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*Research Article*

# Existence and Estimates of Positive Solutions for Some Singular Fractional Boundary Value Problems

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We establish the existence and uniqueness of a positive solution $u$ for the following fractional boundary value problem: $D^{\alpha}u(x) = -a(x)u^{\sigma}(x)$, $x \in (0,1)$ with the conditions $\lim_{x\to 0^+} x^{2-\alpha}u(x) = 0$, $u(1) = 0$, where $1 < \alpha \leq 2$, $\sigma \in (-1,1)$, and $a$ is a nonnegative continuous function on $(0,1)$ that may be singular at $x = 0$ or $x = 1$. We also give the global behavior of such a solution.

## 1. Introduction

Recently, the theory of fractional differential equations has been developed very quickly and the investigation for the existence of solutions of these differential equations has attracted considerable attention of researchers in the last few years (see [1–11] and the references therein).

Fractional differential equations arise in various fields of science and engineering such as control, porous media, electrochemistry, viscoelasticity, and electromagnetism. They also serve as an excellent tool for the description of hereditary properties of various materials and processes (see [12–14]). In consequence, the subject of fractional differential equations is gaining much importance. Motivated by the surge in the development of this subject, we consider the following singular Dirichlet problem:

$$\begin{gathered} D^{\alpha}u(x) = -a(x)u^{\sigma}(x), \quad x \in (0,1), \\ \lim_{x\to 0^+} x^{2-\alpha}u(x) = 0, \quad u(1) = 0, \end{gathered} \tag{1}$$

where $1 < \alpha \leq 2$, $-1 < \sigma < 1$, and $a$ is a nonnegative continuous function on $(0,1)$ that may be singular at $x = 0$ or $x = 1$. Then we study the existence and exact asymptotic behavior of positive solutions for this problem.

We recall that, for a measurable function $v$, the Riemann-Liouville fractional integral $I_{\beta}v$ and the Riemann-Liouville derivative $D^{\beta}v$ of order $\beta > 0$ are, respectively, defined by

$$\begin{aligned} I_{\beta}v(x) &= \frac{1}{\Gamma(\beta)}\int_0^x (x-t)^{\beta-1}v(t)\,dt, \\ D^{\beta}v(x) &= \frac{1}{\Gamma(n-\beta)}\left(\frac{d}{dx}\right)^n \int_0^x (x-t)^{n-\beta-1}v(t)\,dt \\ &= \left(\frac{d}{dx}\right)^n I_{n-\beta}v(x), \end{aligned} \tag{2}$$

provided that the right hand sides are pointwise defined on $(0,1]$. Here $n = [\beta] + 1$ and $[\beta]$ means the integer part of the number $\beta$ and $\Gamma$ is the Euler Gamma function.

Moreover, we have the following well-known properties (see [3, 13, 15]):

(i) $I_{\beta}I_{\gamma}v(x) = I_{\beta+\gamma}v(x)$ for $x \in [0,1]$, $v \in L^1((0,1])$, $\beta + \gamma \geq 1$;

(ii) $D^{\beta}I_{\beta}v(x) = v(x)$ for a.e. $x \in [0,1]$, where $v \in L^1((0,1])$, $\beta > 0$;

(iii) if $v \in C((0,1)) \cap L^1((0,1))$ and $D^\beta v(x) = 0$, then $v(x) = \sum_{j=1}^{n} c_j t^{\beta-j}$, where $(c_1, c_2, \ldots, c_n) \in \mathbb{R}^n$ and $n$ is the smallest integer greater than or equal to $\beta$.

Several results are obtained for fractional differential equations with different boundary conditions, but none of them deal with the existence of a positive solution to problem (1).

Our aim in this paper is to establish the existence and uniqueness of a positive solution $u \in C_{2-\alpha}([0,1])$ for problem (1) with a precise asymptotic behavior, where $C_{2-\alpha}([0,1])$ is the set of all functions $f$ such that $t \to t^{2-\alpha} f(t)$ is continuous on $[0,1]$.

To state our result, we need some notations. We will use $\mathcal{K}$ to denote the set of Karamata functions $L$ defined on $(0,\eta]$ by

$$L(t) := c \exp\left(\int_t^\eta \frac{z(s)}{s} ds\right), \tag{3}$$

for some $\eta > 1$, where $c > 0$ and $z \in C([0,\eta])$ such that $z(0) = 0$. It is clear that a function $L$ is in $\mathcal{K}$ if and only if $L$ is a positive function in $C^1((0,\eta])$ such that

$$\lim_{t \to 0^+} \frac{tL'(t)}{L(t)} = 0. \tag{4}$$

For two nonnegative functions $f$ and $g$ defined on a set $S$, the notation $f(x) \approx g(x)$, $x \in S$, means that there exists $c > 0$ such that $(1/c)f(x) \le g(x) \le cf(x)$ for all $x \in S$. We denote by $x^+ = \max(x, 0)$ for $x \in \mathbb{R}$ and by $B^+((0,1))$ the set of all nonnegative measurable functions on $(0,1)$.

Throughout this paper, we assume that $a$ is nonnegative on $(0,1)$ and satisfies the following condition:

$(H_0)$ $a \in C((0,1))$ such that for $t \in (0,1)$

$$a(t) \approx t^{-\lambda} L_1(t) (1-t)^{-\mu} L_2(1-t), \tag{5}$$

where $\lambda \le \alpha + (2-\alpha)(1-\sigma)$, $\mu \le \alpha$, $L_1, L_2 \in \mathcal{K}$ satisfying

$$\int_0^\eta \frac{L_1(t)}{t^{\lambda+(2-\alpha)\sigma-1}} dt < \infty, \qquad \int_0^\eta \frac{L_2(t)}{t^{\mu-\alpha+1}} dt < \infty. \tag{6}$$

In the sequel, we introduce the function $\theta$ defined on $(0,1)$ by

$$\theta(x) = x^{\min(1,(2-\lambda+(\alpha-2)\sigma)/(1-\sigma))} \left(\widetilde{L}_1(x)\right)^{1/(1-\sigma)} \times (1-x)^{\min(1,(\alpha-\mu)/(1-\sigma))} \left(\widetilde{L}_2(1-x)\right)^{1/(1-\sigma)}, \tag{7}$$

where

$$\widetilde{L}_1(x) = \begin{cases} 1, & \text{if } \lambda < \alpha - (\alpha-1) \times (1-\sigma), \\ \displaystyle\int_x^\eta \frac{L_1(s)}{s} ds, & \text{if } \lambda = \alpha - (\alpha-1) \times (1-\sigma), \\ L_1(x), & \text{if } \alpha - (\alpha-1)(1-\sigma) < \lambda < \alpha + (2-\alpha)(1-\sigma), \\ \displaystyle\int_0^x \frac{L_1(s)}{s} ds, & \text{if } \lambda = \alpha + (2-\alpha) \times (1-\sigma), \end{cases}$$

$$\widetilde{L}_2(x) = \begin{cases} 1, & \text{if } \mu < \alpha + \sigma - 1, \\ \displaystyle\int_x^\eta \frac{L_2(s)}{s} ds, & \text{if } \mu = \alpha + \sigma - 1, \\ L_2(x), & \text{if } \alpha + \sigma - 1 < \mu < \alpha, \\ \displaystyle\int_0^x \frac{L_2(s)}{s} ds, & \text{if } \mu = \alpha. \end{cases} \tag{8}$$

Our main result is the following.

**Theorem 1.** *Let $\sigma \in (-1,1)$ and assume that $a$ satisfies $(H_0)$. Then problem (1) has a unique positive solution $u \in C_{2-\alpha}([0,1])$ satisfying for $x \in (0,1)$,*

$$u(x) \approx x^{\alpha-2}\theta(x). \tag{9}$$

*Remark 2.* Note that, for $x \in (0,1)$, we have

$$x^{\alpha-2}\theta(x) \approx x^{\min(\alpha-1,(\alpha-\lambda)/(1-\sigma))} \times \left(\widetilde{L}_1(x)\right)^{1/(1-\sigma)} (1-x)^{\min(1,(\alpha-\mu)/(1-\sigma))} \times \left(\widetilde{L}_2(1-x)\right)^{1/(1-\sigma)}. \tag{10}$$

This implies in particular that, for $1 < \alpha < 2$ and $\alpha < \lambda \le \alpha + (2-\alpha)(1-\sigma)$, the solution $u$ blows up at $x = 0$ and for $\lambda < \alpha$, $\lim_{x \to 0^+} u(x) = 0$.

This paper is organized as follows. Some preliminary lemmas are stated and proved in the next section, involving some already known results on Karamata functions. In Section 3, we give the proof of Theorem 1.

## 2. Technical Lemmas

To let the paper be self-contained, we begin this section by recapitulating some properties of Karamata regular variation theory. The following is due to [16, 17].

**Lemma 3.** *The following hold.*

(i) *Letting $L \in \mathcal{K}$ and $\varepsilon > 0$, then one has*

$$\lim_{t \to 0^+} t^\varepsilon L(t) = 0. \tag{11}$$

(ii) *Let $L_1, L_2 \in \mathcal{K}$ and $p \in \mathbb{R}$. Then one has $L_1 + L_2 \in \mathcal{K}$, $L_1 L_2 \in \mathcal{K}$, and $L_1^p \in \mathcal{K}$.*

*Example 4.* Let $m$ be a positive integer. Let $c > 0$, $(\mu_1, \mu_2, \ldots, \mu_m) \in \mathbb{R}^m$, and $d$ be a sufficiently large positive real number such that the function

$$L(t) = c\prod_{k=1}^{m}\left(\log_k\left(\frac{d}{t}\right)\right)^{-\mu_k} \tag{12}$$

is defined and positive on $(0, \eta]$, for some $\eta > 1$, where $\log_k x = \log \circ \log \circ \cdots \circ \log x$ ($k$ times). Then $L \in \mathcal{K}$.

Applying Karamata's theorem (see [16, 17]), we get the following.

**Lemma 5.** *Let $\mu \in \mathbb{R}$ and $L$ be a function in $\mathcal{K}$ defined on $(0, \eta]$. One has the following:*

(i) *if $\mu < -1$, then $\int_0^\eta s^\mu L(s)ds$ diverges and $\int_t^\eta s^\mu L(s)ds \sim_{t\to 0^+} -t^{1+\mu}L(t)/(\mu+1)$;*

(ii) *if $\mu > -1$, then $\int_0^\eta s^\mu L(s)ds$ converges and $\int_0^t s^\mu L(s)ds \sim_{t\to 0^+} t^{1+\mu}L(t)/(\mu+1)$.*

**Lemma 6.** *Let $L \in \mathcal{K}$ be defined on $(0, \eta]$. Then one has*

$$\lim_{t\to 0^+}\frac{L(t)}{\int_t^\eta (L(s)/s)\,ds} = 0. \tag{13}$$

*If further $\int_0^\eta (L(s)/s)ds$ converges, then one has*

$$\lim_{t\to 0^+}\frac{L(t)}{\int_0^t (L(s)/s)\,ds} = 0. \tag{14}$$

*Proof.* We distinguish two cases.

*Case 1.* We suppose that $\int_0^\eta (L(s)/s)ds$ converges. Since the function $t \to L(t)/t$ is nonincreasing in $(0, \omega]$, for some $\omega < \eta$, it follows that, for each $t \le \omega$, we have

$$L(t) \le \int_0^t \frac{L(s)}{s}ds. \tag{15}$$

It follows that $\lim_{t\to 0^+} L(t) = 0$. So we deduce (13).

Now put

$$\varphi(t) = \frac{L(t)}{t}, \quad \text{for } t \in (0, \eta). \tag{16}$$

Using that $\lim_{t\to 0^+}(t\varphi'(t)/\varphi(t)) = -1$, we obtain

$$\int_0^t \varphi(s)\,ds \sim_{t\to 0^+} -\int_0^t s\varphi'(s)\,ds = -t\varphi(t) + \int_0^t \varphi(s)\,ds. \tag{17}$$

This implies that

$$\int_0^t \frac{L(s)}{s}ds \sim_{t\to 0^+} -L(t) + \int_0^t \frac{L(s)}{s}ds. \tag{18}$$

So (14) holds.

*Case 2.* We suppose that $\int_0^\eta (L(s)/s)ds$ diverges. We have, for some $\omega < \eta$,

$$\int_t^\omega \varphi(s)\,ds \sim_{t\to 0^+} t\varphi(t) - \omega\varphi(\omega) + \int_t^\omega \varphi(s)\,ds. \tag{19}$$

This implies that

$$\int_t^\omega \frac{L(s)}{s}ds \sim_{t\to 0^+} L(t) - \omega\varphi(\omega) + \int_t^\omega \frac{L(s)}{s}ds. \tag{20}$$

This proves (13) and completes the proof. □

*Remark 7.* Let $L \in \mathcal{K}$ be defined on $(0, \eta]$; then using (4) and (13), we deduce that

$$t \longrightarrow \int_t^\eta \frac{L(s)}{s}ds \in \mathcal{K}. \tag{21}$$

If further $\int_0^\eta (L(s)/s)ds$ converges, we have by (13) that

$$t \longrightarrow \int_0^t \frac{L(s)}{s}ds \in \mathcal{K}. \tag{22}$$

**Lemma 8.** *Given $1 < \alpha \le 2$ and $\varphi \in C([0,1])$, then the unique continuous solution of*

$$\begin{gathered} D^\alpha u(x) = -\varphi(x), \quad x \in (0,1), \\ \lim_{x\to 0} x^{2-\alpha}u(x) = 0, \quad u(1) = 0 \end{gathered} \tag{23}$$

*is given by*

$$u(x) = G_\alpha\varphi(x) := \int_0^1 G_\alpha(x,t)\,\varphi(t)\,dt, \tag{24}$$

*where*

$$G_\alpha(x,t) = \frac{1}{\Gamma(\alpha)}\left[x^{\alpha-1}(1-t)^{\alpha-1} - \left((x-t)^+\right)^{\alpha-1}\right] \tag{25}$$

*is Green's function for the boundary value problem* (23).

*Proof.* Since $\varphi \in C([0,1])$, then $u_0 = -I_\alpha\varphi$ is a solution of the equation $D^\alpha u = -\varphi$. Hence $D^\alpha(u + I_\alpha\varphi) = 0$. Consequently there exist two constants $c_1, c_2 \in \mathbb{R}$ such that $u(x) + I_\alpha\varphi(x) = c_1x^{\alpha-1} + c_2x^{\alpha-2}$. Using the fact that $\lim_{x\to 0}x^{2-\alpha}u(x) = 0$ and $u(1) = 0$, we obtain $c_2 = 0$ and $c_1 = I_\alpha\varphi(1)$. So

$$\begin{aligned} u(x) &= \frac{1}{\Gamma(\alpha)}x^{\alpha-1}\int_0^1 (1-t)^{\alpha-1}\varphi(t)\,dt \\ &\quad - \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\varphi(t)\,dt \\ &= \int_0^1 G_\alpha(x,t)\,\varphi(t)\,dt. \end{aligned} \tag{26}$$

□

In the following, we give some estimates on the Green function $G_\alpha(x,y)$. So, we need the following lemma.

**Lemma 9.** *For $\lambda, \mu \in (0,\infty)$ and $t \in [0,1]$ one has*

$$\min\left(1, \frac{\mu}{\lambda}\right)\left(1-t^{\lambda}\right) \leq 1-t^{\mu} \leq \max\left(1, \frac{\mu}{\lambda}\right)\left(1-t^{\lambda}\right). \quad (27)$$

**Proposition 10.** *On $(0,1)\times(0,1)$, one has*

(i) $G_\alpha(x,t) \approx x^{\alpha-2}(1-t)^{\alpha-2}\ \min(x,t)\ (1-\max(x,t))$;

(ii) *there exist two constants $c_1, c_2 > 0$ such that*

$$c_1 x^{\alpha-1} t(1-t)^{\alpha-1}(1-x) \leq G_\alpha(x,t) \leq c_2 x^{\alpha-2} t(1-t)^{\alpha-1}. \quad (28)$$

*Proof.* (i) For $x, t \in (0,1)\times(0,1)$ we have

$$G_\alpha(x,t) = \frac{(1-t)^{\alpha-1}x^{\alpha-1}}{\Gamma(\alpha)}\left[1-\left(\frac{(x-t)^{+}}{x(1-t)}\right)^{\alpha-1}\right]. \quad (29)$$

Since $(x-t)^{+}/x(1-t) \in (0,1)$ for $x, t \in (0,1)$, then by applying Lemma 9 with $\mu = \alpha-1$ and $\lambda = 1$, we obtain

$$\begin{aligned} G_\alpha(x,t) &\approx x^{\alpha-1}(1-t)^{\alpha-1}\left(1-\frac{(x-t)^{+}}{x(1-t)}\right) \\ &= x^{\alpha-2}(1-t)^{\alpha-2}\min(x,t)(1-\max(x,t)). \end{aligned} \quad (30)$$

(ii) Using the following inequalities for $x, t \in [0,1]$,

$$x(1-x)t(1-t) \leq \min(x,t)(1-\max(x,t)) \leq t(1-t), \quad (31)$$

we deduce the result from (i). □

As a consequence of Proposition 10, we obtain the following.

**Corollary 11.** *Let $f \in B^{+}((0,1))$ and put $G_\alpha f(x) := \int_0^1 G_\alpha(x,t)f(t)dt$ for $x \in (0,1]$. Then*

$$\begin{aligned} &G_\alpha f(x) < \infty \text{ for } x \in (0,1) \\ &\text{iff} \int_0^1 t(1-t)^{\alpha-1} f(t)\,dt < \infty. \end{aligned} \quad (32)$$

**Proposition 12.** *Given $1<\alpha<2$ and $f$ such that the function $t \rightarrow t(1-t)^{\alpha-1}f(t)$ is continuous and integrable on $(0,1)$, then $G_\alpha f$ is the unique solution in $C_{2-\alpha}([0,1])$ of the following boundary value problem:*

$$\begin{aligned} &D^{\alpha}u(x) = -f(x), \quad x \in (0,1), \\ &\lim_{x\to 0^{+}} x^{2-\alpha}u(x) = 0, \quad u(1) = 0. \end{aligned} \quad (33)$$

*Proof.* From Corollary 11, the function $G_\alpha f$ is defined on $(0,1)$ and by Proposition 10, we have

$$G_\alpha|f|(x) \leq c_2 x^{\alpha-2}\int_0^1 t(1-t)^{\alpha-1}|f(t)|\,dt, \quad (34)$$

which implies that $I_{2-\alpha}(G_\alpha|f|)$ is bounded on $(0,1)$. Now, using Fubini's theorem, we have

$$\begin{aligned} &I_{2-\alpha}(G_\alpha f)(x) \\ &= \frac{1}{\Gamma(2-\alpha)}\int_0^x (x-t)^{1-\alpha}G_\alpha f(t)\,dt \\ &= \frac{1}{\Gamma(2-\alpha)}\int_0^1\left(\int_0^x (x-t)^{1-\alpha}G_\alpha(t,s)\,dt\right)f(s)\,ds. \end{aligned} \quad (35)$$

On the other hand, we have

$$\begin{aligned} &\frac{1}{\Gamma(2-\alpha)}\int_0^x (x-t)^{1-\alpha}G_\alpha(t,s)\,dt \\ &= \frac{1}{\Gamma(2-\alpha)\Gamma(\alpha)}\left[(1-s)^{\alpha-1}\int_0^x (x-t)^{1-\alpha}t^{\alpha-1}dt\right. \\ &\qquad \left. - \int_0^x (x-t)^{1-\alpha}\left((t-s)^{+}\right)^{\alpha-1}dt\right] \\ &= x(1-s)^{\alpha-1} \\ &\quad - \frac{1}{\Gamma(2-\alpha)\Gamma(\alpha)}\int_0^x (x-t)^{1-\alpha} \\ &\quad \times\left((t-s)^{+}\right)^{\alpha-1}dt. \end{aligned} \quad (36)$$

Now, suppose that $s \leq x$; then we have

$$\begin{aligned} &\int_0^x (x-t)^{1-\alpha}\left((t-s)^{+}\right)^{\alpha-1}dt \\ &\quad = \int_s^x (x-t)^{1-\alpha}(t-s)^{\alpha-1}dt. \end{aligned} \quad (37)$$

By considering the substitution $t = s+\theta(x-s)$, we obtain

$$\int_s^x (x-t)^{1-\alpha}(t-s)^{\alpha-1}dt = \Gamma(\alpha)\Gamma(2-\alpha)(x-s). \quad (38)$$

Moreover if $x \leq s$ and $t \in (0,x)$, we have $\int_0^x (x-t)^{1-\alpha}((t-s)^{+})^{\alpha-1}dt = 0$.

So, it follows that

$$\begin{aligned} &\frac{1}{\Gamma(2-\alpha)}\int_0^x (x-t)^{1-\alpha}G_\alpha(t,s)\,dt \\ &\quad = x(1-s)^{\alpha-1}-(x-s)^{+}. \end{aligned} \quad (39)$$

This implies that

$$\begin{aligned}
I_{2-\alpha}(G_\alpha f)(x) &= \int_0^1 \left[x(1-s)^{\alpha-1}-(x-s)^+\right] f(s)\,ds \\
&= x\int_0^x \left((1-s)^{\alpha-1}-1\right) f(s)\,ds \\
&\quad + \int_0^x sf(s)\,ds + x\int_x^1 (1-s)^{\alpha-1} f(s)\,ds, \\
D^\alpha(G_\alpha f)(x) &= \frac{d^2}{dx^2}\left(I_{2-\alpha}(G_\alpha f)\right)(x) \\
&= -f(x), \quad \text{for } x\in(0,1).
\end{aligned} \tag{40}$$

Moreover, using part (i) of Proposition 10 and the dominated convergence theorem, we conclude that $\lim_{x\to 0^+} x^{2-\alpha} G_\alpha f(x)=0$ and $G_\alpha f(1)=0$.

Finally, we prove the uniqueness. Let $u,v\in C_{2-\alpha}([0,1])$ be two solutions of (33) and put $w=v-u$. Then $w\in C_{2-\alpha}([0,1])\subset L^1((0,1))\cap C((0,1))$ and $D^\alpha w=0$. Hence, it follows that $w(x)=c_1x^{\alpha-1}+c_2x^{\alpha-2}$. Using the fact that $\lim_{x\to0^+}x^{2-\alpha}w(x)=w(1)=0$, we conclude that $w=0$ and so $u=v$.

In the sequel, we assume that $\beta\le 2$ and $\gamma\le\alpha$ and we put

$$b(t)=t^{-\beta}L_3(t)(1-t)^{-\gamma}L_4(1-t), \tag{41}$$

where $L_3,L_4\in\mathcal{K}$ satisfy

$$\int_0^\eta \frac{L_3(t)}{t^{\beta-1}}dt<\infty, \qquad \int_0^\eta t^{\alpha-1-\gamma}L_4(t)\,dt<\infty. \tag{42}$$

So, we aim to give some estimates on the potential function $G_\alpha b(x)$.

We define the Karamata functions $\psi_\beta$, $\phi_\gamma$ by

$$\psi_\beta(x)=\begin{cases}\displaystyle\int_0^x\frac{L_3(t)}{t}dt, & \text{if } \beta=2,\\ L_3(x), & \text{if } 1<\beta<2,\\ \displaystyle\int_x^\eta\frac{L_3(t)}{t}dt, & \text{if } \beta=1,\\ 1, & \text{if } \beta<1,\end{cases} \tag{43}$$

$$\phi_\gamma(x)=\begin{cases}\displaystyle\int_0^x\frac{L_4(t)}{t}dt, & \text{if } \gamma=\alpha,\\ L_4(x), & \text{if } \alpha-1<\gamma<\alpha,\\ \displaystyle\int_x^\eta\frac{L_4(t)}{t}dt, & \text{if } \gamma=\alpha-1,\\ 1, & \text{if } \gamma<\alpha-1.\end{cases} \tag{44}$$

□

Then, we have the following.

**Proposition 13.** *For $x\in(0,1)$,*

$$G_\alpha b(x)\approx x^{\min(\alpha-1,\alpha-\beta)}(1-x)^{\min(1,\alpha-\gamma)}\psi_\beta(x)\phi_\gamma(1-x). \tag{45}$$

*Proof.* Using Proposition 10, we have

$$\begin{aligned}
x^{2-\alpha}G_\alpha b(x) &\approx \int_0^1 (1-t)^{\alpha-2-\gamma}t^{-\beta}\min(x,t) \\
&\qquad\times(1-\max(x,t))\,L_3(t)\,L_4(1-t)\,dt \\
&\approx (1-x)\int_0^x (1-t)^{\alpha-2-\gamma}t^{1-\beta} \\
&\qquad\times L_3(t)\,L_4(1-t)\,dt \\
&\quad + x\int_x^1 (1-t)^{\alpha-1-\gamma}t^{-\beta} \\
&\qquad\times L_3(t)\,L_4(1-t)\,dt \\
&= (1-x)\,I(x)+xJ(x).
\end{aligned} \tag{46}$$

For $0<x\le 1/2$, we have $I(x)\approx\int_0^x t^{1-\beta}L_3(t)dt$. So, using Lemma 5 and hypothesis (42), we deduce that

$$I(x)\approx\begin{cases}x^{2-\beta}L_3(x), & \text{if } \beta<2,\\ \displaystyle\int_0^x\frac{L_3(t)}{t}dt, & \text{if } \beta=2.\end{cases} \tag{47}$$

Now, we have

$$\begin{aligned}
J(x) &\approx \int_x^{1/2} t^{-\beta}L_3(t)\,dt \\
&\quad + \int_{1/2}^1 (1-t)^{\alpha-1-\gamma}L_4(1-t)\,dt \\
&\approx 1+\int_x^{1/2} t^{-\beta}L_3(t)\,dt,
\end{aligned} \tag{48}$$

which implies by Lemma 5 that

$$J(x)\approx\begin{cases}x^{1-\beta}L_3(x), & \text{if } 1<\beta\le 2,\\ \displaystyle\int_x^\eta\frac{L_3(t)}{t}dt, & \text{if } \beta=1,\\ 1, & \text{if } \beta<1.\end{cases} \tag{49}$$

Hence, it follows by Lemma 6 and hypothesis (42) that, for $0<x\le 1/2$, we get

$$x^{2-\alpha}G_\alpha b(x)\approx\begin{cases}\displaystyle\int_0^x\frac{L_3(t)}{t}dt & \text{if } \beta=2,\\ x^{2-\beta}L_3(x) & \text{if } 1<\beta<2,\\ \displaystyle x\int_x^\eta\frac{L_3(t)}{t}dt & \text{if } \beta=1,\\ x & \text{if } \beta<1.\end{cases} \tag{50}$$

That is

$$G_\alpha b(x)\approx x^{\min(\alpha-1,\alpha-\beta)}\psi_\beta(x). \tag{51}$$

Now, for $1/2 \le x < 1$, we use again Lemma 5 and hypothesis (42) to deduce that

$$\begin{aligned} I(x) &\approx \int_0^{1/2} t^{1-\beta} L_3(t)\,dt \\ &\quad + \int_{1/2}^{x} (1-t)^{\alpha-2-\gamma} L_4(1-t)\,dt \\ &\approx 1 + \int_{1/2}^{x} (1-t)^{\alpha-2-\gamma} L_4(1-t)\,dt \\ &\approx \begin{cases} (1-x)^{\alpha-1-\gamma} L_4(1-x), & \text{if } \alpha-1<\gamma\le\alpha, \\ \displaystyle\int_{1-x}^{\eta} \frac{L_4(t)}{t}\,dt, & \text{if } \gamma=\alpha-1, \\ 1, & \text{if } \gamma<\alpha-1, \end{cases} \\ J(x) &\approx \int_0^{1-x} t^{\alpha-1-\gamma} L_4(t)\,dt \\ &\approx \begin{cases} (1-x)^{\alpha-\gamma} L_4(1-x), & \text{if } \gamma<\alpha, \\ \displaystyle\int_0^{1-x} \frac{L_4(t)}{t}\,dt, & \text{if } \gamma=\alpha. \end{cases} \end{aligned} \tag{52}$$

Hence, it follows from Lemma 3 that, for $x \in [1/2, 1)$, we get

$$x^{2-\alpha} G_\alpha b(x) \approx \begin{cases} \displaystyle\int_0^{1-x} \frac{L_4(t)}{t}\,dt, & \text{if } \gamma=\alpha, \\ (1-x)^{\alpha-\gamma} L_4(1-x), & \text{if } \alpha-1<\gamma<\alpha, \\ (1-x)\displaystyle\int_{1-x}^{\eta} \frac{L_4(t)}{t}\,dt, & \text{if } \gamma=\alpha-1, \\ 1-x, & \text{if } \gamma<\alpha-1. \end{cases} \tag{53}$$

That is

$$x^{2-\alpha}\ G_\alpha b(x) \approx (1-x)^{\min(1,\alpha-\gamma)} \phi_\gamma(1-x). \tag{54}$$

This together with (51) implies that, for $x \in (0,1)$, we have

$$G_\alpha b(x) \approx x^{\min(\alpha-1,\alpha-\beta)} (1-x)^{\min(1,\alpha-\gamma)} \psi_\beta(x)\, \phi_\gamma(1-x). \tag{55}$$

□

## 3. Proof of Theorem 1

In order to prove Theorem 1, we need the following Lemma.

**Lemma 14.** *Assume that the function a satisfies $(H_0)$ and put $\omega(t) = a(t)t^{(\alpha-2)\sigma}(\theta(t))^\sigma$ for $t \in (0,1)$. Then one has, for $x \in (0,1)$,*

$$G_\alpha \omega(x) \approx x^{\alpha-2}\theta(x). \tag{56}$$

*Proof.* Put $r = \min(\alpha-1, (\alpha-\lambda)/(1-\sigma))$ and $s = \min(1, (\alpha-\mu)/(1-\sigma))$. Then for $t \in (0,1)$, we have

$$\begin{aligned} \omega(t) &= t^{-\lambda+r\sigma} L_1(t) \left(\widetilde{L}_1(t)\right)^{\sigma/(1-\sigma)} \\ &\quad \times (1-t)^{-\mu+s\sigma} L_2(1-t) \left(\widetilde{L}_2(1-t)\right)^{\sigma/(1-\sigma)}. \end{aligned} \tag{57}$$

Let $\beta = \lambda - r\sigma$, $\gamma = \mu - s\sigma$, $L_3(t) = L_1(t)(\widetilde{L}_1(t))^{\sigma/(1-\sigma)}$, and $L_4(t) = L_2(t)(\widetilde{L}_2(t))^{\sigma/(1-\sigma)}$. Then, using Proposition 13, we obtain by a simple computation that

$$x^{2-\alpha} G_\alpha(\omega)(x) \approx \theta(x). \tag{58}$$

□

*Proof of Theorem 1.* From Lemma 14, there exists $M > 1$ such that, for each $x \in (0,1)$,

$$\frac{1}{M}\theta(x) \le x^{2-\alpha} G_\alpha \omega(x) \le M\theta(x), \tag{59}$$

where $\omega(t) = a(t)t^{(\alpha-2)\sigma}\theta^\sigma(t)$.

Put $c_0 = M^{1/(1-|\sigma|)}$ and let

$$\Lambda = \left\{ v \in C([0,1]) : \frac{1}{c_0}\theta \le v \le c_0\theta \right\}. \tag{60}$$

In order to use a fixed point theorem, we denote $\widetilde{a}(t) = a(t)t^{(\alpha-2)\sigma}$ and we define the operator $T$ on $\Lambda$ by

$$Tv(x) = x^{2-\alpha} G_\alpha(\widetilde{a}v^\sigma)(x). \tag{61}$$

For this choice of $c_0$, we can easily prove that, for $v \in \Lambda$, we have $Tv \le c_0\theta$ and $Tv \ge (1/c_0)\theta$.

Now, we have

$$\begin{aligned} Tv(x) &= \frac{x^{2-\alpha}}{\Gamma(\alpha)} \int_0^1 G_\alpha(x,t)\,\widetilde{a}(t)\, v^\sigma(t)\,dt \\ &= \frac{x^{2-\alpha}}{\Gamma(\alpha)} \int_0^1 \Big[ x^{\alpha-1}(1-t)^{\alpha-1} \\ &\qquad - \left((x-t)^+\right)^{\alpha-1} \Big]\, \widetilde{a}(t)\, v^\sigma(t)\,dt. \end{aligned} \tag{62}$$

Since the function $(x,t) \to x^{\alpha-1}(1-t)^{\alpha-1} - ((x-t)^+)^{\alpha-1}$ is continuous on $[0,1]\times[0,1]$ and by Proposition 10, Corollary 11, and Lemma 14, the function $t \to t(1-t)^{\alpha-1}\,\widetilde{a}(t)\theta^\sigma(t)$ is integrable on $(0,1)$, we deduce that the operator $T$ is compact from $\Lambda$ to itself. It follows by the Schauder fixed point theorem that there exists $v \in \Lambda$ such that $Tv = v$. Put $u(x) = x^{\alpha-2}v(x)$. Then $u \in C_{2-\alpha}([0,1])$ and $u$ satisfies the equation

$$u(x) = G_\alpha(au^\sigma)(x). \tag{63}$$

Since the function $t \to t(1-t)^{\alpha-1}a(t)u^\sigma(t)$ is continuous and integrable on $(0,1)$, then by Proposition 12, the function $u$ is a positive continuous solution of problem (1).

Finally, let us prove that $u$ is the unique positive continuous solution satisfying (9). To this aim, we assume that (1) has two positive solutions $u, v \in C_{2-\alpha}([0,1])$ satisfying (9) and consider the nonempty set $J = \{m \ge 1 : 1/m \le u/v \le m\}$ and put $c = \inf J$. Then $c \ge 1$ and we have $(1/c)v \le u \le cv$. It follows that $u^\sigma \le c^{|\sigma|}v^\sigma$ and consequently

$$\begin{aligned} &-D^\alpha\left(c^{|\sigma|}v - u\right) = a\left(c^{|\sigma|}v^\sigma - u^\sigma\right) \ge 0, \\ &\lim_{t\to 0^+} x^{2-\alpha}\left(c^{|\sigma|}v - u\right)(t) = 0, \\ &\left(c^{|\sigma|}v - u\right)(1) = 0, \end{aligned} \tag{64}$$

which implies by Proposition 12 that $c^{|\sigma|}v - u = G_\alpha(a(c^{|\sigma|}v^\sigma - u^\sigma)) \geq 0$. By symmetry, we also obtain that $v \leq c^{|\sigma|}u$. Hence, $c^{|\sigma|} \in J$ and $c \leq c^{|\sigma|}$. Since $|\sigma| < 1$, then $c = 1$ and consequently $u = v$. □

*Example 15.* Let $\sigma \in (-1, 1)$ and $a$ be a positive continuous function on $(0, 1)$ such that

$$a(t) \approx t^{-\lambda}(1-t)^{-\mu}\log\left(\frac{2}{1-t}\right), \tag{65}$$

where $\lambda < \alpha + (2-\alpha)(1-\sigma)$ and $\mu < \alpha$. Then, using Theorem 1, problem (1) has a unique positive continuous solution $u$ satisfying the following estimates:

$$\begin{aligned} u(x) \approx{} & x^{\min(\alpha-1,(\alpha-\lambda)/(1-\sigma))}\left(\widetilde{L}_1(x)\right)^{1/(1-\sigma)} \\ & \times (1-x)^{\min(1,(\alpha-\mu)/(1-\sigma))}\left(\widetilde{L}_2(1-x)\right)^{1/(1-\sigma)}, \end{aligned} \tag{66}$$

where

$$\begin{aligned} \widetilde{L}_1(x) &= \begin{cases} 1, & \text{if } \lambda \neq \alpha - (\alpha-1)(1-\sigma), \\ \log\left(\dfrac{2}{x}\right), & \text{if } \lambda = \alpha - (\alpha-1)(1-\sigma), \end{cases} \\ \widetilde{L}_2(x) &= \begin{cases} 1, & \text{if } \mu < \alpha+\sigma-1, \\ \left(\log\left(\dfrac{2}{x}\right)\right)^2, & \text{if } \mu = \alpha+\sigma-1, \\ \log\left(\dfrac{2}{x}\right), & \text{if } \alpha+\sigma-1 < \mu < \alpha. \end{cases} \end{aligned} \tag{67}$$

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## References

[1] S. Abbas, M. Benchohra, and G. M. N'Guérékata, *Topics in Fractional Differential Equations*, Developments in Mathematics, Springer, New York, NY, USA, 2012.

[2] R. P. Agarwal, D. O'Regan, and S. Staněk, "Positive solutions for Dirichlet problems of singular nonlinear fractional differential equations," *Journal of Mathematical Analysis and Applications*, vol. 371, no. 1, pp. 57–68, 2010.

[3] Z. Bai and H. Lü, "Positive solutions for boundary value problem of nonlinear fractional differential equation," *Journal of Mathematical Analysis and Applications*, vol. 311, no. 2, pp. 495–505, 2005.

[4] J. Caballero, J. Harjani, and K. Sadarangani, "Positive solutions for a class of singular fractional boundary value problems," *Computers & Mathematics with Applications*, vol. 62, no. 3, pp. 1325–1332, 2011.

[5] J. Deng and L. Ma, "Existence and uniqueness of solutions of initial value problems for nonlinear fractional differential equations," *Applied Mathematics Letters*, vol. 23, no. 6, pp. 676–680, 2010.

[6] N. Kosmatov, "A singular boundary value problem for nonlinear differential equations of fractional order," *Journal of Applied Mathematics and Computing*, vol. 29, no. 1-2, pp. 125–135, 2009.

[7] Y. Liu, w. Zhang, and X. Liu, "A sufficient condition for the existence of a positive solution for a nonlinear fractional differential equation with the Riemann-Liouville derivative," *Applied Mathematics Letters*, vol. 25, no. 11, pp. 1986–1992, 2012.

[8] T. Qiu and Z. Bai, "Existence of positive solutions for singular fractional differential equations," *Electronic Journal of Differential Equations*, vol. 149, p. 19, 2008.

[9] S. Staněk, "The existence of positive solutions of singular fractional boundary value problems," *Computers & Mathematics with Applications*, vol. 62, no. 3, pp. 1379–1388, 2011.

[10] S. Zhang, "Positive solutions to singular boundary value problem for nonlinear fractional differential equation," *Computers & Mathematics with Applications*, vol. 59, no. 3, pp. 1300–1309, 2010.

[11] Y. Zhao, S. Sun, Z. Han, and Q. Li, "Positive solutions to boundary value problems of nonlinear fractional differential equations," *Abstract and Applied Analysis*, vol. 2011, Article ID 390543, 16 pages, 2011.

[12] K. Diethelm and A. D. Freed, "On the solution of nonlinear fractional order differential equations used in the modelling of viscoplasticity," in *SCientific Computing in Chemical Engineering II-Computational Fluid Dynamics, Reaction Engineering and Molecular Properties*, F. Keil, W. Mackens, and H. Voss, Eds., pp. 217–224, Springer, Heidelberg, Germany, 1999.

[13] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, The Netherlands, 2006.

[14] W. Lin, "Global existence theory and chaos control of fractional differential equations," *Journal of Mathematical Analysis and Applications*, vol. 332, no. 1, pp. 709–726, 2007.

[15] I. Podlubny, *Fractional Differential Equations*, vol. 198 of *Mathematics in Science and Engineering*, Academic Press, San Diego, Calif, USA, 1999.

[16] V. Marić, *Regular Variation and Differential Equations*, vol. 1726 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 2000.

[17] E. Seneta, *Regularly Varying Functions*, vol. 508 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1976.