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Research Article

Weighted Composition Operator from Mixed Norm Space to Bloch-Type Space on the Unit Ball

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We discuss the boundedness and compactness of the weighted composition operator from mixed norm space to Bloch-type space on the unit ball of C^n.

1. Introduction

Let $H(B_n)$ be the class of all holomorphic functions on B_n and $S(B_n)$ the collection of all the holomorphic self-mappings of B_n, where B_n is the unit ball in the n-dimensional complex space C^n. Let dv denote the Lebesegue measure on B_n normalized so that $v(B_n) = 1$ and $d\sigma$ the normalized rotation invariant measure on the boundary $S = \partial B_n$ of B_n. For $f \in H(B_n)$, let

$$\Re f(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z) \tag{1}$$

be the radial derivative of f.

A positive continuous function μ on $[0,1)$ is called normal (see, e.g., [1]) if there exist three constants $0 \le \delta < 1$, and $0 < a < b < \infty$, such that for $r \in [\delta, 1)$

$$\frac{\mu(r)}{(1-r)^a} \downarrow 0, \qquad \frac{\mu(r)}{(1-r)^b} \uparrow \infty, \qquad r \longrightarrow 1. \tag{2}$$

In the rest of this paper we always assume that μ is normal on $[0,1)$, and from now on if we say that a function $\mu : B_n \to [0,\infty)$ is normal we will also suppose that it is radial on B_n, that is, $\mu(z) = \mu(|z|)$ for $z \in B_n$.

Let $0 < p \le \infty$, $0 < q \le \infty$, and μ be normal on $[0,1)$. f is said to belong to the mixed norm space $L(p,q,\mu)$ if f is a measurable function on B_n and $\|f\|_{p,q,\mu} < \infty$, where

$$\begin{gathered}
\|f\|_{p,q,\mu} = \left\{\int_0^1 r^{2n-1}(1-r)^{-1}\mu^p(r)\, M_q^p(r,f)\, dr\right\}^{1/p} \\
(0 < p < \infty,\ 0 < q \le \infty), \\
\|f\|_{\infty,q,\mu} = \sup_{0\le r<1} \mu(r)\, M_q(r,f), \\
M_\infty(r,f) = \sup_{\zeta\in S} |f(r\zeta)|, \\
M_q(r,f) = \left\{\int_S |f(r\zeta)|^q d\sigma(\zeta)\right\}^{1/q}, \quad (0 < q < \infty).
\end{gathered} \tag{3}$$

If $0 < p = q < \infty$, then $L(p,q,\mu)$ is just the space $L^p(\mu) = \{f$ is measurable function on $B_n : \int_{B_n} |f(z)|^p(\mu^p(z)/(1-|z|))dv(z) < \infty\}$.

Let $H(p,q,\mu) = L(p,q,\mu) \cap H(B_n)$. If $0 < p = q < \infty$, then $H(p,q,\mu)$ is just the weighted Bergman space $L_a^p(\mu)$. In particular, $H(p,q,\mu)$ is Bergman space $L_a^p(\mu)$ if $0 < p = q < \infty$ and $\mu(r) = (1-r)^{1/p}$. Otherwise, if $p = q = 2$ and $\mu(r) = (1-r)^{\beta/2}$ $(\beta < 0)$, then $H(p,q,\mu(r))$ is the Dirichlet-type space.

For $0 < p, q < \infty$, $-1 < \gamma < 1$, let $\mu(r) = r^{-(2n-1)/p}(1-r)^{(\gamma+1)/p}$; it is easy to see that the mixed norm space $H(p,q,\mu)$, written by $H_{p,q,\gamma}$, consists of all $f \in H(B_n)$ such that

$$\|f\|_{H_{p,q,\gamma}} = \left\{\int_0^1 M_q^p(f,r)(1-r)^\gamma dr\right\}^{1/p} < \infty. \tag{4}$$

Now $f \in H(B_n)$ is said to belong to Bloch-type space $\mathscr{B}_\mu$ if

$$\|f\|_{\mu,1} = \sup_{z\in B_n}\mu(z)\left|\nabla f(z)\right| < \infty, \tag{5}$$

where $\nabla f(z) = (\partial f(z)/\partial z_1, \ldots, \partial f(z)/\partial z_n)$ is the complex gradient of f.

It is clear that $\mathscr{B}_\mu$ is a Banach space with norm $\|f\|_{\mathscr{B}_\mu} = |f(0)| + \|f\|_{\mu,1}$. For $f \in H(B_n)$, we denote

$$\|f\|_{\mu,2} = \sup_{z\in B_n}\mu(z)\left|\Re f(z)\right|, \qquad \|f\|_{\mu,3} = \sup_{z\in B_n} Q_f^\mu(z), \tag{6}$$

where

$$Q_f^\mu(z) = \sup_{u\in C^n\setminus\{0\}} \frac{\left|\langle \nabla f(z), \overline{u}\rangle\right|}{\sqrt{G_z^\mu(u,u)}},$$

$$G_z^\mu(u,u) = \frac{1}{\mu^2(z)}\left\{\frac{\mu^2(z)}{\sigma_\mu^2(|z|)}|u|^2 + \left(1 - \frac{\mu^2(z)}{\sigma_\mu^2(|z|)}\right)\frac{|\langle z,u\rangle|^2}{|z|^2}\right\} \quad (z \neq 0), \tag{7}$$

$$G_0^\mu(u,u) = \frac{|u|^2}{\mu^2(0)},$$

$$\frac{1}{\sigma_\mu(t)} = \frac{1}{\mu(0)} + \int_0^t \frac{d\tau}{(1-\tau)^{1/2}\mu(\tau)} \quad (0 \le t < 1).$$

It was proved that $\|f\|_{\mu,1}$, $\|f\|_{\mu,2}$, and $\|f\|_{\mu,3}$ are equivalent for $f \in \mathscr{B}_\mu(B_n)$ in [2, 3].

Let $\varphi \in S(B_n)$, $\psi \in H(B_n)$; the composition operator C_φ induced by φ is defined by

$$\left(C_\varphi f\right)(z) = f(\varphi(z)), \quad f \in H(B_n), z \in B_n, \tag{8}$$

and the weighted composition operator $T_{\psi,\varphi}$ is defined by

$$T_{\psi,\varphi}(f) = \psi f \circ \varphi \tag{9}$$

for $f \in H(B_n)$. We can regard this operator as a generalization of a multiplication operator M_ψ and a composition operator C_φ. That is, when $\varphi(z) \equiv z$, we obtain $T_{\psi,\varphi}f(z) = M_\psi f(z) = \psi(z)f(z)$ and when $\psi(z) \equiv 1$ we obtain $T_{\psi,\varphi}f(z) = C_\varphi f(z) = f(\varphi(z))$.

It is interesting to provide a function theoretic characterization when ψ and φ induce a bounded or compact weighted composition operator between some spaces of holomorphic functions on B_n. Recently, this operator is well studied by many papers; see, for example, [3–17] and their references therein. In particular, Stević [18] gave some conditions of weighted composition operators between mixed-norm spaces and H_α^∞ spaces on the unit ball. Zhou and Chen [19] discussed weighted composition operators from $F(p,q,s)$ to Bloch-type spaces on the unit ball. More recently, the weighted composition operator from Bers-type space to Bloch-type space on the unit ball was studied in [6]. Now in this paper, we will continue this line of research and characterize the boundedness and compactness of the weighted composition operator $T_{\psi,\varphi}$ acting from mixed-norm spaces $H_{p,q,\gamma}$ to Bloch-type space $\mathscr{B}_\mu$ on the unit ball of C^n. The paper is organized as follows. In Section 2, we give some lemmas. The main results are given in Section 3.

Throughout the remainder of this paper, C will denote a positive constant; the exact value of which will vary from one appearance to the next. The notation $A \asymp B$ means that there is a positive constant C such that $B/C \le A \le CB$.

2. Some Lemmas

Lemma 1. *Assume that $0 < p, q < \infty$, $-1 < \gamma < \infty$, and $f \in H_{p,q,\gamma}$. Then there is a positive constant C which is independent of f such that*

$$\left|f(z)\right| \le C\frac{\|f\|_{H_{p,q,\gamma}}}{\left(1-|z|^2\right)^{n/q+(\gamma+1)/p}}, \tag{10}$$

$$\left|\Re f(z)\right| \le C\frac{\|f\|_{H_{p,q,\gamma}}}{\left(1-|z|^2\right)^{n/q+1+(\gamma+1)/p}}. \tag{11}$$

Proof. We first prove (10). By the monotonicity of the integral means and [20, Theorem 1.12] we have that

$$\begin{aligned}\|f\|_{H_{p,q,\gamma}}^p &\ge \int_{(1+|z|)/2}^{(3+|z|)/4} M_q^p(f,r)(1-r)^\gamma dr \\ &\ge CM_q^p\left(f,\frac{1+|z|}{2}\right)\int_{(1+|z|)/2}^{(3+|z|)/4}(1-r)^\gamma dr \\ &\ge CM_q^p\left(f,\frac{1+|z|}{2}\right)\left(1-|z|^2\right)^{\gamma+1} \\ &\ge C\left(1-|z|^2\right)^{\gamma+1+(pn)/q}\left|f(z)\right|^p,\end{aligned} \tag{12}$$

from which the desired result (10) follows.

Next we prove (11). By the monotonicity of the integral means, using the well-known asymptotic formula (e.g., [21, Theorem 2]), we obtain that

$$\int_0^1 M_q^p(f,r)(1-r)^\gamma dr \asymp \left|f(0)\right|^p + \int_0^1 M_q^p(\Re f,r)(1-r)^{\gamma+p}dr. \tag{13}$$

By [20, Theorem 1.12], it follows that

$$\begin{aligned}\|f\|_{H_{p,q,\gamma}}^{p} &\geq \int_{(1+|z|)/2}^{1} M_q^p(f,r)(1-r)^{\gamma}dr \\ &\geq C\int_{(1+|z|)/2}^{1} M_q^p(\mathfrak{R}f,r)(1-r)^{\gamma+p}dr \\ &\geq CM_q^p\left(\mathfrak{R}f,\frac{1+|z|}{2}\right)\int_{(1+|z|)/2}^{1}(1-r)^{\gamma+p}dr \\ &\geq CM_q^p\left(\mathfrak{R}f,\frac{1+|z|}{2}\right)\left(1-|z|^2\right)^{\gamma+1+p} \\ &\geq C\left(1-|z|^2\right)^{\gamma+1+p+(pn)/q}|\mathfrak{R}f(z)|^p.\end{aligned} \tag{14}$$

Then the desired result (11) follows. This completes the proof. □

From the above lemma, when $f\in H_{p,q,\gamma}$, then

$$f\in\mathscr{B}^{n/q+1+(\gamma+1)/p},\quad \|f\|_{\mathscr{B}^{n/q+(\gamma+1)/p+1}}\leq C\|f\|_{H_{p,q,\gamma}}. \tag{15}$$

For $z\in B_n$, $u\in C^n$, denote the Bergman metric of B_n by

$$H_z(u,u)=\frac{\left(1-|z|^2\right)|u|^2+|\langle z,u\rangle|^2}{\left(1-|z|^2\right)^2}. \tag{16}$$

Lemma 2. *Let $v(r)=(1-r^2)^{n/q+(\gamma+1)/p+1}$ and $\varphi\in S(B_n)$. Then*

$$G_{\varphi(z)}^{v}(J\varphi(z)z,J\varphi(z)z)\leq\frac{CH_{\varphi(z)}(J\varphi(z)z,J\varphi(z)z)}{\left(1-|\varphi(z)|^2\right)^{2(n/q+(\gamma+1)/p)}} \tag{17}$$

for all $z\in B_n$, where $J\varphi(z)$ denotes the Jacobian matrix of $\varphi(z)$ and

$$J\varphi(z)z=\left(\sum_{k=1}^{n}\frac{\partial\varphi_1}{\partial z_k}z_k,\ldots,\sum_{k=1}^{n}\frac{\partial\varphi_n}{\partial z_k}z_k\right)^T. \tag{18}$$

Proof. Let $\alpha=n/q+(\gamma+1)/p$. If $\varphi(z)=0$, the desired result is obvious. If $\varphi(z)\neq0$, from the definition of σ_v,

$$\frac{1}{\sigma_v(r)}=1+\int_0^r\frac{dt}{(1-t)^{1/2}(1-t^2)^{\alpha+1}}\asymp\frac{\left(1-r^2\right)^{1/2}}{v(r)},\quad 0\leq r<1. \tag{19}$$

Thus

$$\begin{aligned}&G_{\varphi(z)}^{v}(J\varphi(z),J\varphi(z)z)\\ &=\frac{1}{v^2(|\varphi(z)|)}\times\left[\frac{v^2(|\varphi(z)|)}{\sigma_v^2(|\varphi(z)|)}|J\varphi(z)z|^2+\left(1-\frac{v^2(|\varphi(z)|)}{\sigma_v^2(|\varphi(z)|)}\right)\frac{|\langle\varphi(z),J\varphi(z)z\rangle|^2}{|\varphi(z)|^2}\right]\\ &=\frac{1}{v^2(|\varphi(z)|)}\times\left[\frac{v^2(|\varphi(z)|)}{\sigma_v^2(|\varphi(z)|)}\left(|J\varphi(z)z|^2-\frac{|\langle\varphi(z),J\varphi(z)z\rangle|^2}{|\varphi(z)|^2}\right)+\frac{|\langle\varphi(z),J\varphi(z)z\rangle|^2}{|\varphi(z)|^2}\right]\\ &\leq\frac{C}{v^2(|\varphi(z)|)}\times\left[\left(1-|\varphi(z)|^2\right)\left(|J\varphi(z)z|^2-\frac{|\langle\varphi(z),J\varphi(z)z\rangle|^2}{|\varphi(z)|^2}\right)+\frac{|\langle\varphi(z),J\varphi(z)z\rangle|^2}{|\varphi(z)|^2}\right]\\ &=\frac{C}{v^2(|\varphi(z)|)}\times\left[\left(1-|\varphi(z)|^2\right)\left(|J\varphi(z)z|^2+|\langle\varphi(z),J\varphi(z)z\rangle|^2\right)\right]\\ &=\frac{C\left(1-|\varphi(z)|^2\right)^2}{v^2(|\varphi(z)|)}H_{\varphi(z)}(J\varphi(z)z,J\varphi(z)z)\\ &=\frac{CH_{\varphi(z)}(J\varphi(z)z,J\varphi(z)z)}{\left(1-|\varphi(z)|^2\right)^{2(n/q+(\gamma+1)/p)}}.\end{aligned} \tag{20}$$

The desired result follows from (20). The proof is completed. □

The proof of the next lemma is standard; see, for example, [4, Proposition 3.11]. Hence, it is omitted.

Lemma 3. *Assume that $0<p,q<\infty$, $-1<\gamma<\infty$, μ is a normal function, and $\varphi\in S(B_n)$, $\psi\in H(B_n)$. Then $T_{\psi,\varphi}:H_{p,q,\gamma}\to\mathscr{B}_\mu$ is compact if and only if for any bounded sequence $\{f_k\}_{k\in\mathbb{N}}$ in $H_{p,q,\gamma}$ which converges to zero uniformly on compact subsets of B_n as $k\to\infty$; then $\|T_{\psi,\varphi}f_k\|_{\mathscr{B}_\mu}\to0$, as $k\to\infty$.*

Lemma 4. *For $\beta > -1$ and $m > 1 + \beta$, one has*

$$\int_0^1 \frac{(1-r)^\beta}{(1-\rho r)^m} dr \le C(1-\rho)^{1+\beta-m}, \quad 0 < \rho < 1. \tag{21}$$

Proof.

$$\begin{aligned}\int_0^1 \frac{(1-r)^\beta}{(1-\rho r)^m} dr &= \int_0^1 \frac{(1-r)^\beta}{(1-\rho r)^{m-\beta}(1-\rho r)^\beta} dr \\ &\le \int_0^1 \frac{(1-r)^\beta}{(1-\rho r)^{m-\beta}(1-r)^\beta} dr \\ &= \int_0^1 \frac{1}{(1-\rho r)^{m-\beta}} dr \\ &= \frac{1}{p(1+\beta-m)}(1-\rho)^{1+\beta-m} \\ &= C(1-\rho)^{1+\beta-m}.\end{aligned} \tag{22}$$

This completes the proof. □

3. The Boundedness and Compactness of $T_{\psi,\varphi}: H_{p,q,\gamma} \to \mathscr{B}_\mu$

Theorem 5. *Assume that $0 < p, q < \infty$, $-1 < \gamma < \infty$, μ is a normal function, and $\varphi \in S(B_n)$, $\psi \in H(B_n)$. Then $T_{\psi,\varphi} : H_{p,q,\gamma} \to \mathscr{B}_\mu$ is bounded if and only if*

$$M_1 := \sup_{z\in B_n} \frac{\mu(z)\left|\Re\psi(z)\right|}{\left(1-|\varphi(z)|^2\right)^{n/q+(\gamma+1)/p}} < \infty, \tag{23}$$

$$\begin{aligned}M_2 := &\sup_{z\in B_n} \frac{\mu(z)\left|\psi(z)\right|}{\left(1-|\varphi(z)|^2\right)^{n/q+(\gamma+1)/p}} \\ &\times \left\{H_{\varphi(z)}(J\varphi(z)z, J\varphi(z)z)\right\}^{1/2} < \infty.\end{aligned} \tag{24}$$

Proof

Sufficiency. Assume that (23) and (24) hold. Then for any $f \in H_{p,q,\gamma}$, if $J\varphi(z)z \ne 0$ for $z \in B_n$, by Lemma 1 and Lemma 2, it follows that

$$\begin{aligned}&\left\|T_{\psi,\varphi} f(z)\right\|_{\mathscr{B}_\mu} \\ &\quad = \sup_{z\in B_n} \mu(z)\left|\Re\left(T_{\psi,\varphi} f\right)(z)\right| \\ &\le \sup_{z\in B_n} \mu(z)\left|\Re\psi(z)\right|\left|f(\varphi(z))\right| \\ &\quad + \sup_{z\in B_n} \mu(z)\left|\psi(z)\right|\left|\Re(f\circ\varphi)(z)\right| \\ &\le \sup_{z\in B_n} \frac{\mu(z)\left|\Re\psi(z)\right|\|f\|_{H_{p,q,\gamma}}}{\left(1-|\varphi(z)|^2\right)^{n/q+(\gamma+1)/p}} \\ &\quad + \sup_{z\in B_n} \mu(z)\left|\psi(z)\right|\left|\left\langle \nabla f(\varphi(z)), \overline{J\varphi(z)z}\right\rangle\right| \\ &\le M_1\|f\|_{H_{p,q,\gamma}} \\ &\quad + \sup_{z\in B_n}\Bigg(\Big(C\mu(z)\left|\psi(z)\right|\left\{H_{\varphi(z)}(J\varphi(z)z, J\varphi(z)z)\right\}^{1/2} \\ &\qquad\qquad \times \left|\left\langle \nabla f(\varphi(z)), \overline{J\varphi(z)z}\right\rangle\right|\Big) \\ &\qquad\qquad \times \Big(\left(1-|\varphi(z)|^2\right)^{q/n+(\gamma+1)/p} \\ &\qquad\qquad\quad \times \sqrt{G^{\nu}_{\varphi(z)}(J\varphi(z)z, J\varphi(z)z)}\Big)^{-1}\Bigg) \\ &\le M_1\|f\|_{H_{p,q,\gamma}} + CM_2\|f\|_{\mathscr{B}_{(1-r^2)^{q/n+(\gamma+1)/p+1}}} \le C\|f\|_{H_{p,q,\gamma}}.\end{aligned} \tag{25}$$

When $J\varphi(z)z = 0$ for $z \in B_n$. From (23) we can easily obtain

$$\mu(z)\left|\Re\left(T_{\psi,\varphi}(f)\right)(z)\right| \le M_1\|f\|_{H_{p,q,\gamma}}. \tag{26}$$

Combining (25) and (26), the boundedness of $T_{\psi,\varphi} : H_{p,q,\gamma} \to \mathscr{B}_\mu$ follows.

Necessity. Suppose that $T_{\psi,\varphi} : H_{p,q,\gamma} \to \mathscr{B}_\mu$ is bounded. Firstly, we assume that $w \in B_n$ and $\varphi(w) = r_w e_1$, where $r_w = |\varphi(w)|$ and $e_1 = (1, 0, 0, \ldots, 0)$.

If $\sqrt{(1-r_w^2)(|\eta_2|^2 + \cdots + |\eta_n|^2)} \le |\eta_1|$, where $J\varphi(w)w = (\eta_1, \ldots, \eta_n)^T$, choose the function

$$f_w(z) = \frac{z_1 - r_w}{1 - r_w z_1}\left(\frac{1-r_w^2}{(1-r_w z_1)^2}\right)^{n/q+(\gamma+1)/p}. \tag{27}$$

By [20, Theorem 1.12] and Lemma 4 we have that

$$\begin{aligned}M_q(f_w, r) &= \left(\int_S |f_w(r\zeta)|^q d\sigma(\zeta)\right)^{1/q} \\ &\le \left(\int_S \left(\frac{1-r_w^2}{(1-r_w r\zeta_1)^2}\right)^{n+q(\gamma+1)/p} d\sigma(\zeta)\right)^{1/q} \\ &\le C\frac{(1-r_w^2)^{n/q+(\gamma+1)/p}}{(1-rr_w^2)^{n/q+2(\gamma+1)/p}},\end{aligned}$$

$$\begin{aligned}\|f_w\|^p_{H_{p,q,\gamma}} &= \int_0^1 M_q^p(f_w,r)(1-r)^\gamma dr\\ &\le C\left(1-r_w^2\right)^{pn/q+\gamma+1}\int_0^1\frac{(1-r)^\gamma}{\left(1-rr_w^2\right)^{pn/q+2(\gamma+1)}}dr\\ &\le C\left(1-r_w^2\right)^{pn/q+\gamma+1}\frac{1}{\left(1-r_w^2\right)^{pn/q+\gamma+1}}\le C.\end{aligned} \tag{28}$$

Then $f_w\in H_{p,q,\gamma}$ and $\|f_w\|_{H_{p,q,\gamma}}\le C$. Moreover, $f_w(\varphi(w))=0$ and

$$\nabla f_w(\varphi(w))=\left(\frac{1}{\left(1-r_w^2\right)^{n/q+(\gamma+1)/p+1}},0,\dots,0\right). \tag{29}$$

Thus

$$\begin{aligned}\left\|T_{\psi,\varphi}f_w\right\|_{\mathscr{B}_\mu} &\ge \mu(w)\left|\Re(\psi f\circ\varphi)(w)\right|\\ &\ge \mu(w)|\psi(w)|\left|\Re(f\circ\varphi)(w)\right|\\ &\quad-\mu(w)\left|\Re\psi(w)\right|\left|f_w(\varphi(w))\right|\\ &=\mu(w)|\psi(w)|\left|\left\langle\nabla f_w(\varphi(w)),\overline{J\varphi(w)w}\right\rangle\right|\\ &=\frac{\mu(w)|\psi(w)||\eta_1|}{\left(1-r_w^2\right)^{n/q+(\gamma+1)/p+1}}.\end{aligned} \tag{30}$$

By the definition of $H_{\varphi(w)}(J\varphi(w)w,J\varphi(w)w)$ and (30) it follows that

$$\begin{aligned}&\frac{\mu(w)|\psi(w)|\left\{H_{\varphi(w)}(J\varphi(w)w,J\varphi(w)w)\right\}^{1/2}}{\left(1-|\varphi(w)|^2\right)^{n/q+(\gamma+1)/p}}\\ &=\Big(\mu(w)|\psi(w)|\\ &\quad\times\left\{\left(1-|\varphi(w)|^2\right)|J\varphi(w)w|^2+\left|\langle\varphi(w),J\varphi(w)w\rangle\right|^2\right\}^{1/2}\Big)\\ &\quad\times\left(\left(1-|\varphi(w)|^2\right)^{n/q+(\gamma+1)/p+1}\right)^{-1}\\ &=\frac{\mu(w)|\psi(w)|\left\{\left(1-r_w^2\right)\left(|\eta_2|^2+\cdots+|\eta_n|^2\right)+|\eta_1|^2\right\}^{1/2}}{\left(1-|\varphi(w)|^2\right)^{n/q+(\gamma+1)/p+1}}\\ &\le\frac{\sqrt{2}\mu(w)|\psi(w)||\eta_1|}{\left(1-r_w^2\right)^{n/q+(\gamma+1)/p+1}}\le C\left\|T_{\psi,\varphi}f_w\right\|_{\mathscr{B}_\mu}\le C.\end{aligned} \tag{31}$$

This shows that when $\sqrt{(1-r_w^2)(|\eta_2|^2+\cdots+|\eta_n|^2)}\le|\eta_1|$, (24) follows.

On the other hand, if $\sqrt{(1-r_w^2)(|\eta_2|^2+\cdots+|\eta_n|^2)}>|\eta_1|$. For $j=2,\dots,n$, let $\theta_j=\arg\eta_j$ and $a_j=e^{-i\theta_j}$, when $\eta_j\ne0$; otherwise $a_j=0$ when $\eta_j=0$. Take

$$f_w(z)=\frac{a_2z_2+\cdots+a_nz_n}{\left(1-r_wz_1\right)^{n/q+(\gamma+1)/p+1}}. \tag{32}$$

By [20, Theorem 1.12] and Lemma 4 we obtain that

$$\begin{aligned}M_q(f_w,r)&\le\left\{\int_S\frac{\left(|\zeta_2|+\cdots+|\zeta_n|\right)^q}{\left|1-r_wr\zeta_1\right|^{n+q(\gamma+1)/p+q}}d\sigma(\zeta)\right\}^{1/q}\\ &\le\left\{\int_S\frac{C\left(|\zeta_2|^2+\cdots+|\zeta_n|^2\right)^{q/2}}{\left|1-r_wr\zeta_1\right|^{n+q(\gamma+1)/p+q}}d\sigma(\zeta)\right\}^{1/q}\\ &=\left\{\int_S\frac{C\left(1-|\zeta_1|^2\right)^{q/2}}{\left|1-r_wr\zeta_1\right|^{n+q(\gamma+1)/p+q}}d\sigma(\zeta)\right\}^{1/q}\\ &\le C\left\{\int_S\frac{1}{\left|1-r_wr\zeta_1\right|^{n+q(\gamma+1)/p+q/2}}d\sigma(\zeta)\right\}^{1/q}\\ &\le\frac{C}{\left(1-rr_w^2\right)^{(\gamma+1)/p+1/2}}.\\ \|f_w\|^p_{H_{p,q,\gamma}}&=\int_0^1M_q^p(f_w,r)(1-r)^\gamma dr\\ &\le C\int_0^1\frac{(1-r)^\gamma}{\left(1-rr_w^2\right)^{\gamma+1+p/2}}dr\\ &\le C\left(1-r_w^2\right)^{p/2}\le C.\end{aligned} \tag{33}$$

Hence $f_w\in H_{p,q,\gamma}$ and $\|f_w\|_{H_{p,q,\gamma}}\le C$. Moreover $f_w(\varphi(w))=0$ and

$$\begin{aligned}&\nabla f_w(\varphi(w))\\ &=\left(0,\frac{a_2}{\left(1-r_w^2\right)^{n/q+(\gamma+1)/p+1}},\dots,\frac{a_n}{\left(1-r_w^2\right)^{n/q+(\gamma+1)/p+1}}\right).\end{aligned} \tag{34}$$

Similar to the proof of (30), we obtain that

$$\frac{\mu(w)|\psi(w)|\left(|\eta_2|+\cdots+|\eta_n|\right)}{\left(1-r_w^2\right)^{n/q+(\gamma+1)/p+1}}\le C\left\|T_{\psi,\varphi}f_w\right\|_{\mathscr{B}_\mu}. \tag{35}$$

It follows from (35) that

$$\begin{aligned}&\frac{\mu(w)|\psi(w)|}{\left(1-|\varphi(w)|^2\right)^{n/q+(\gamma+1)/p}}\left\{H_{\varphi(w)}(J\varphi(w)w,J\varphi(w)w)\right\}^{1/2}\\ &=\Big(\mu(w)|\psi(w)|\\ &\quad\times\left\{\left(1-|\varphi(w)|^2\right)|J\varphi(w)w|^2+\left|\langle\varphi(w),J\varphi(w)w\rangle\right|^2\right\}^{1/2}\Big)\end{aligned}$$

$$
\begin{aligned}
&\times\left(\left(1-|\varphi(w)|^2\right)^{n/q+(\gamma+1)/p+1}\right)^{-1}\\
&=\frac{\mu(w)|\psi(w)|\left\{\left(1-r_w^2\right)\left(|\eta_2|^2+\cdots+|\eta_n|^2\right)+|\eta_1|^2\right\}^{1/2}}{\left(1-|\varphi(w)|^2\right)^{n/q+(\gamma+1)/p+1}}\\
&\leq\frac{\mu(w)|\psi(w)|\left\{2\left(1-r_w^2\right)\left(|\eta_2|^2+\cdots+|\eta_n|^2\right)\right\}^{1/2}}{\left(1-|\varphi(w)|^2\right)^{n/q+(\gamma+1)/p+1}}\\
&\leq C\frac{\mu(w)|\psi(w)|\sqrt{2\left(1-r_w^2\right)}\left(|\eta_2|+\cdots+|\eta_n|\right)}{\left(1-r_w^2\right)^{n/q+(\gamma+1)/p+1}}\\
&\leq\left\|T_{\psi,\varphi}f_w\right\|_{\mathscr{B}_\mu}.
\end{aligned}
\tag{36}
$$

That is, when $\sqrt{(1-r_w^2)(|\eta_2|^2+\cdots+|\eta_n|^2)} > |\eta_1|$, (24) follows. Combining the above two cases, the desired result (24) holds.

For the general situation, we can use some unitary transform U_w to make $\varphi(w)=r_we_1U_w$ and we can prove (11) by taking the function $g_w=f_w\circ U_w^{-1}$. By the linearity of the unitary transform U_w, $|\zeta|=|U_w^{-1}\zeta|$, and $d\sigma$ the normalized rotation invariant measure on the boundary S, we get that

$$
\begin{aligned}
\|g_w\|_{H_{p,q,\gamma}}^p&=\int_0^1\left(\int_S|g_w(r\zeta)|^qd\sigma(\zeta)\right)^{p/q}(1-r)^\gamma dr\\
&=\int_0^1\left(\int_S\left|f_w\left(U_w^{-1}(r\zeta)\right)\right|^qd\sigma(\zeta)\right)^{p/q}(1-r)^\gamma dr\\
&=\int_0^1\left(\int_S\left|f_w\left(rU_w^{-1}(\zeta)\right)\right|^qd\sigma(\zeta)\right)^{p/q}(1-r)^\gamma dr\\
&=\int_0^1\left(\int_S\left|f_w(r\eta)\right|^qd\sigma(\eta)\right)^{p/q}(1-r)^\gamma dr\\
&=\|f_w\|_{H_{p,q,\gamma}}^p.
\end{aligned}
\tag{37}
$$

Next we prove (23). Set the function

$$
h_w(z)=\frac{\left(1-|w|^2\right)^{b-(\gamma+1)/p}}{(1-\langle z,w\rangle)^{n/q+b}}
\tag{38}
$$

for fixed $w\in B_n$ and $b>(\gamma+1)/p$. Then,

$$
\begin{aligned}
M_q(h_w(z),r)&=\left(\int_{\partial B_n}|h_w(r\zeta)|^qd\sigma(\zeta)\right)^{1/q}\\
&=\left(\int_{\partial B_n}\frac{\left(1-|w|^2\right)^{(b-(\gamma+1)/p)q}}{|1-\langle r\zeta,w\rangle|^{(n/q+b)q}}d\sigma(\zeta)\right)^{1/q}.
\end{aligned}
\tag{39}
$$

By [20, Theorem 1.12], it follows that

$$
M_p(h_w(z),r)\leq\frac{\left(1-|w|^2\right)^{b-(\gamma+1)/p}}{\left(1-r|w|^2\right)^b}.
\tag{40}
$$

Applying Lemma 4 we have that

$$
\begin{aligned}
\|h_w\|_{H_{p,q,\gamma}}^q&=\int_0^1M_q^p(h_w,r)(1-r)^\gamma dr\\
&\leq C\int_0^1\frac{\left(1-|w|^2\right)^{pb-(\gamma+1)}}{\left(1-r|w|^2\right)^{pb}}(1-r)^\gamma dr\\
&=C\left(1-|w|^2\right)^{pb-(\gamma+1)}\int_0^1\frac{(1-r)^\gamma}{\left(1-r|w|^2\right)^{pb}}dr\\
&\leq C\left(1-|w|^2\right)^{pb-(\gamma+1)}\left(1-|w|^2\right)^{(\gamma+1)-pb}=C.
\end{aligned}
\tag{41}
$$

Therefore $h_w\in H_{p,q,\gamma}$, and $\sup_{w\in B_n}\|h_w\|_{H_{p,q,\gamma}}\leq C$. Besides,

$$
h_{\varphi(w)}(\varphi(w))=\left(\frac{1}{1-|\varphi(w)|^2}\right)^{n/q+(\gamma+1)/p},
\tag{42}
$$

$$
\begin{aligned}
&\nabla h_{\varphi(w)}(\varphi(w))\\
&\quad=\left(\frac{n}{q}+b\right)\left(\frac{\overline{\varphi_1(w)}}{\left(1-|\varphi(w)|^2\right)^{n/q+(\gamma+1)/p+1}},\ldots,\right.\\
&\qquad\left.\frac{\overline{\varphi_n(w)}}{\left(1-|\varphi(w)|^2\right)^{n/q+(\gamma+1)/p+1}}\right).
\end{aligned}
\tag{43}
$$

Therefore,

$$
\begin{aligned}
\infty&>\left\|T_{\psi,\varphi}\left(h_{\varphi(w)}\right)\right\|_{\mathscr{B}_\mu}\geq\mu(w)\left|\Re\left(\psi h_{\varphi(w)}\circ\varphi\right)(w)\right|\\
&=\mu(w)\left|\Re\psi(w)h_{\varphi(w)}(\varphi(w))+\psi(w)\Re\left(h_{\varphi(w)}\circ\varphi\right)(w)\right|\\
&\geq\frac{\mu(w)\left|\Re\psi(w)\right|}{\left(1-|\varphi(w)|^2\right)^{n/q+(\gamma+1)/p}}\\
&\quad-\mu(w)|\psi(w)|\left|\Re\left(h_{\varphi(w)}\circ\varphi\right)(w)\right|.
\end{aligned}
\tag{44}
$$

It follows from (43) and (24) that

$$\begin{aligned}
&\mu(w)|\psi(w)|\left|\Re\left(h_{\varphi(w)}\circ\varphi\right)(w)\right| \\
&= \mu(w)|\psi(w)|\left|\left\langle \nabla h_{\varphi(w)}(\varphi(w)), \overline{J\varphi(w)w}\right\rangle\right| \\
&= \left(\frac{n}{q}+b\right)\frac{\mu(w)|\psi(w)||\langle\varphi(w), J\varphi(w)w\rangle|}{\left(1-|\varphi(w)|^2\right)^{n/q+(\gamma+1)/p+1}} \\
&\le \left(\frac{n}{q}+b\right)\frac{\mu(w)|\psi(w)|}{\left(1-|\varphi(w)|^2\right)^{n/q+(\gamma+1)/p}} \\
&\quad\times\left\{H_{\varphi(w)}(J\varphi(w)w, J\varphi(w)w)\right\}^{1/2} \\
&\le CM_2 < \infty.
\end{aligned} \tag{45}$$

Combining (44) and (45), the desired result (23) holds. This completes the proof. □

Theorem 6. *Assume that* $0 < p, q < \infty$, $-1 < \gamma < \infty$, μ *is a normal function, and* $\varphi \in S(B_n)$, $\psi \in H(B_n)$. *Then* $T_{\psi,\varphi} : H_{p,q,\gamma} \to \mathscr{B}_\mu$ *is compact if and only if the followings are all satisfied:*

(a) $\psi \in \mathscr{B}_\mu$ *and* $\psi\varphi_l \in \mathscr{B}_\mu$ *for* $l \in \{1, \ldots, n\}$;

(b)

$$\lim_{|\varphi(z)|\to 1}\frac{\mu(z)|\Re\psi(z)|}{\left(1-|\varphi(z)|^2\right)^{n/q+(\gamma+1)/p}} = 0; \tag{46}$$

(c)

$$\begin{aligned}
&\lim_{|\varphi(z)|\to 1}\frac{\mu(z)|\psi(z)|}{\left(1-|\varphi(z)|^2\right)^{n/q+(\gamma+1)/p}} \\
&\quad\times\left\{H_{\varphi(z)}(J\varphi(z)z, J\varphi(z)z)\right\}^{1/2} = 0.
\end{aligned} \tag{47}$$

Proof

Sufficiency. Suppose that (a), (b), and (c) hold. Then for any $\varepsilon > 0$, there is $\delta > 0$, such that

$$\begin{gathered}
\frac{\mu(z)|\Re\psi(z)|}{\left(1-|\varphi(z)|^2\right)^{n/q+(\gamma+1)/p}} < \varepsilon, \\
\frac{\mu(z)|\psi(z)|}{\left(1-|\varphi(z)|^2\right)^{n/q+(\gamma+1)/p}}\left\{H_{\varphi(z)}(J\varphi(z)z, J\varphi(z)z)\right\}^{1/2} < \varepsilon,
\end{gathered} \tag{48}$$

when $|\varphi(z)| > \delta$.

Let $\{f_k\}_{k\in\mathbb{N}}$ be any sequence which converges to 0 uniformly on compact subsets of B_n satisfying $\|f_k\|_{H_{p,q,\gamma}} \le 1$. Then f_k and $\Re f_k$ converge to 0 uniformly on $K = \{w \in B_n : |w| \le \delta\}$. Hence

$$\begin{aligned}
&\sup_{z\in B_n}\mu(z)\left|\Re\left(T_{\psi,\varphi}f_k\right)(z)\right| \\
&\quad= \sup_{\varphi(z)\in K}\mu(z)\left|\Re\left(T_{\psi,\varphi}f_k\right)(z)\right| \\
&\qquad+ \sup_{\varphi(z)\in B_n\setminus K}\mu(z)\left|\Re\left(T_{\psi,\varphi}f_k\right)(z)\right|.
\end{aligned} \tag{49}$$

If $\varphi(z) \in B_n \setminus K$ and $J\varphi(z)z \ne 0$, by Lemma 1 and Lemma 2, we have

$$\begin{aligned}
&\mu(z)\left|\Re\left(T_{\psi,\varphi}f_k\right)(z)\right| \\
&\le \mu(z)|\psi(z)||\Re(f_k\circ\varphi)(z)| + \mu(z)|\Re\psi(z)||f_k(\varphi(z))| \\
&\le \Big(C\mu(z)|\psi(z)|\left\{H_{\varphi(z)}(J\varphi(z)z, J\varphi(z)z)\right\}^{1/2} \\
&\qquad\times\left|\left\langle\nabla f_k(\varphi(z)), \overline{J\varphi(z)z}\right\rangle\right|\Big) \\
&\quad\times\left(\left(1-|\varphi(z)|^2\right)^{n/q+(\gamma+1)/p}\sqrt{G^{\nu}_{\varphi(z)}(J\varphi(z)z, J\varphi(z)z)}\right)^{-1} \\
&\quad+\varepsilon\|f_k\|_{H_{p,q,\gamma}} \\
&\le C\varepsilon\|f_k\|_{\mathscr{B}_{(1-r^2)^{n/q+(\gamma+1)/p+1}}} + \varepsilon\|f_k\|_{H_{p,q,\gamma}} \le C\varepsilon.
\end{aligned} \tag{50}$$

When $J\varphi(z)z = 0$,

$$\mu(z)\left|\Re\left(T_{\psi,\varphi}f_k\right)(z)\right| \le \varepsilon\|f_k\|_{H_{p,q,\gamma}} \le \varepsilon. \tag{51}$$

Combining (50) and (51) we obtain that

$$\sup_{\varphi(z)\in B_n\setminus K}\mu(z)\left|\Re\left(T_{\psi,\varphi}f_k\right)(z)\right| \le C\varepsilon. \tag{52}$$

If $\varphi(z) \in K$, by (a), we have that

$$\begin{aligned}
&\mu(z)\left|\Re\left(T_{\psi,\varphi}f_k\right)(z)\right| \\
&\le \mu(z)|\psi(z)||\Re(f_k\circ\varphi)(z)| + \mu(z)|\Re\psi(z)||f_k(\varphi(z))| \\
&\le \mu(z)|\psi(z)|\left|\left\langle\nabla f_k(\varphi(z)), \overline{J\varphi(z)z}\right\rangle\right| \\
&\quad+|f_k(\varphi(z))|\,\|\psi\|_{\mathscr{B}_\mu} \\
&\le |\nabla f_k(\varphi(z))|\sum_{l=1}^{n}\left(\mu(z)|\psi(z)||\Re\varphi_l(z)|\right) \\
&\quad+|f_k(\varphi(z))|\,\|\psi\|_{\mathscr{B}_\mu}
\end{aligned}$$

$$\begin{aligned}
&\le |\nabla f_k(\varphi(z))| \\
&\quad \times \sum_{l=1}^{n} \left(\mu(z)|\psi(z)||\Re\varphi_l(z)| - \mu(z)|\varphi_l(z)||\Re\psi(z)|\right. \\
&\qquad \left. +\mu(z)|\Re\psi(z)|\right) + |f_k(\varphi(z))| \|\psi\|_{\mathscr{B}_\mu} \\
&\le |\nabla f_k(\varphi(z))| \\
&\quad \times \sum_{l=1}^{n} \left(\mu(z)|\psi(z)\Re\varphi_l(z) + \Re\psi(z)\varphi_l(z)|\right. \\
&\qquad \left. +\mu(z)|\Re\psi(z)|\right) + |f_k(\varphi(z))| \|\psi\|_{\mathscr{B}_\mu} \\
&\le |\nabla f_k(\varphi(z))| \sum_{l=1}^{n} \left(\|\psi\varphi_l\|_{\mathscr{B}_\mu} + \|\psi\|_{\mathscr{B}_\mu}\right) \\
&\quad + |f_k(\varphi(z))| \|\psi\|_{\mathscr{B}_\mu} \\
&\longrightarrow 0, \quad k \longrightarrow \infty.
\end{aligned} \tag{53}$$

Combining (49), (52), (53), and Lemma 4, it follows that the $T_{\psi,\varphi} : H_{p,q,\gamma} \to \mathscr{B}_\mu$ is compact.

Necessity. Assume that $T_{\psi,\varphi} : H_{p,q,\gamma} \to \mathscr{B}_\mu$ is compact. It is obvious that $T_{\psi,\varphi} : H_{p,q,\gamma} \to \mathscr{B}_\mu$ is bounded. Then taking $f(z) = 1 \in H_{p,q,\gamma}$ and by the boundedness of $T_{\psi,\varphi} : H_{p,q,\gamma} \to \mathscr{B}_\mu$, it follows that

$$\begin{aligned}
&\|T_{\psi,\varphi} f(z)\|_{\mathscr{B}_\mu} \\
&\quad = \sup_{z\in B_n} \mu(z) \left|\Re\left(T_{\psi,\varphi} f\right)(z)\right| \\
&\quad = \sup_{z\in B_n} \mu(z) \left|\Re\psi(z) f(\varphi(z)) + \psi(z)\Re(f\circ\varphi)(z)\right| \\
&\quad = \sup_{z\in B_n} \mu(z) |\Re\psi(z)| < \infty.
\end{aligned} \tag{54}$$

This shows that $\psi \in \mathscr{B}_\mu$.

On the other hand, for $l \in \{1, \ldots, n\}$, take the function $f(z) = z_l \in H_{p,q,\gamma}$. By the boundedness of $T_{\psi,\varphi} : H_{p,q,\gamma} \to \mathscr{B}_\mu$, we get that

$$\begin{aligned}
&\|T_{\psi,\varphi} f(z)\|_{\mathscr{B}_\mu} \\
&\quad = \sup_{z\in B_n} \mu(z) \left|\Re\psi(z) f(\varphi(z)) + \psi(z)\Re(f\circ\varphi)(z)\right| \\
&\quad = \sup_{z\in B_n} \mu(z) \left|\Re\psi(z)\varphi_l(z) + \psi(z)\Re\varphi_l(z)\right| \\
&\quad = \sup_{z\in B_n} \mu(z) \left|\Re(\psi\varphi_l)(z)\right| < \infty.
\end{aligned} \tag{55}$$

That is, $\psi\varphi_l \in \mathscr{B}_\mu$ for $l \in \{1, \ldots, n\}$. Hence we obtain (a).

Next we prove (b) and (c). Let $\{z_k\}_{k\in\mathbb{N}}$ be a sequence in B_n such that $|\varphi(z_k)| \to 1$ as $k \to \infty$. We can still suppose $\varphi(z_k) = r_k e_1$, where $r_k = |\varphi(z_k)|$ and e_1 is the vector $(1, 0, 0, \ldots, 0)$. That is, $|r_k| \to 1$, $k \to \infty$.

If $\sqrt{(1-r_k^2)(|\eta_2|^2 + \cdots + |\eta_n|^2)} \le |\eta_1|$, where $J\varphi(z_k)z_k = (\eta_1, \ldots, \eta_n)^T$. Let

$$f_k(z) = \frac{z_1 - r_k}{1 - r_k z_1} \left\{\frac{1 - r_k^2}{(1 - r_k z_1)^2}\right\}^{n/q+(\gamma+1)/p}. \tag{56}$$

From Theorem 5 we know that $f_k \in H_{p,q,\gamma}$, and we notice that f_k converges to 0 uniformly on compact subsets of B_n when $k \to \infty$. By Lemma 3 we have $\lim_{k\to\infty}\|T_{\psi,\varphi} f_k(z)\|_{\mathscr{B}_\mu} = 0$. Then by a similar proof of (30) in Theorem 5 we have

$$\frac{\mu(z_k)|\psi(z_k)||\eta_1|}{(1-r_k^2)^{n/q+(\gamma+1)/p+1}} \le \|T_{\psi,\varphi} f_k(z)\|_{\mathscr{B}_\mu} \longrightarrow 0, \quad k \longrightarrow \infty. \tag{57}$$

And similar to the proofs of (31) and (57) we get that

$$\begin{aligned}
&\frac{\mu(z_k)|\psi(z_k)|}{\left(1 - |\varphi(z_k)|^2\right)^{n/q+(\gamma+1)/p}} \left\{H_{\varphi(z_k)}\left(J\varphi(z_k)z_k, J\varphi(z_k)z_k\right)\right\}^{1/2} \\
&\quad \le \frac{\sqrt{2}\mu(z_k)|\psi(z_k)||\eta_1|}{(1-r_k^2)^{n/q+(\gamma+1)/p+1}} \longrightarrow 0, \quad k \longrightarrow \infty.
\end{aligned} \tag{58}$$

On the other hand, we consider the case of $\sqrt{(1-r_k^2)(|\eta_2|^2 + \cdots + |\eta_n|^2)} > |\eta_1|$. For $j = 2, \ldots, n$, let $\theta_j = \arg\eta_j$ and $a_j = e^{-i\theta_j}$, when $\eta_j \ne 0$; otherwise $a_j = 0$ when $\eta_j = 0$. Take

$$f_k(z) = \frac{(a_2 z_2 + \cdots + a_n z_n)\left(1 - r_k^2\right)}{(1 - r_k z_1)^{n/q+(\gamma+1)/p+2}}. \tag{59}$$

Then $f_k \in H_{p,q,\gamma}$, $k \in \mathbb{N}$, and f_k converges to 0 uniformly on compact subsets of B_n when $k \to \infty$. By Lemma 3 we have $\lim_{k\to\infty}\|T_{\psi,\varphi} f_k(z)\|_{\mathscr{B}_\mu} = 0$. Notice that $f_k(\varphi(z_k)) = 0$ and

$$\begin{aligned}
&\nabla f_w(\varphi(z_k)) \\
&\quad = \left(0, \frac{a_2}{(1-r_k^2)^{n/q+(\gamma+1)/p+1}}, \ldots, \frac{a_n}{(1-r_k^2)^{n/q+(\gamma+1)/p+1}}\right).
\end{aligned} \tag{60}$$

By a similar proof of (30), it follows that

$$\begin{aligned}
&\frac{\mu(z_k)|\psi(z_k)|(|\eta_2| + \cdots + |\eta_n|)}{(1-r_k^2)^{\alpha+1}} \le \|T_{\psi,\varphi} f_k\|_{\mathscr{B}_\mu} \longrightarrow 0, \\
&\qquad k \longrightarrow \infty.
\end{aligned} \tag{61}$$

And similar to the proofs of (31) and (61), we obtain

$$\frac{\mu(z_k)|\psi(z_k)|}{\left(1-|\varphi(z_k)|^2\right)^{\alpha}}\left\{H_{\varphi(z_k)}\left(J\varphi(z_k)z_k, J\varphi(z_k)z_k\right)\right\}^{1/2} \leq C\frac{\mu(z_k)|\psi(z_k)|\sqrt{2(1-r_k^2)}\left(|\eta_2|+\cdots+|\eta_n|\right)}{\left(1-r_k^2\right)^{n/q+(\gamma+1)/p+1}} \longrightarrow 0 \quad k\longrightarrow\infty. \tag{62}$$

Combining (58) and (62), (47) holds under the two cases.

For the general situation, if there exists $\varphi(z_k)$ such that $\varphi(z_k) \neq |\varphi(z_k)|e_1$, then there is a unitary transformation U_k such that $\varphi(z_k) = r_k e_1 U_k$, $k \in \{1,2,\ldots,n\}$. And we can prove (47) by taking the function sequence $g_k = f_k \circ U_k^{-1}$ and the details are omitted.

Next we prove (46). Let $\{z_k\}_{k\in\mathbb{N}}$ be a sequence in B_n such that $|\varphi(z_k)| \to 1$ as $k \to \infty$. Choose

$$h_k(z) = \frac{\left(1-|\varphi(z_k)|^2\right)^{b-(\gamma+1)/p}}{\left(1-\langle z,\varphi(z_k)\rangle\right)^{n/q+b}}. \tag{63}$$

Then $h_k \in H_{p,q,\gamma}$, $k \in \mathbb{N}$, and $\sup_{k\in\mathbb{N}}\|h_k\|_{H_{p,q,\gamma}} \leq C$. It is obvious that $h_k \to 0$ uniformly on compact subsets of B_n as $k \to \infty$. By Lemma 3 we have that $\lim_{k\to\infty}\|T_{\psi,\varphi}(h_k)(z)\|_{\mathscr{B}_\mu} = 0$. Then by the similar proof of (44) we obtain

$$\left\|T_{\psi,\varphi}(h_k)(z)\right\|_{\mathscr{B}_\mu} \geq \frac{\mu(z_k)\left|\Re\psi(z_k)\right|}{\left(1-|\varphi(z_k)|^2\right)^{n/q+(\gamma+1)/p}} - \mu(z_k)|\psi(z_k)|\left|\Re(h_k\circ\varphi)(z_k)\right|. \tag{64}$$

From the similar proof of (45) it follows that

$$\mu(z_k)|\psi(z_k)|\left|\Re(h_k\circ\varphi)(z_k)\right| \leq \left(\frac{n}{q}+b\right)\frac{\mu(z_k)|\psi(z_k)|}{\left(1-|\varphi(z_k)|^2\right)^{n/q+(\gamma+1)/p}} \times\left\{H_{\varphi(z_k)}\left(J\varphi(z_k)z_k, J\varphi(z_k)z_k\right)\right\}^{1/2} \longrightarrow 0, \quad k\longrightarrow\infty. \tag{65}$$

Combining (64) and (65) we obtain (46). This completes the proof. □

Corollary 7. *Assume that $0 < p,q < \infty$, $-1 < \gamma < \infty$, μ is a normal function, and $\varphi \in S(B_n)$. Then $C_\varphi : H_{p,q,\gamma} \to \mathscr{B}_\mu$ is bounded if and only if*

$$\sup_{z\in B_n}\frac{\mu(z)\left\{H_{\varphi(z)}\left(J\varphi(z)z, J\varphi(z)z\right)\right\}^{1/2}}{\left(1-|\varphi(z)|^2\right)^{n/q+(\gamma+1)/p}} < \infty. \tag{66}$$

Corollary 8. *Assume that $0 < p,q < \infty$, $-1 < \gamma < \infty$, μ is a normal function, and $\varphi \in S(B_n)$. Then $C_\varphi : H_{p,q,\gamma} \to \mathscr{B}_\mu$ is compact if and only if*

$$\lim_{|\varphi(z)|\to 1}\frac{\mu(z)\left\{H_{\varphi(z)}\left(J\varphi(z)z, J\varphi(z)z\right)\right\}^{1/2}}{\left(1-|\varphi(z)|^2\right)^{n/q+(\gamma+1)/p}} = 0. \tag{67}$$

And $\varphi_l \in \mathscr{B}_\mu$ for $l \in \{1,\ldots,n\}$.

Corollary 9. *Assume that $0 < p,q < \infty$, $-1 < \gamma < \infty$, μ is a normal function, and $\psi \in H(B_n)$. Then $M_\psi : H_{p,q,\gamma} \to \mathscr{B}_\mu$ is bounded if and only if*

$$\sup_{z\in B_n}\frac{\mu(z)\left|\Re\psi(z)\right|}{\left(1-|z|^2\right)^{n/q+(\gamma+1)/p}} < \infty, \qquad \sup_{z\in B_n}\frac{\mu(z)|\psi(z)|}{\left(1-|z|^2\right)^{n/q+(\gamma+1)/p+1}} < \infty. \tag{68}$$

Corollary 10. *Assume that $0 < p,q < \infty$, $-1 < \gamma < \infty$, μ is a normal function, and $\psi \in H(B_n)$. Then $M_\psi : H_{p,q,\gamma} \to \mathscr{B}_\mu$ is compact if and only if the following are all satisfied:*

(a) *$\psi \in \mathscr{B}_\mu$ and $\psi z_l \in \mathscr{B}_\mu$ for any $l \in \{1,\ldots,n\}$;*

(b)

$$\lim_{|z|\to 1}\frac{\mu(z)\left|\Re\psi(z)\right|}{\left(1-|z|^2\right)^{n/q+(\gamma+1)/p}} = 0; \tag{69}$$

(c)

$$\lim_{|z|\to 1}\frac{\mu(z)|\psi(z)|}{\left(1-|z|^2\right)^{n/q+(\gamma+1)/p+1}} = 0. \tag{70}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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