(H_1) \quad |\arg(\sigma(A))| > \frac{\alpha\pi}{2}, \quad 0 < \alpha < 1,$$

$$(H_2) \quad \rho \left[\sum_{i=1}^m C_i \right] < 1, \tag{44}$$

$$(H_7) \quad \sup_{|z| \leq 1} \rho \left[(I - zN)^{-1} (L + zM) \right] < 1.$$

Proof. From Theorem 6, we only need to prove that the following equality holds:

$$(s^\alpha I - A)^{-1} \sum_{i=1}^m (B_i + s^\alpha C_i) = (I - zN)^{-1} (L + zM), \tag{45}$$

$$\Re(s) \geq 0, \quad |z| \leq 1.$$

In fact, it is easy to obtain

$$I - zN = \left[I - \frac{1 - s^\alpha}{1 + s^\alpha} (I - A)^{-1} (I + A) \right]$$

$$= \left[(1 + s^\alpha) I - (1 - s^\alpha) (I - A)^{-1} (I + A) \right] (1 + s^\alpha)^{-1}$$

$$= (I - A)^{-1} \left[(I - A) (1 + s^\alpha) - (1 - s^\alpha) (I + A) \right]$$

$$\times (1 + s^\alpha)^{-1}$$

$$= 2(I - A)^{-1} (s^\alpha I - A) (1 + s^\alpha)^{-1}. \tag{46}$$

For $\Re e(s) \geq 0$, it follows from (H_1) and (43) that $|z| \leq 1$; then, matrix $(I - zN)$ is invertible. Hence,

$$\begin{aligned} L + zM &= \left[(I - A)^{-1} \sum_{i=1}^m (B_i + C_i) \right. \\ &\quad \left. + \frac{1 - s^\alpha}{1 + s^\alpha} (I - A)^{-1} \sum_{i=1}^m (B_i - C_i) \right] \\ &= (I - A)^{-1} \left[\sum_{i=1}^m (B_i + C_i) + \frac{1 - s^\alpha}{1 + s^\alpha} \sum_{i=1}^m (B_i - C_i) \right] \\ &= (I - A)^{-1} \left[\sum_{i=1}^m (B_i + C_i) (1 + s^\alpha) \right. \\ &\quad \left. + (1 - s^\alpha) \sum_{i=1}^m (B_i - C_i) \right] (1 + s^\alpha)^{-1} \\ &= 2(I - A)^{-1} \sum_{i=1}^m (B_i + s^\alpha C_i) (1 + s^\alpha)^{-1}. \end{aligned} \tag{47}$$

Combining (46) and (47) yields that (45) holds. Therefore, we complete the proof. \square

Next, the asymptotic stability criteria for two special cases of system (1) are presented.

(i) When $C_i = 0, i = 1, 2, \dots, m$, system (1) reduces to Caputo fractional-order linear retarded type differential difference systems with multiple delays as follows:

$$\begin{aligned} D^\alpha x(t) &= Ax(t) + \sum_{i=1}^m B_i x(t - \tau_i), \quad t \geq 0, \\ x(t) &= \varphi(t), \quad t \in [-\tau, 0], \end{aligned} \tag{48}$$

where $D^\alpha x(t)$ denotes an α order Caputo fractional derivative of $x(t), 0 < \alpha < 1, A, B_i$ are $n \times n$ constant matrices, $\tau_i (i = 1, 2, \dots, m)$ are real constants with $0 < \tau_1 \leq \tau_2 \leq \dots \leq \tau_m = \tau, \varphi \in C([- \tau, 0], \mathbb{R}^n)$, and $C([- \tau, 0], \mathbb{R}^n)$ denotes space of continuous functions mapping the interval $[-\tau, 0]$ into \mathbb{R}^n .

Similar to Lemma 1, if the initial function $\varphi \in C([- \tau, 0], \mathbb{R}^n)$, then there exists a unique continuous solution for system (48) on $[0, +\infty)$.

Corollary 10. *The zero solution $x(t) \equiv 0$ of system (48) is delay-independent globally asymptotically stable if the following conditions are satisfied:*

$$\begin{aligned} (H_1) \quad &|\arg(\sigma(A))| > \frac{\alpha\pi}{2}, \quad 0 < \alpha < 1, \\ (H_8) \quad &\rho \left[(s^\alpha I - A)^{-1} \sum_{i=1}^m B_i \xi_i \right] < 1, \\ &\Re e(s) = 0, \quad \xi_i \in \mathbb{C}, \quad |\xi_i| = 1. \end{aligned} \tag{49}$$

(ii) When $\tau_i = 0, i = 1, 2, \dots, m$, system (1) reduces to Caputo fractional-order linear autonomous differential systems:

$$\begin{aligned} \left[I - \sum_{i=1}^m C_i \right] D^\alpha x(t) &= \left[A + \sum_{i=1}^m B_i \right] x(t), \quad t \geq 0, \\ x(0) &= x_0, \end{aligned} \tag{50}$$

where $D^\alpha x(t)$ denotes an α order Caputo fractional derivative of $x(t), 0 < \alpha < 1$, and A, B_i, C_i are $n \times n$ constant matrices.

An application of the results in [23, 24] yields the following conclusion.

Corollary 11. *The zero solution $x(t) \equiv 0$ of system (50) is globally asymptotically stable if the following conditions are satisfied:*

$$\begin{aligned} (H_9) \quad &\det \left[I - \sum_{i=1}^m C_i \right] \neq 0, \\ (H_{10}) \quad &|\arg(\sigma(G))| > \frac{\alpha\pi}{2}, \end{aligned} \tag{51}$$

$$G = \left(I - \sum_{i=1}^m C_i \right)^{-1} \left(A + \sum_{i=1}^m B_i \right).$$

Remark 12. When $\det[I - \sum_{i=1}^m C_i] = 0$, system (50) reduces to linear fractional singular differential system. The stability analysis of linear fractional singular (delay) differential systems will become our future investigative works.

4. An Illustrative Example

The following example is presented to illustrate the effectiveness and applicability of the proposed stability criteria.

Example 1. Consider system (1) with

$$\begin{aligned} A &= \begin{bmatrix} -4 & 2 \\ 4 & -10 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}, \\ B_2 &= \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} \frac{2}{3} & -\frac{1}{6} \\ \frac{1}{3} & -\frac{1}{6} \end{bmatrix}, \\ C_2 &= \begin{bmatrix} -\frac{1}{6} & \frac{1}{6} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}, \quad \alpha = \frac{1}{2}, \\ \tau_1 &= 1, \quad \tau_2 = 2, \quad \tau = 2. \end{aligned} \tag{52}$$

The initial function is given by $\varphi(t) = t, t \in [-2, 0]$. Let

$$\begin{aligned} \Theta &= (s^\alpha I - A)^{-1} (\xi_1 B_1 + \xi_2 B_2 + s^\alpha \xi_1 C_1 + s^\alpha \xi_2 C_2), \\ \Re e(s) &= 0, \quad |\xi_1| = |\xi_2| = 1. \end{aligned} \tag{53}$$

By computation, the eigenvalues of matrix Θ give

$$\begin{aligned}\lambda_1(\Theta) &= (s^\alpha + 6)^{-1} \left(\xi_1 + 2\xi_2 + \frac{1}{2}s^\alpha \xi_1 + \frac{1}{3}s^\alpha \xi_2 \right), \\ \lambda_2(\Theta) &= (s^\alpha + 8)^{-1} \left(2\xi_1 + \xi_2 + \frac{1}{3}s^\alpha \xi_1 + \frac{1}{2}s^\alpha \xi_2 \right).\end{aligned}\quad (54)$$

Note that $\sigma(A) = \{-6, -8\}$; then, we have

$$|\arg(\sigma(A))| = \pi > \frac{\alpha\pi}{2}, \quad \alpha = \frac{1}{2}.\quad (55)$$

It is not difficult to verify that

$$\begin{aligned}\rho[\xi_1 C_1 + \xi_2 C_2] &< 1, \quad |\xi_1| \leq 1, \quad |\xi_2| \leq 1, \\ \rho[\Theta] &< 1, \quad \Re(s) = 0, \quad |\xi_1| = 1, \quad |\xi_2| = 1.\end{aligned}\quad (56)$$

Thus, conditions (H_1) , (H_3) , and (H_4) are satisfied. Therefore, it follows from Theorem 7 that the zero solution $x(t) \equiv 0$ of system (1) with the coefficient matrices (52) is delay-independent globally asymptotically stable.

In fact, the characteristic equation of system (1) with the coefficient matrices (52) can be expressed as

$$\det \left[s^\alpha I - A - \sum_{i=1}^m B_i e^{-s\tau_i} - s^\alpha \sum_{i=1}^m C_i e^{-s\tau_i} \right] = 0.\quad (57)$$

Obviously, the characteristic equation (57) includes the transcendental terms. It is very difficult that one precisely solves the roots of (57). An application of Theorem 7 yields that the zero solution $x(t) \equiv 0$ of system (1) with (52) is delay-independent globally asymptotically stable.

5. Conclusions

In this paper, the delay-independent asymptotic stability of linear fractional-order linear neutral differential systems with multiple discrete delays has been discussed. We have synchronously taken into account the factors of such systems including Caputo's fractional-order derivative, state delays. The asymptotic stability criteria have been derived based on the algebraic approach and matrix theory, which ensure the asymptotic stability for all time-delay parameters. By applying these stability criteria, one can avoid solving the roots of transcendental equations. The results obtained are computationally flexible and efficient. In fact, the characteristic equation of system (1) with (52) includes the transcendental terms. Generally, it is very difficult that one precisely solves the roots of characteristic equation. In Example 1, we analyse the distribution of characteristic roots when the coefficient matrices satisfy the appropriate conditions. We only need to check the spectrum range under conditions (H_1) , (H_4) , and (H_5) . An application of Theorem 7 yields that the zero solution $x(t) \equiv 0$ of system (1) is delay-independent globally asymptotically stable. Example 1 shows that Theorem 7 is computationally flexible and efficient. The stability analysis of linear fractional singular (delay) differential systems will become our future investigative works.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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