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Research Article

A Modified Mixed Ishikawa Iteration for Common Fixed Points of Two Asymptotically Quasi Pseudocontractive Type Non-Self-Mappings

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A new modified mixed Ishikawa iterative sequence with error for common fixed points of two asymptotically quasi pseudocontractive type non-self-mappings is introduced. By the flexible use of the iterative scheme and a new lemma, some strong convergence theorems are proved under suitable conditions. The results in this paper improve and generalize some existing results.

1. Introduction

Let E be a real Banach space with its dual E^* and let C be a nonempty, closed, and convex subset of E. The mapping $J : E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\| \cdot \|x^*\|, \|x\| = \|x^*\|\}, \quad x \in E. \tag{1}$$

Let $T : C \to E$ be a mapping. We denote the fixed point set of T by $F(T)$; that is, $F(T) = \{x \in C : x = Tx\}$. Recall that a mapping $T : C \to E$ is said to be nonexpansive if, for each $x, y \in C$,

$$\|Tx - Ty\| \le \|x - y\|. \tag{2}$$

T is said to be asymptotically nonexpansive if there exists a sequence $k_n \subseteq [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \le k_n \|x - y\|, \quad \forall x, y \in C. \tag{3}$$

A sequence of self-mappings $\{T_i\}_{i=1}^{\infty}$ on C is said to be uniform Lipschitzian with the coefficient L if, for any $i = 1, 2, \ldots$, the following holds:

$$\|T_i^n x - T_i^n y\| \le L \|x - y\|, \quad \forall x, y \in C. \tag{4}$$

T is said to be asymptotically pseudocontractive if there exist $k_n \subseteq [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ and $j(x - y) \in J(x - y)$ such that

$$\langle T^n x - T^n y, j(x - y) \rangle \le k_n \|x - y\|^2, \quad \forall x, y \in C. \tag{5}$$

It is obvious to see that every nonexpansive mapping is asymptotically nonexpansive and every asymptotically nonexpansive mapping is asymptotically pseudocontractive. Goebel and Kirk [1] introduced the class of asymptotically nonexpansive mappings in 1972. The class of asymptotically pseudocontractive mappings was introduced by Schu [2] and has been studied by various authors for its generalized mappings in Hilbert spaces, Banach spaces, or generalized topological vector spaces by using the modified Mann or Ishikawa iteration methods (see, e.g.,[3–21]).

In 2003, Chidume et al. [22] studied fixed points of an asymptotically nonexpansive non-self-mapping $T : C \to E$ and the strong convergence of an iterative sequence $\{x_n\}$ generated by

$$x_{n+1} = P\left(\left(1 - \alpha_n\right) x_n + \alpha_n T(PT)^{n-1} x_n\right), \quad n \ge 1, \ x_1 \in C, \tag{6}$$

where $P : E \to C$ is a nonexpansive retraction.

In 2011, Zegeye et al. [23] proved a strong convergence of Ishikawa scheme to a uniformly L-Lipschitzian and asymptotically pseudocontractive mappings in the intermediate sense which satisfies the following inequality (see [24]):

$$\limsup_{n\to\infty}\sup_{x,y\in C}\left(\langle T^n x - T^n y, x-y\rangle - k_n\|x-y\|^2\right) \le 0, \quad \forall x,y\in C, \tag{7}$$

where $k_n \subseteq [1,\infty)$ with $k_n \to 1$ as $n\to\infty$.

Motivated and inspired by the above results, in this paper, we introduce a new modified mixed Ishikawa iterative sequence with error for common fixed points of two more generalized asymptotically quasi pseudocontractive type non-self-mappings. By the flexible use of the iterative scheme and a new lemma (i.e., Lemma 6 in this paper), under suitable conditions, we prove some strong convergence theorems. Our results extend and improve many results of other authors to a certain extent, such as [6, 8, 14–23].

2. Preliminaries

Definition 1. Let C be a nonempty closed convex subset of a real Banach space E. C is said to be a nonexpansive retract (with P) of E if there exists a nonexpansive mapping $P : E \to C$ such that, for all $x\in C$, $Px = x$. And P is called a nonexpansive retraction.

Let $T : C \to E$ be a non-self-mapping (maybe self-mapping). T is called uniformly L-Lipschitzian (with P) if there exists a constant $L>0$ such that

$$\left\|T(PT)^{n-1}x - T(PT)^{n-1}y\right\| \le L\|x-y\|, \quad \forall x,y\in C,\ n\ge 1. \tag{8}$$

T is said to be asymptotically pseudocontractive (with P) if there exist $k_n \subseteq [1,\infty)$ with $k_n\to 1$ as $n\to\infty$ and $\forall x,y\in C$, $\exists j(x-y)\in J(x-y)$ such that

$$\left\langle T(PT)^{n-1}x - T(PT)^{n-1}y, j(x-y)\right\rangle \le k_n\|x-y\|^2. \tag{9}$$

T is said to be an asymptotically pseudocontractive type (with P) if there exist $k_n \subseteq [1,\infty)$ with $k_n\to 1$ as $n\to\infty$ and $\forall x,y\in C$, $j(x-y)\in J(x-y)$ such that

$$\limsup_{n\to\infty}\sup_{x,y\in C}\liminf_{j(x-y)\in J(x-y)}\left(\left\langle T(PT)^{n-1}x - T(PT)^{n-1}y, j(x-y)\right\rangle - k_n\|x-y\|^2\right) \le 0. \tag{10}$$

T is said to be an asymptotically quasi pseudocontractive type (with P) if $F(T)\neq\emptyset$, for $p\in F(T)$, there exist $k_n\subseteq[1,\infty)$ with $k_n\to 1$ as $n\to\infty$, and, $\forall x\in C$, $j(x-p)\in J(x-p)$ such that

$$\limsup_{n\to\infty}\sup_{x\in C}\liminf_{j(x-p)\in J(x-p)}\left(\left\langle T(PT)^{n-1}x - p, j(x-y)\right\rangle - k_n\|x-p\|^2\right) \le 0. \tag{11}$$

Remark 2. It is clear that every asymptotically pseudocontractive mapping (with P) is asymptotically pseudocontractive type (with P) and every asymptotically pseudocontractive type (with P) is asymptotically quasi pseudocontractive type (with P). If $T : C\to C$ is a self-mapping, then we can choose $P = I$ as the identical mapping and we can get the usual definition of asymptotically pseudocontractive mapping, and so forth.

Definition 3. Let C be a nonexpansive retract (with P) of E, let $T_1, T_2 : C\to E$ be two uniformly L-Lipschitzian non-self-mappings and let T_1 be an asymptotically quasi pseudocontractive type (with P).

The sequence $\{x_n\}$ is called the new modified mixed Ishikawa iterative sequence with error (with P), if $\{x_n\}$ is generated by

$$\begin{aligned} x_{n+1} &= P\left((1-\alpha_n-\gamma_n)x_n + \alpha_n T_1(PT_1)^{n-1}\times\left((1-\beta_n)y_n + \beta_n T_1(PT_1)^{n-1}y_n\right) + \gamma_n u_n\right),\\ y_n &= P\left((1-\alpha'_n-\gamma'_n)x_n + \alpha'_n T_2(PT_2)^{n-1}\times\left((1-\beta'_n)x_n + \beta'_n T_2(PT_2)^{n-1}x_n\right) + \gamma'_n v_n\right), \end{aligned} \tag{12}$$

where $x_1\in C$ is arbitrary, $\{u_n\}$ and $\{v_n\}\subset C$ are bounded, and $\alpha_n,\beta_n,\gamma_n,\alpha'_n,\beta'_n,\gamma'_n\in[0,1]$, $n=1,2,\ldots$.

If $\alpha'_n=\beta'_n=\gamma'_n=0$, (12) turns to

$$x_{n+1} = P\left((1-\alpha_n-\gamma_n)x_n + \alpha_n T_1(PT_1)^{n-1}\times\left((1-\beta_n)x_n + \beta_n T_1(PT_1)^{n-1}x_n\right) + \gamma_n u_n\right), \tag{13}$$

and it is called the new modified mixed Mann iterative sequence with error (with P).

If $\gamma_n=\gamma'_n=0$, (12) becomes

$$\begin{aligned} x_{n+1} &= P\left((1-\alpha_n)x_n + \alpha_n T_1(PT_1)^{n-1}\times\left((1-\beta_n)y_n + \beta_n T_1(PT_1)^{n-1}y_n\right)\right),\\ y_n &= P\left((1-\alpha'_n)x_n + \alpha'_n T_2(PT_2)^{n-1}\times\left((1-\beta'_n)x_n + \beta'_n T_2(PT_2)^{n-1}x_n\right)\right), \end{aligned} \tag{14}$$

and it is called the new modified mixed Ishikawa iterative sequence (with P).

If $\beta_n=\beta'_n=0$, (14) turns to

$$\begin{aligned} x_{n+1} &= P\left((1-\alpha_n)x_n + \alpha_n T_1(PT_1)^{n-1}y_n\right),\\ y_n &= P\left((1-\alpha'_n)x_n + \alpha'_n T_2(PT_2)^{n-1}x_n\right), \end{aligned} \tag{15}$$

and it is called the new mixed Ishikawa iterative sequence (with P).

If $T_1=T_2=T : C\to C$ is a self-mapping and $P=I$ is the identical mapping, then (15) is just the modified Ishikawa iterative sequence

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T^n y_n,\\ y_n &= (1-\alpha'_n)x_n + \alpha'_n T^n x_n. \end{aligned} \tag{16}$$

If $\alpha_n' = 0$, (15) becomes (6), obviously. So, iterative method (12) is greatly generalized.

The following lemmas will be needed in what follows to prove our main results.

Lemma 4 (see [19]). *Let E be a real Banach space. Then, for all $x, y \in E$, $j(x+y) \in J(x+y)$, the following inequality holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle x, j(x+y)\rangle. \tag{17}$$

Lemma 5 (see [6, 7]). *Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be three sequences of nonnegative numbers satisfying the recursive inequality:*

$$a_{n+1} \leq (1+b_n)a_n + c_n, \quad \forall n \geq n_0, \tag{18}$$

where n_0 is some nonnegative integer. If $\Sigma_{n=1}^{\infty} b_n < \infty$, $\Sigma_{n=1}^{\infty} c_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

Lemma 6. *Suppose that $\phi : [0,+\infty) \to [0,+\infty)$ is a strictly increasing function with $\phi(0) = 0$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\lambda_n\}$ $(0 \leq \lambda_n \leq 1)$ be four sequences of nonnegative numbers satisfying the recursive inequality:*

$$a_{n+1} \leq (1+b_n)a_n - \lambda_n\phi(a_{n+1}) + c_n, \quad \forall n \geq n_0, \tag{19}$$

where n_0 is some nonnegative integer. If $\Sigma_{n=1}^{\infty} b_n < \infty$, $\Sigma_{n=1}^{\infty} c_n < \infty$, $\Sigma_{n=1}^{\infty} \lambda_n = \infty$, then $\lim_{n\to\infty} a_n = 0$.

Proof. From (19), we get

$$a_{n+1} \leq (1+b_n)a_n + c_n, \quad \forall n \geq n_0. \tag{20}$$

By Lemma 5, we know that $\lim_{n\to\infty} a_n = a \geq 0$ exists. Let $M = \sup_{1\leq n\leq\infty}\{a_n\} < \infty$. Now we show $a = 0$. Otherwise, if $a > 0$, then $\exists n_1 \geq n_0$, such that $a_{n+1} \geq (1/2)a > 0$ when $n \geq n_1$. Because ϕ is a strictly increasing function, so $\phi(a_{n+1}) \geq \phi((1/2)a) > 0$. From (19) again, we have

$$\begin{aligned}
0 &< \phi\left(\frac{1}{2}a\right)\sum_{n=1}^{\infty}\lambda_n \\
&= \phi\left(\frac{1}{2}a\right)\sum_{n=1}^{n_1}\lambda_n + \phi\left(\frac{1}{2}a\right)\sum_{n=n_1+1}^{\infty}\lambda_n \\
&\leq \phi\left(\frac{1}{2}a\right)\sum_{n=1}^{n_1}\lambda_n + \sum_{n=n_1+1}^{\infty}\lambda_n\phi(a_{n+1}) \\
&\leq \phi\left(\frac{1}{2}a\right)\sum_{n=1}^{n_1}\lambda_n + \sum_{n=n_1+1}^{\infty}(a_n - a_{n+1}) \\
&\quad + \sum_{n=n_1+1}^{\infty} b_n a_n + \sum_{n=n_1+1}^{\infty} c_n \\
&\leq \phi\left(\frac{1}{2}a\right)\sum_{n=1}^{n_1}\lambda_n + a_{n_1+1} + M\sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} c_n < \infty.
\end{aligned} \tag{21}$$

This is a contradiction with the given condition $\Sigma_{n=1}^{\infty}\lambda_n = \infty$. Therefore $\lim_{n\to\infty} a_n = 0$. □

Lemma 7. *Suppose that $\phi : [0,+\infty) \to [0,+\infty)$ is a strictly increasing function with $\phi(0) = 0$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\lambda_n\}$ $(0 \leq \lambda_n \leq 1)$, $\{\varepsilon_n\}$ be five sequences of nonnegative numbers satisfying the recursive inequality:*

$$a_{n+1} \leq (1+b_n)a_n - \lambda_n\phi(a_{n+1}) + c_n + \lambda_n\varepsilon_n, \quad \forall n \geq n_0, \tag{22}$$

where n_0 is some nonnegative integer. If $\Sigma_{n=1}^{\infty} b_n < \infty$, $\Sigma_{n=1}^{\infty} c_n < \infty$, $\Sigma_{n=1}^{\infty}\lambda_n = \infty$, $\lim_{n\to\infty}\varepsilon_n = 0$, then $\lim_{n\to\infty} a_n = 0$.

Proof. Firstly, we show $\liminf_{n\to\infty} a_n = a = 0$. If $a > 0$, then, for arbitrary $r \in (0, a)$, $\exists n_1 \geq n_0$, such that $a_{n+1} \geq r > 0$ when $n \geq n_1$. Because ϕ is a strictly increasing function and $\lim_{n\to\infty}\varepsilon_n = 0$, so $\phi(a_{n+1}) \geq \phi(r) > 0$ and $\varepsilon_n \leq (1/2)\phi(r)$ when $n \geq n_1$. From (22), we have

$$\begin{aligned}
a_{n+1} &\leq (1+b_n)a_n - \lambda_n\phi(a_{n+1}) + c_n + \lambda_n\frac{1}{2}\phi(a_{n+1}) \\
&= (1+b_n)a_n - \frac{1}{2}\lambda_n\phi(a_{n+1}) + c_n, \quad \forall n \geq n_1.
\end{aligned} \tag{23}$$

By Lemma 6, we get $0 = \lim_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = a > 0$. This is contradictory. So, $\liminf_{n\to\infty} a_n = 0$.

Secondly, $\forall\varepsilon > 0$, from the given conditions in Lemma 7, $\exists n_2 \geq n_0$, when $\forall n \geq n_2$, we have

$$\varepsilon_n \leq \phi(\varepsilon), \qquad \sum_{n=n_2}^{\infty} b_n \leq \ln 2, \qquad \sum_{n=n_2}^{\infty} c_n \leq \varepsilon. \tag{24}$$

On the other hand, since $\liminf_{n\to\infty} a_n = 0$, $\exists N \geq n_2$ such that $a_N \leq \varepsilon$. Now we claim

$$a_k \leq \left(\varepsilon + \sum_{n=N}^{k-1} c_n\right)\exp\left(\sum_{n=N}^{k-1} b_n\right), \quad \forall k \geq N. \tag{25}$$

In fact, when $k = N$, (25) holds. Suppose that (25) holds for k dose not for $k+1$. Then

$$a_{k+1} > \left(\varepsilon + \sum_{n=N}^{k} c_n\right)\exp\left(\sum_{n=N}^{k} b_n\right). \tag{26}$$

Furthermore, $a_{k+1} > \varepsilon$, $\phi(a_{k+1}) > \phi(\varepsilon)$. But by (22), (24), and the inductive hypothesis, we have

$$\begin{aligned}
a_{n+1} &\leq (1+b_n)a_n - \lambda_n\phi(a_{n+1}) + c_n + \lambda_n\varepsilon_n \\
&\leq (1+b_n)a_n - \lambda_n\phi(\varepsilon) + c_n + \lambda_n\phi(\varepsilon) \\
&\leq (1+b_n)\left(\varepsilon + \sum_{n=N}^{k-1} c_n\right)\exp\left(\sum_{n=N}^{k-1} b_n\right) + c_n \\
&\leq \left(\varepsilon + \sum_{n=N}^{k-1} c_n\right)\exp\left(\sum_{n=N}^{k} b_n\right) + c_n \\
&\leq \left(\varepsilon + \sum_{n=N}^{k} c_n\right)\exp\left(\sum_{n=N}^{k} b_n\right).
\end{aligned} \tag{27}$$

This is a contradiction with (26). So, (25) holds. Whereupon,

$$\limsup_{k\to\infty} a_k \le \left(\varepsilon + \sum_{n=N}^{\infty} c_n\right)\exp\left(\sum_{n=N}^{\infty} b_n\right) \le 2(\varepsilon+\varepsilon) = 4\varepsilon. \tag{28}$$

Therefore, $\limsup_{k\to\infty} a_k = 0 = \lim_{n\to\infty} a_n$. □

3. Main Results

Now, we are in a position to state and prove the main results of this paper.

Theorem 8. *Let C be nonexpansive retract (with P) of a real Banach space E. Assume that $T_1, T_2 : C \to E$ are two uniformly L-Lipschitzian non-self-mappings (with P) and T_1 is an asymptotically quasi pseudocontractive type with coefficient numbers $\{k_n\} \subset [1, +\infty) : k_n \to 1$ satisfying $F = F(T_1) \cap F(T_2) \neq \emptyset$. Suppose that $\{u_n\}, \{v_n\} \subset C$ are two bounded sequences; $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha_n'\}, \{\beta_n'\}, \{\gamma_n'\} \subset [0, 1]$ are six number sequences satisfying the following:*

(C1) $\sum_{n=1}^{\infty}\alpha_n = +\infty$, $\sum_{n=1}^{\infty}\alpha_n^2 < +\infty$, $\sum_{n=1}^{\infty}\alpha_n(k_n - 1) < +\infty$;

(C2) $\alpha_n + \gamma_n \le 1$, $\alpha_n' + \gamma_n' \le 1$, $\sum_{n=1}^{\infty}\gamma_n < +\infty$;

(C3) $\sum_{n=1}^{\infty}\alpha_n\beta_n < +\infty$, $\sum_{n=1}^{\infty}\alpha_n\alpha_n' < +\infty$, $\sum_{n=1}^{\infty}\alpha_n\gamma_n' < +\infty$.

If $x_1 \in C$ is arbitrary, then the iterative sequence $\{x_n\}$ generated by (12) converges strongly to the fixed point $x^* \in F$ if and only if there exists a strictly increasing function $\phi : [0, +\infty) \to [0, +\infty)$ with $\phi(0) = 0$ such that

$$\limsup_{n\to\infty} \inf_{j(x_{n+1}-x^*)\in J(x_{n+1}-x^*)} \Big[\Big\langle T_1(PT_1)^{n-1}x_{n+1} - x^*, j(x_{n+1}-x^*)\Big\rangle - k_n\|x_{n+1}-x^*\|^2 + \phi(\|x_{n+1}-x^*\|)\Big] \le 0. \tag{29}$$

Proof. (Adequacy). Let

$$\varepsilon_n' = \inf_{j(x_{n+1}-x^*)\in J(x_{n+1}-x^*)} \Big[\Big\langle T_1(PT_1)^{n-1}x_{n+1} - x^*, j(x_{n+1}-x^*)\Big\rangle - k_n\|x_{n+1}-x^*\|^2 + \phi(\|x_{n+1}-x^*\|)\Big],$$
$$\varepsilon_n = \max\{\varepsilon_n', 0\} + \frac{1}{n}. \tag{30}$$

Then there exists $j(x_{n+1} - x^*) \in J(x_{n+1} - x^*)$ such that

$$\Big\langle T_1(PT_1)^{n-1}x_{n+1} - x^*, j(x_{n+1}-x^*)\Big\rangle - k_n\|x_{n+1}-x^*\|^2 + \phi(\|x_{n+1}-x^*\|) \le \varepsilon_n. \tag{31}$$

From (29), we know that $\limsup_{n\to\infty}\varepsilon_n' \le 0$. So, $\lim_{n\to\infty}\varepsilon_n = 0$.

Now, from the given conditions and (12), we can let

$$\sigma_n = (1-\beta_n)y_n + \beta_n T_1(PT_1)^{n-1}y_n,$$
$$\delta_n = (1-\beta_n')x_n + \beta_n' T_2(PT_2)^{n-1}x_n, \tag{32}$$

and $M = \sup_{n\ge1}\{\|\mu_n - x^*\|, \|\nu_n - x^*\|\} < \infty$. Then

$$\begin{aligned}
\|\delta_n - x^*\| &\le \beta_n'\|T_2(PT_2)x_n - x^*\| + (1-\beta_n')\|x_n - x^*\| \\
&\le \beta_n' L\|x_n - x^*\| + \|x_n - x^*\|; \\
\|y_n - x^*\| &\le (1-\alpha_n'-\gamma_n')\|x_n - x^*\| \\
&\quad + \alpha_n' L\|\delta_n - x^*\| + \gamma_n'\|\nu_n - x^*\| \\
&\le \|x_n - x^*\| + \alpha_n'\beta_n' L^2\|x_n - x^*\| \\
&\quad + \alpha_n' L\|x_n - x^*\| + \gamma_n' M \\
&= (1 + \alpha_n'\beta_n' L^2 + \alpha_n' L)\|x_n - x^*\| + \gamma_n' M \\
&\le (1 + L + L^2)\|x_n - x^*\| + M; \\
\|\sigma_n - x^*\| &\le \beta_n\|T_1(PT_1)^{n-1}y_n - x^*\| \\
&\quad + (1-\beta_n)\|y_n - x^*\| \\
&\le \beta_n L\|y_n - x^*\| + \|y_n - x^*\| \\
&\le (1+L)(1+L+L^2)\|x_n - x^*\| + (1+L)M; \\
\|y_n - x_{n+1}\| &\le \alpha_n L\|\sigma_n - x^*\| + \alpha_n\|x_n - x^*\| \\
&\quad + \alpha_n' L\|\delta_n - x^*\| + \alpha_n'\|x_n - x^*\| \\
&\quad + (\gamma_n + \gamma_n')\|x_n - x^*\| + (\gamma_n + \gamma_n')M \\
&\le \alpha_n L\big[(1+L)(1+L+L^2)\|x_n - x^*\| \\
&\qquad + (1+L)M\big] \\
&\quad + \alpha_n' L\big[(1+\beta_n' L)\|x_n - x^*\|\big] \\
&\quad + (\alpha_n + \alpha_n' + \gamma_n + \gamma_n')\|x_n - x^*\| + (\gamma_n + \gamma_n')M \\
&\le \big[\alpha_n L(1+L)(1+L+L^2) + \alpha_n' L(1+\beta_n' L) \\
&\qquad + \alpha_n + \alpha_n' + \gamma_n + \gamma_n'\big]\|x_n - x^*\| \\
&\quad + (\alpha_n L(1+L) + \gamma_n + \gamma_n')M; \\
\|\sigma_n - x_{n+1}\| &\le \|y_n - x_{n+1}\| + \beta_n\|T_1(PT_1)^{n-1}y_n - y_n\| \\
&\le s_n\|x_n - x^*\| + t_n,
\end{aligned} \tag{33}$$

where

$$s_n = \alpha_n L(1+L)\left(1+L+L^2\right) + \alpha_n' L\left(1+\beta_n' L\right) + \alpha_n + \alpha_n' + \gamma_n + \gamma_n' + \beta_n(1+L)\left(1+L+L^2\right); \qquad (34)$$
$$t_n = \left[\alpha_n L(1+L) + \gamma_n + \gamma_n' + \beta_n(1+L)\right] M.$$

So, by Lemma 4,

$$\begin{aligned} 2\alpha_n &\left\langle T_1(PT_1)^{n-1}\sigma_n - T_1(PT_1)^{n-1}x_{n+1}, j(x_{n+1}-x^*)\right\rangle \\ &\le 2\alpha_n L\|x_{n+1}-x^*\|\,\|\sigma_n - x_{n+1}\| \\ &\le 2\alpha_n L\|x_{n+1}-x^*\|\left[s_n\|x_n-x^*\| + t_n\right]; \end{aligned} \qquad (35)$$

$$\begin{aligned} &\|x_{n+1}-x^*\|^2 \\ &\le (1-\alpha_n-\gamma_n)^2\|x_n-x^*\|^2 \\ &\quad + 2\alpha_n\left\langle T_1(PT_1)^{n-1}\sigma_n - x^*, j(x_{n+1}-x^*)\right\rangle \\ &\quad + 2\gamma_n\langle \mu_n - x^*, j(x_{n+1}-x^*)\rangle \\ &\le (1-\alpha_n-\gamma_n)^2\|x_n-x^*\|^2 \\ &\quad + 2\alpha_n\left\langle T_1(PT_1)^{n-1}\sigma_n - T_1(PT_1)^{n-1}x_{n+1}, j(x_{n+1}-x^*)\right\rangle \\ &\quad + 2\alpha_n\left\langle T_1(PT_1)^{n-1}x_{n+1} - x^*, j(x_{n+1}-x^*)\right\rangle \\ &\quad + 2\gamma_n M\|x_{n+1}-x^*\|. \end{aligned} \qquad (36)$$

For the third in (36), we have

$$\begin{aligned} 2\alpha_n &\left\langle T_1(PT_1)^{n-1}x_{n+1} - x^*, j(x_{n+1}-x^*)\right\rangle \\ &= 2\alpha_n d_n + 2\alpha_n\left[k_n\|x_{n+1}-x^*\|^2 - \phi(\|x_{n+1}-x^*\|)\right] \\ &\le 2\alpha_n\varepsilon_n + 2\alpha_n\left[k_n\|x_{n+1}-x^*\|^2 - \phi(\|x_{n+1}-x^*\|)\right], \end{aligned} \qquad (37)$$

where

$$\begin{aligned} d_n &= \left\langle T_1(PT_1)^{n-1}x_{n+1} - x^*, j(x_{n+1}-x^*)\right\rangle \\ &\quad - k_n\|x_{n+1}-x^*\|^2 + \phi(\|x_{n+1}-x^*\|) \le \varepsilon_n. \end{aligned} \qquad (38)$$

Substituting (35) into (36), we get

$$\begin{aligned} \|x_{n+1}-x^*\|^2 \le{}& (1-\alpha_n)^2\|x_n-x^*\|^2 + 2\alpha_n\varepsilon_n \\ &+ 2\alpha_n k_n\|x_{n+1}-x^*\|^2 - 2\alpha_n\phi(\|x_{n+1}-x^*\|) \\ &+ 2\alpha_n L(s_n\|x_n-x^*\| + t_n)\|x_{n+1}-x^*\| \\ &+ 2\gamma_n M\|x_{n+1}-x^*\|. \end{aligned} \qquad (39)$$

Let $a_n = \|x_n - x^*\|^2$, $\varphi(t) = 2\phi(\sqrt{t})$, and

$$\begin{aligned} \xi_n &= L\alpha_n s_n \\ &= L^2\alpha_n^2(1+L)\left(1+L+L^2\right) \\ &\quad + \alpha_n\alpha_n' L^2\left(1+\beta_n' L\right) + \alpha_n^2 L + \alpha_n\alpha_n' L + L\alpha_n\gamma_n \\ &\quad + L\alpha_n\gamma_n' + L\alpha_n\beta_n(1+L)\left(1+L+L^2\right), \end{aligned} \qquad (40)$$

$$\begin{aligned} \rho_n &= L\alpha_n t_n + M\gamma_n \\ &= \left[\alpha_n^2 L^2(1+L) + L\alpha_n\gamma_n + L\alpha_n\gamma_n' + \alpha_n\beta_n\left(L+L^2\right)\right] M \\ &\quad + \gamma_n M. \end{aligned} \qquad (41)$$

Then (39) becomes

$$\begin{aligned} a_{n+1} \le{}& (1-\alpha_n)^2 a_n + 2\alpha_n\varepsilon_n + 2\alpha_n k_n a_{n+1} - \alpha_n\varphi(a_{n+1}) \\ &+ 2\left(\xi_n\|x_n-x^*\| + \rho_n\right)\|x_{n+1}-x^*\|. \end{aligned} \qquad (42)$$

By using $2ab \le a^2 + b^2$, we have

$$\begin{aligned} a_{n+1} \le{}& (1-\alpha_n)^2 a_n + 2\alpha_n\varepsilon_n + 2\alpha_n k_n a_{n+1} \\ &- \alpha_n\varphi(a_{n+1}) + \xi_n(a_n + a_{n+1}) + \rho_n(1 + a_{n+1}) \\ ={}& \left(1 - 2\alpha_n + \alpha_n^2 + \xi_n\right) a_n + (2\alpha_n k_n + \xi_n + \rho_n) a_{n+1} \\ &- \alpha_n\varphi(a_{n+1}) + 2\alpha_n\varepsilon_n + \rho_n. \end{aligned} \qquad (43)$$

From (40), (41), and the given conditions, we know

$$\sum_{n=1}^{\infty}\alpha_n^2 < +\infty, \qquad \sum_{n=1}^{\infty}\xi_n < +\infty, \qquad \sum_{n=1}^{\infty}\rho_n < +\infty. \qquad (44)$$

Then, $\lim_{n\to\infty}(2\alpha_n k_n + \xi_n + \rho_n) = 0$. Therefore $\exists n_0$, when $n \ge n_0$, $2\alpha_n k_n + \xi_n + \rho_n \le 1/2$. Let

$$b_n = \frac{1 - 2\alpha_n + \alpha_n^2 + \xi_n}{1 - 2\alpha_n k_n - \xi_n - \rho_n} - 1 = \frac{2\alpha_n(k_n - 1) + \alpha_n^2 + 2\xi_n + \rho_n}{1 - 2\alpha_n k_n - \xi_n - \rho_n};$$
$$c_n = \frac{\rho_n}{1 - 2\alpha_n k_n - \xi_n - \rho_n}. \qquad (45)$$

So, when $n \ge n_0$, we get

$$0 \le b_n \le 2\left[2\alpha_n(k_n - 1) + \alpha_n^2 + 2\xi_n + \rho_n\right], \quad 0 \le c_n \le 2\rho_n. \qquad (46)$$

From (44) and the given conditions, we have $\sum_{n=n_0}^{\infty} b_n < +\infty$, $\sum_{n=n_0}^{\infty} c_n < +\infty$. On the other hand, from (43), we have

$$a_{n+1} \le (1 + b_n) a_n - \alpha_n\varphi(a_{n+1}) + 4\alpha_n\varepsilon_n + c_n, \quad \forall n \ge n_0. \qquad (47)$$

By Lemma 7, we at last get

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \|x_n - x^*\|^2 = 0; \quad (48)$$

for example, $\lim_{n\to\infty} x_n = x^* \in F = F(T_1) \cap F(T_2)$.

(Necessity). Suppose that $\lim_{n\to\infty} x_n = x^* \in F$. Then we can choose an arbitrary continuous strictly increasing function $\phi : [0,+\infty) \to [0,+\infty)$ with $\phi(0) = 0$, such as $\phi(t) = t$. We can get $\lim_{n\to\infty}\phi(\|x_{n+1} - x^*\|) = 0$.

Because T_1 is an asymptotically quasi pseudocontractive type (with P), by (11) in Definition 1, for any $p \in F(T_1) \supseteq F$, we have

$$\limsup_{n\to\infty} \sup_{x\in C} \liminf_{j(x-p)\in J(x-p)} \left(\left\langle T(PT)^{n-1}x - p, j(x-y)\right\rangle - k_n\|x - p\|^2\right) \le 0. \quad (49)$$

So,

$$\begin{aligned} \limsup_{n\to\infty} \inf_{j(x_{n+1}-x^*)\in J(x_{n+1}-x^*)} &\Big[\left\langle T_1(PT_1)^{n-1}x_{n+1} - x^*, j(x_{n+1} - x^*)\right\rangle - k_n\|x_{n+1} - x^*\|^2 + \phi(\|x_{n+1} - x^*\|)\Big] \\ = \limsup_{n\to\infty} \inf_{j(x_{n+1}-x^*)\in J(x_{n+1}-x^*)} &\Big[\left\langle T_1(PT_1)^{n-1}x_{n+1} - x^*, j(x_{n+1} - x^*)\right\rangle - k_n\|x_{n+1} - x^*\|^2\Big] \\ + \lim_{n\to\infty}\phi(\|x_{n+1} - x^*\|) \le 0 + 0 = 0; \end{aligned} \quad (50)$$

that is, (29) holds. This completes the proof of Theorem 8. □

Combining with Theorem 8 and Definition 3, we have some results as follows.

Theorem 9. *Let C be nonexpansive retract (with P) of a real Banach space E. Assume that $T_1, T_2 : C \to E$ are two uniformly L-Lipschitzian non-self-mappings (with P) and T_1 is an asymptotically quasi pseudocontractive type with coefficient numbers $\{k_n\} \subset [1,+\infty) : k_n \to 1$ satisfying $F = F(T_1) \cap F(T_2) \ne \emptyset$. Suppose that $\{\alpha_n\}, \{\beta_n\}, \{\alpha'_n\}, \{\beta'_n\} \subset [0,1]$ are four number sequences satisfying the following:*

(C1) $\sum_{n=1}^{\infty}\alpha_n = +\infty$, $\sum_{n=1}^{\infty}\alpha_n^2 < +\infty$, $\sum_{n=1}^{\infty}\alpha_n(k_n - 1) < +\infty$;

(C2) $\sum_{n=1}^{\infty}\alpha_n\beta_n < +\infty$, $\sum_{n=1}^{\infty}\alpha_n\alpha'_n < +\infty$.

If $x_1 \in C$ is arbitrary, then the iterative sequence $\{x_n\}$ generated by (14) converges strongly to the fixed point $x^* \in F$ if and only if there exists a strictly increasing function $\phi : [0,+\infty) \to [0,+\infty)$ with $\phi(0) = 0$ such that (29) holds.

Theorem 10. *Let C be nonexpansive retract (with P) of a real Banach space E. Assume that $T_1, T_2 : C \to E$ are two uniformly L-Lipschitzian non-self-mappings (with P) and T_1 is an asymptotically quasi pseudocontractive type with coefficient numbers $\{k_n\} \subset [1,+\infty) : k_n \to 1$ satisfying $F = F(T_1) \cap F(T_2) \ne \emptyset$. Suppose that $\{\alpha_n\}, \{\alpha'_n\} \subset [0,1]$ are two number sequences satisfying the following:*

(C1) $\sum_{n=1}^{\infty}\alpha_n = +\infty$, $\sum_{n=1}^{\infty}\alpha_n^2 < +\infty$, $\sum_{n=1}^{\infty}\alpha_n(k_n - 1) < +\infty$;

(C2) $\sum_{n=1}^{\infty}\alpha_n\alpha'_n < +\infty$.

If $x_1 \in C$ is arbitrary, then the iterative sequence $\{x_n\}$ generated by (15) converges strongly to the fixed point $x^* \in F$ if and only if there exists a strictly increasing function $\phi : [0,+\infty) \to [0,+\infty)$ with $\phi(0) = 0$ such that (29) holds.

Theorem 11. *Let C be a nonempty closed convex subset of a real Banach space E. Assume that $T : C \to C$ is uniformly L-Lipschitzian self-mappings and asymptotically quasi pseudocontractive type with coefficient numbers $\{k_n\} \subset [1,+\infty) : k_n \to 1$ satisfying $F = F(T) \ne \emptyset$. Suppose that $\{\alpha_n\}, \{\alpha'_n\} \subset [0,1]$ are two number sequences satisfying the following:*

(C1) $\sum_{n=1}^{\infty}\alpha_n = +\infty$, $\sum_{n=1}^{\infty}\alpha_n^2 < +\infty$, $\sum_{n=1}^{\infty}\alpha_n(k_n - 1) < +\infty$;

(C2) $\sum_{n=1}^{\infty}\alpha_n\alpha'_n < +\infty$.

If $x_1 \in C$ is arbitrary, then the iterative sequence $\{x_n\}$ generated by (16) converges strongly to the fixed point $x^* \in F$ if and only if there exists a strictly increasing function $\phi : [0,+\infty) \to [0,+\infty)$ with $\phi(0) = 0$ such that (29) holds.

Remark 12. Our research and results in this paper have the following several advantaged characteristics. (a) The iterative scheme is the new modified mixed Ishikawa iterative scheme with error on two mappings T_1, T_2. (b) The common fixed point $x^* \in F = F(T_1) \cap F(T_2)$ is studied. (c) The research object is the very generalized asymptotically quasi pseudocontractive type (with P) non-self-mapping. (d) The tool used by us is the very powerful tool: Lemma 7. So, our results here extend and improve many results of other authors to a certain extent, such as [6, 8, 14–23], and the proof methods are very different from the previous.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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