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Research Article

Homogenization of Parabolic Equations with an Arbitrary Number of Scales in Both Space and Time

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The main contribution of this paper is the homogenization of the linear parabolic equation $\partial_t u^\varepsilon(x,t) - \nabla \cdot (a(x/\varepsilon^{q_1}, \ldots, x/\varepsilon^{q_n}, t/\varepsilon^{r_1}, \ldots, t/\varepsilon^{r_m})\nabla u^\varepsilon(x,t)) = f(x,t)$ exhibiting an arbitrary finite number of both spatial and temporal scales. We briefly recall some fundamentals of multiscale convergence and provide a characterization of multiscale limits for gradients, in an evolution setting adapted to a quite general class of well-separated scales, which we name by jointly well-separated scales (see appendix for the proof). We proceed with a weaker version of this concept called very weak multiscale convergence. We prove a compactness result with respect to this latter type for jointly well-separated scales. This is a key result for performing the homogenization of parabolic problems combining rapid spatial and temporal oscillations such as the problem above. Applying this compactness result together with a characterization of multiscale limits of sequences of gradients we carry out the homogenization procedure, where we together with the homogenized problem obtain n local problems, that is, one for each spatial microscale. To illustrate the use of the obtained result, we apply it to a case with three spatial and three temporal scales with $q_1 = 1$, $q_2 = 2$, and $0 < r_1 < r_2$.

1. Introduction

In this paper, we study the homogenization of

$$\partial_t u^\varepsilon(x,t) - \nabla \cdot \left(a\left(\frac{x}{\varepsilon^{q_1}}, \ldots, \frac{x}{\varepsilon^{q_n}}, \frac{t}{\varepsilon^{r_1}}, \ldots, \frac{t}{\varepsilon^{r_m}} \right) \nabla u^\varepsilon(x,t) \right) = f(x,t) \quad \text{in } \Omega_T,$$
$$u^\varepsilon(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T),$$
$$u^\varepsilon(x,0) = u^0(x) \quad \text{in } \Omega, \tag{1}$$

where $0 < q_1 < \cdots < q_n$ and $0 < r_1 < \cdots < r_m$. Here $\Omega_T = \Omega \times (0,T)$, where Ω is an open bounded subset of $\mathbb{R}^N$ with smooth boundary and a is periodic with respect to the unit cube $Y = (0,1)^N$ in $\mathbb{R}^N$ in the n first variables and with respect to the unit interval $S = (0,1)$ in the remaining m variables. The homogenization of (1) consists in studying the asymptotic behavior of the solutions u^ε as ε tends to zero and finding the limit equation which admits the limit u of this sequence as its unique solution. The main contribution of this paper is the proof of a homogenization result for (1), that is, for parabolic problems with an arbitrary finite number of scales in both space and time.

Parabolic problems with rapid oscillations in one spatial and one temporal scale were investigated already in [1] using asymptotic expansions. Techniques of two-scale convergence type, see, for example, [2–4], for this kind of problems were first introduced in [5]. One of the main contributions in [5] is a compactness result for a more restricted class of test functions compared with usual two-scale convergence, which has a key role in the homogenization procedure. In [6], a similar result for an arbitrary number of well-separated spatial scales is proven and the type of convergence in question is formalized under the name of very weak multiscale convergence.

A number of recent papers address various kinds of parabolic homogenization problems applying techniques related to those introduced in [5]. [7] treats a monotone parabolic problem with the same choices of scales as in [5] in the more general setting of Σ-convergence. In [8], the case with two fast temporal scales is treated with one of them identical to a single fast spatial scale. These results with the same choice of scales are extended to a more general class of differential operators in [9] and in [10], the two fast spatial scales are fixed to be $\varepsilon_1 = \varepsilon$, $\varepsilon_2 = \varepsilon^2$, while only one fast temporal scale appears. Significant progress was made in [11], where the case with an arbitrary number of temporal scales is treated and none of them has to coincide with the single fast spatial scale. A first study of parabolic problems where the number of fast spatial and temporal scales both exceeds one is found in [12], where the fast spatial scales are $\varepsilon_1 = \varepsilon$, $\varepsilon_2 = \varepsilon^2$ and the rapid temporal scales are chosen as $\varepsilon'_1 = \varepsilon^2$, $\varepsilon'_2 = \varepsilon^4$, and $\varepsilon'_3 = \varepsilon^5$. Similar techniques have also been recently applied to hyperbolic problems. In [13] the two fast spatial scales are well separated and the fast temporal scale coincides with the slower of the fast spatial scales and in [14] the set of scales is the same as in [8, 9]. Clearly all of these previous results include strong restrictions on the choices of scales. Our aim here is to provide a unified approach with the choices of scales in the examples above as special cases. The homogenization procedure for (1) covers arbitrary numbers of spatial and temporal scales and any reasonable choice of the exponents $q_1, \dots, q_n$ and $r_1, \dots, r_m$ defining the fast spatial and temporal scales, respectively. The key to this is the result on very weak multiscale convergence proved in Theorem 7 which adapts the original concept in [6] to the appropriate evolution setting. Let us note that techniques used for the proof of the special case with $\varepsilon_1 = \varepsilon$, $\varepsilon_2 = \varepsilon^2$ in [10] do not apply to the case with arbitrary numbers of scales studied here.

The present paper is organized as follows. In Section 2 we briefly recall the concepts of multiscale convergence and evolution multiscale convergence and give a characterization of gradients with respect to this latter type of convergence under a certain well-separatedness assumption. In Section 3 we consider very weak multiscale convergence in the evolution setting and give the key compactness result employed in the homogenization of (1), which is carried out in Section 4. In this final section, we also illustrate how this general homogenization result can be used by applying it to the particular case governed by $a(x/\varepsilon, x/\varepsilon^2, t/\varepsilon^{r_1}, t/\varepsilon^{r_2})$ where $0 < r_1 < r_2$.

Notation. $F_\sharp(Y)$ is the space of all functions in $F_{\text{loc}}(\mathbb{R}^N)$ that are Y-periodic repetitions of some function in $F(Y)$. We denote $Y_k = Y$ for $k = 1, \dots, n$, $Y^n = Y_1 \times \cdots \times Y_n$, $y^n = y_1, \dots, y_n$, $dy^n = dy_1 \dots dy_n$, $S_j = S$ for $j = 1, \dots, m$, $S^m = S_1 \times \cdots \times S_m$, $s^m = s_1, \dots, s_m$, $ds^m = ds_1 \dots ds_m$, and $\mathcal{Y}_{n,m} = Y^n \times S^m$. Moreover, we let $\varepsilon_k(\varepsilon)$, $k = 1, \dots, n$, and $\varepsilon'_j(\varepsilon)$, $j = 1, \dots, m$, be strictly positive functions such that $\varepsilon_k(\varepsilon)$ and $\varepsilon'_j(\varepsilon)$ go to zero when ε does. More explanations of standard notations for homogenization theory are found in [15].

2. Multiscale Convergence

Our approach for the homogenization procedure in Section 4 is based on the two-scale convergence method, first introduced in [2] and generalized to include several scales in [16]. Following [16], we say that a sequence $\{u^\varepsilon\}$ in $L^2(\Omega)$ $(n+1)$-scale converges to $u_0 \in L^2(\Omega \times Y^n)$ if

$$\int_\Omega u^\varepsilon(x)\, v\left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_n}\right) dx \longrightarrow \int_\Omega \int_{Y^n} u_0(x, y^n)\, v(x, y^n)\, dy^n\, dx \tag{2}$$

for any $v \in L^2(\Omega; C_\sharp(Y^n))$ and we write

$$u^\varepsilon(x) \overset{n+1}{\rightharpoonup} u_0(x, y^n). \tag{3}$$

This type of convergence can be adapted to the evolution setting; see, for example, [12]. We give the following definition of evolution multiscale convergence.

Definition 1. A sequence $\{u^\varepsilon\}$ in $L^2(\Omega_T)$ is said to $(n+1, m+1)$-scale converge to $u_0 \in L^2(\Omega_T \times \mathcal{Y}_{n,m})$ if

$$\int_{\Omega_T} u^\varepsilon(x,t)\, v\left(x, t, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_n}, \frac{t}{\varepsilon'_1}, \dots, \frac{t}{\varepsilon'_m}\right) dx\, dt \longrightarrow \int_{\Omega_T} \int_{\mathcal{Y}_{n,m}} u_0(x, t, y^n, s^m)\, v(x, t, y^n, s^m)\, dy^n\, ds^m\, dx\, dt \tag{4}$$

for any $v \in L^2(\Omega_T; C_\sharp(\mathcal{Y}_{n,m}))$. We write

$$u^\varepsilon(x,t) \overset{n+1,m+1}{\rightharpoonup} u_0(x, t, y^n, s^m). \tag{5}$$

Normally, some assumptions are made on the relation between the scales. We say that the scales in a list $\{\varepsilon_1, \dots, \varepsilon_n\}$ are separated if

$$\lim_{\varepsilon \to 0} \frac{\varepsilon_{k+1}}{\varepsilon_k} = 0 \tag{6}$$

for $k = 1, \dots, n-1$ and that the scales are well-separated if there exists a positive integer l such that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon_k}\left(\frac{\varepsilon_{k+1}}{\varepsilon_k}\right)^l = 0 \tag{7}$$

for $k = 1, \dots, n-1$.

We also need the concept in the following definition.

Definition 2. Let $\{\varepsilon_1, \dots, \varepsilon_n\}$ and $\{\varepsilon'_1, \dots, \varepsilon'_m\}$ be lists of well-separated scales. Collect all elements from both lists in one common list. If from possible duplicates, where by duplicates we mean scales which tend to zero equally fast, one member of each such pair is removed and the list in order of magnitude of all the remaining elements is well-separated, the lists $\{\varepsilon_1, \dots, \varepsilon_n\}$ and $\{\varepsilon'_1, \dots, \varepsilon'_m\}$ are said to be jointly well-separated.

In the remark below, we give some further comments on the concept introduced in Definition 2.

Remark 3. To include also the temporal scales alongside with the spatial scales allows us to study a much richer class of homogenization problems such as all the cases included in (1). For a more technically formulated definition and some examples, see Section 2.4 in [17]. Note that the lists $\{\varepsilon^{q_1},\dots,\varepsilon^{q_n}\}$ and $\{\varepsilon^{r_1},\dots,\varepsilon^{r_m}\}$ of spatial and temporal scales, respectively, in (1) are jointly well-separated for any choice of $0<q_1<\cdots<q_n$ and $0<r_1<\cdots<r_m$.

Below we provide a characterization of evolution multiscale limits for gradients, which will be used in the proof of the homogenization result in Section 4. Here $W_2^1(0,T;H_0^1(\Omega),L^2(\Omega))$ is the space of all functions in $L^2(0,T;H_0^1(\Omega))$ such that the time derivative belongs to $L^2(0,T;H^{-1}(\Omega))$; see, for example, Chapter 23 in [18].

Theorem 4. *Let $\{u^\varepsilon\}$ be a bounded sequence in $W_2^1(0,T;H_0^1(\Omega),L^2(\Omega))$ and suppose that the lists $\{\varepsilon_1,\dots,\varepsilon_n\}$ and $\{\varepsilon'_1,\dots,\varepsilon'_m\}$ are jointly well-separated. Then there exists a subsequence such that*

$$u^\varepsilon(x,t)\longrightarrow u(x,t)\quad \text{in } L^2(\Omega_T),$$
$$u^\varepsilon(x,t)\rightharpoonup u(x,t)\quad \text{in } L^2(0,T;H_0^1(\Omega)), \tag{8}$$

$$\nabla u^\varepsilon(x,t)\overset{n+1,m+1}{\rightharpoonup}\nabla u(x,t)+\sum_{j=1}^{n}\nabla_{y_j}u_j(x,t,y^j,s^m), \tag{9}$$

where $u\in W_2^1(0,T;H_0^1(\Omega),L^2(\Omega))$, $u_1\in L^2(\Omega_T\times S^m;H_\sharp^1(Y_1)/\mathbb{R})$ and $u_j\in L^2(\Omega_T\times\mathcal{Y}_{j-1,m};H_\sharp^1(Y_j)/\mathbb{R})$ for $j=2,\dots,n$.

Proof. See Theorem 2.74 in [17] and the appendix of this paper. □

3. Very Weak Multiscale Convergence

A first compactness result of very weak convergence type was presented in [5] for the purpose of homogenizing linear parabolic equations with fast oscillations in one spatial scale and one temporal scale. A compactness result for the case with oscillations in n well-separated spatial scales was proven in [6], where the notion of very weak convergence was introduced. It states that for any bounded sequence $\{u^\varepsilon\}$ in $H_0^1(\Omega)$ and the scales in the list $\{\varepsilon_1,\dots,\varepsilon_n\}$ well-separated it holds up to subsequence that

$$\begin{aligned}&\int_\Omega\frac{u^\varepsilon(x)}{\varepsilon_n}v\left(x,\frac{x}{\varepsilon_1},\dots,\frac{x}{\varepsilon_{n-1}}\right)\varphi\left(\frac{x}{\varepsilon_n}\right)dx\\&\quad\longrightarrow\int_\Omega\int_{Y^n}u_n(x,y^n)\,v(x,y^{n-1})\,\varphi(y_n)\,dy^n\,dx\end{aligned} \tag{10}$$

for any $v\in D(\Omega;C_\sharp^\infty(Y^{n-1}))$ and $\varphi\in C_\sharp^\infty(Y_n)/\mathbb{R}$, where u_n is the same as in the right-hand side of

$$\nabla u^\varepsilon(x)\overset{n+1}{\rightharpoonup}\nabla u(x)+\sum_{j=1}^{n}\nabla_{y_j}u_j(x,y^j), \tag{11}$$

the original time independent version of the gradient characterization in Theorem 4, that is found in [16]. In Theorem 7 below we present a generalized result including oscillations in time with a view to homogenizing (1). First we define very weak evolution multiscale convergence.

Definition 5. We say that a sequence $\{g^\varepsilon\}$ in $L^1(\Omega_T)$ $(n+1,m+1)$-scale converges very weakly to $g_0\in L^1(\Omega_T\times\mathcal{Y}_{n,m})$ if

$$\begin{aligned}&\int_{\Omega_T}g^\varepsilon(x,t)\,v\left(x,\frac{x}{\varepsilon_1},\dots,\frac{x}{\varepsilon_{n-1}}\right)\\&\qquad\times c\left(t,\frac{t}{\varepsilon'_1},\dots,\frac{t}{\varepsilon'_m}\right)\varphi\left(\frac{x}{\varepsilon_n}\right)dx\,dt\\&\quad\longrightarrow\int_{\Omega_T}\int_{\mathcal{Y}_{n,m}}g_0(x,t,y^n,s^m)\,v(x,y^{n-1})\\&\qquad\times c(t,s^m)\,\varphi(y_n)\,dy^n\,ds^m\,dx\,dt\end{aligned} \tag{12}$$

for any $v\in D(\Omega;C_\sharp^\infty(Y^{n-1}))$, $\varphi\in C_\sharp^\infty(Y_n)/\mathbb{R}$ and $c\in D(0,T;C_\sharp^\infty(S^m))$. A unique limit is provided by requiring that

$$\int_{Y_n}g_0(x,t,y^n,s^m)\,dy_n=0. \tag{13}$$

We write

$$g^\varepsilon(x,t)\underset{vw}{\overset{n+1,m+1}{\rightharpoonup}}g_0(x,t,y^n,s^m). \tag{14}$$

The following proposition (see Theorem 3.3 in [16]) is needed for the proof of Theorem 7.

Proposition 6. *Let $v\in D(\Omega;C_\sharp^\infty(Y^n))$ be a function such that*

$$\int_{Y_n}v(x,y^n)\,dy_n=0, \tag{15}$$

and assume that the scales in the list $\{\varepsilon_1,\dots,\varepsilon_n\}$ are well-separated. Then $\{\varepsilon_n^{-1}v(x,x/\varepsilon_1,\dots,x/\varepsilon_n)\}$ is bounded in $H^{-1}(\Omega)$.

We are now ready to state the following theorem which is essential for the homogenization of (1); see also Theorem 7 in [19] and Theorem 2.78 in [17].

Theorem 7. *Let $\{u^\varepsilon\}$ be a bounded sequence in $W_2^1(0,T;H_0^1(\Omega),L^2(\Omega))$ and assume that the lists $\{\varepsilon_1,\dots,\varepsilon_n\}$ and $\{\varepsilon'_1,\dots,\varepsilon'_m\}$ are jointly well-separated. Then there exists a subsequence such that*

$$\frac{u^\varepsilon(x,t)}{\varepsilon_n}\underset{vw}{\overset{n+1,m+1}{\rightharpoonup}}u_n(x,t,y^n,s^m), \tag{16}$$

where, for $n=1$, $u_1\in L^2(\Omega_T\times S^m;H_\sharp^1(Y_1)/\mathbb{R})$ and, for $n=2,3,\dots$, $u_n\in L^2(\Omega_T\times\mathcal{Y}_{n-1,m};H_\sharp^1(Y_n)/\mathbb{R})$ are the same as in Theorem 4.

Proof. We want to prove that for any $v \in D(\Omega; C_\sharp^\infty(Y^{n-1}))$, $c \in D(0,T; C_\sharp^\infty(S^m))$ and $\varphi \in C_\sharp^\infty(Y_n)/\mathbb{R}$,

$$\int_{\Omega_T} \frac{u^\varepsilon(x,t)}{\varepsilon_n} v\left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_{n-1}}\right) \times c\left(t, \frac{t}{\varepsilon_1'}, \dots, \frac{t}{\varepsilon_m'}\right) \varphi\left(\frac{x}{\varepsilon_n}\right) dx\, dt \longrightarrow \int_{\Omega_T} \int_{\mathcal{Y}_{n,m}} u_n(x,t,y^n,s^m)\, v(x,y^{n-1}) \times c(t,s^m)\, \varphi(y_n)\, dy^n\, ds^m\, dx\, dt \tag{17}$$

for some suitable subsequence. First we note that any $\varphi \in C_\sharp^\infty(Y_n)/\mathbb{R}$ can be expressed as

$$\varphi(y_n) = \Delta_{y_n} w(y_n) = \nabla_{y_n} \cdot (\nabla_{y_n} w(y_n)) \tag{18}$$

for some $w \in C_\sharp^\infty(Y_n)/\mathbb{R}$ (see, e.g., Remark 3.2 in [7]). Furthermore, let

$$\psi(y_n) = \nabla_{y_n} w(y_n) \tag{19}$$

and observe that

$$\int_{Y_n} \psi(y_n)\, dy_n = \int_{Y_n} \nabla_{y_n} w(y_n)\, dy_n = 0 \tag{20}$$

because of the Y_n-periodicity of w. By (18), the left-hand side of (17) can be expressed as

$$\begin{aligned} &\int_{\Omega_T} \frac{u^\varepsilon(x,t)}{\varepsilon_n} v\left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_{n-1}}\right) \times c\left(t, \frac{t}{\varepsilon_1'}, \dots, \frac{t}{\varepsilon_m'}\right) (\nabla_{y_n} \cdot \psi)\left(\frac{x}{\varepsilon_n}\right) dx\, dt \\ &= \int_{\Omega_T} u^\varepsilon(x,t)\, v\left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_{n-1}}\right) \times c\left(t, \frac{t}{\varepsilon_1'}, \dots, \frac{t}{\varepsilon_m'}\right) \nabla \cdot \left(\psi\left(\frac{x}{\varepsilon_n}\right)\right) dx\, dt. \end{aligned} \tag{21}$$

Integrating by parts with respect to x, we obtain

$$\begin{aligned} -\int_{\Omega_T} &\nabla u^\varepsilon(x,t) \cdot v\left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_{n-1}}\right) \times c\left(t, \frac{t}{\varepsilon_1'}, \dots, \frac{t}{\varepsilon_m'}\right) \psi\left(\frac{x}{\varepsilon_n}\right) \\ &+ u^\varepsilon(x,t) \nabla_x v\left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_{n-1}}\right) \times c\left(t, \frac{t}{\varepsilon_1'}, \dots, \frac{t}{\varepsilon_m'}\right) \cdot \psi\left(\frac{x}{\varepsilon_n}\right) \\ &+ \sum_{j=1}^{n-1} u^\varepsilon(x,t)\, \varepsilon_j^{-1} \nabla_{y_j} v\left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_{n-1}}\right) \times c\left(t, \frac{t}{\varepsilon_1'}, \dots, \frac{t}{\varepsilon_m'}\right) \cdot \psi\left(\frac{x}{\varepsilon_n}\right) dx\, dt. \end{aligned} \tag{22}$$

To begin with, we consider the first term. Passing to the multiscale limit using Theorem 4, we arrive up to subsequence at

$$-\int_{\Omega_T} \int_{\mathcal{Y}_{n,m}} \left(\nabla u(x,t) + \sum_{j=1}^{n} \nabla_{y_j} u_j(x,t,y^j,s^m)\right) \cdot v(x,y^{n-1})\, c(t,s^m)\, \psi(y_n)\, dy^n\, ds^m\, dx\, dt, \tag{23}$$

and due to (20) all but the last term vanish. We have

$$-\int_{\Omega_T} \int_{\mathcal{Y}_{n,m}} \nabla_{y_n} u_n(x,t,y^n,s^m) \cdot v(x,y^{n-1})\, c(t,s^m)\, \psi(y_n)\, dy^n\, ds^m\, dx\, dt. \tag{24}$$

Moreover, (8) means that the second term of (22) up to a subsequence approaches

$$\begin{aligned} -\int_{\Omega_T} \int_{\mathcal{Y}_{n,m}} &u(x,t) \nabla_x v(x,y^{n-1})\, c(t,s^m) \cdot \psi(y_n)\, dy^n\, ds^m\, dx\, dt \\ = -\int_{\Omega_T} \int_{\mathcal{Y}_{n-1,m}} &u(x,t) \nabla_x v(x,y^{n-1})\, c(t,s^m) \cdot \left(\int_{Y_n} \psi(y_n)\, dy_n\right) dy^{n-1}\, ds^m\, dx\, dt = 0, \end{aligned} \tag{25}$$

where the last equality is a result of (20).

It remains to investigate the last term of (22). We write

$$\begin{aligned} &\sum_{j=1}^{n-1} \int_{\Omega_T} u^\varepsilon(x,t)\, \varepsilon_j^{-1} \nabla_{y_j} v\left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_{n-1}}\right) \times c\left(t, \frac{t}{\varepsilon_1'}, \dots, \frac{t}{\varepsilon_m'}\right) \cdot \psi\left(\frac{x}{\varepsilon_n}\right) dx\, dt \\ &= \sum_{j=1}^{n-1} \frac{\varepsilon_n}{\varepsilon_j} \int_{\Omega_T} u^\varepsilon(x,t)\, \varepsilon_n^{-1} \nabla_{y_j} v\left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_{n-1}}\right) \times c\left(t, \frac{t}{\varepsilon_1'}, \dots, \frac{t}{\varepsilon_m'}\right) \cdot \psi\left(\frac{x}{\varepsilon_n}\right) dx\, dt. \end{aligned} \tag{26}$$

Clearly, $\{\varepsilon_n^{-1} \nabla_{y_j} v(x, x/\varepsilon_1, \dots, x/\varepsilon_{n-1}) \cdot \psi(x/\varepsilon_n)\}$ is bounded in $H^{-1}(\Omega)$ for $j = 1, \dots, n-1$ by Proposition 6. Observing that $\{u^\varepsilon\}$ is assumed to be bounded in $L^2(0,T; H_0^1(\Omega))$, this means that, for any integer $j \in [1, n-1]$, there are constants $C_1, C_2, C_3 > 0$ such that

$$\begin{aligned}
&\left(\frac{\varepsilon_n}{\varepsilon_j}\int_{\Omega_T} u^\varepsilon(x,t)\,\varepsilon_n^{-1}\nabla_{y_j} v\left(x,\frac{x}{\varepsilon_1},\dots,\frac{x}{\varepsilon_{n-1}}\right)\right.\\
&\qquad\left.\times c\left(t,\frac{t}{\varepsilon'_1},\dots,\frac{t}{\varepsilon'_m}\right)\cdot\psi\left(\frac{x}{\varepsilon_n}\right)dx\,dt\right)^2\\
&=\left(\frac{\varepsilon_n}{\varepsilon_j}\right)^2\left(\int_{\Omega_T} u^\varepsilon(x,t)\,\varepsilon_n^{-1}\nabla_{y_j} v\left(x,\frac{x}{\varepsilon_1},\dots,\frac{x}{\varepsilon_{n-1}}\right)\right.\\
&\qquad\left.\times c\left(t,\frac{t}{\varepsilon'_1},\dots,\frac{t}{\varepsilon'_m}\right)\cdot\psi\left(\frac{x}{\varepsilon_n}\right)dx\,dt\right)^2\\
&\le C_1\left(\frac{\varepsilon_n}{\varepsilon_j}\right)^2\int_0^T\left(\int_{\Omega} u^\varepsilon(x,t)\,\varepsilon_n^{-1}\nabla_{y_j} v\left(x,\frac{x}{\varepsilon_1},\dots,\frac{x}{\varepsilon_{n-1}}\right)\right.\\
&\qquad\left.\times c\left(t,\frac{t}{\varepsilon'_1},\dots,\frac{t}{\varepsilon'_m}\right)\cdot\psi\left(\frac{x}{\varepsilon_n}\right)dx\right)^2dt\\
&\le C_1\left(\frac{\varepsilon_n}{\varepsilon_j}\right)^2\\
&\quad\times\int_0^T\left(\left\|\varepsilon_n^{-1}\nabla_{y_j} v\left(\cdot,\frac{\cdot}{\varepsilon_1},\dots,\frac{\cdot}{\varepsilon_{n-1}}\right)\cdot\psi\left(\frac{\cdot}{\varepsilon_n}\right)\right\|_{H^{-1}(\Omega)}\right.\\
&\qquad\left.\times\left\|u^\varepsilon(\cdot,t)\,c\left(t,\frac{t}{\varepsilon'_1},\dots,\frac{t}{\varepsilon'_m}\right)\right\|_{H_0^1(\Omega)}\right)^2dt\\
&\le C_2\left(\frac{\varepsilon_n}{\varepsilon_j}\right)^2\int_0^T\|u^\varepsilon(\cdot,t)\|^2_{H_0^1(\Omega)}dt\\
&=C_2\left(\frac{\varepsilon_n}{\varepsilon_j}\right)^2\|u^\varepsilon\|^2_{L^2(0,T;H_0^1(\Omega))}\le C_3\left(\frac{\varepsilon_n}{\varepsilon_j}\right)^2.
\end{aligned}\tag{27}$$

Hence, all the terms in the sum (26) vanish as $\varepsilon \to 0$ as a result of the separatedness of the scales. Then (24) is all that remains after passing to the limit in (22). Finally, integrating (24) by parts, we obtain

$$\begin{aligned}
&\int_{\Omega_T}\int_{\mathcal{Y}_{n,m}} u_n(x,t,y^n,s^m)\,v(x,y^{n-1})\,c(t,s^m)\,\nabla_{y_n}\\
&\qquad\cdot\psi(y_n)\,dy^n\,ds^m\,dx\,dt\\
&=\int_{\Omega_T}\int_{\mathcal{Y}_{n,m}} u_n(x,t,y^n,s^m)\,v(x,y^{n-1})\\
&\qquad\times c(t,s^m)\,\varphi(y_n)\,dy^n\,ds^m\,dx\,dt,
\end{aligned}\tag{28}$$

which is the right-hand side of (17). □

Remark 8. The notion of very weak multiscale convergence is an alternative type of multiscale convergence. It is remarkable in the sense that it enables us to provide a compactness result of multiscale convergence type for sequences that are not bounded in any Lebesgue space. In fact, it deals with the normally forbidden situation of finding a limit for a quotient, where the denominator goes to zero while the numerator does not. The price to pay for this is that we have to use much smaller class of admissible testfunctions. In the set of modes of multiscale convergence usually applied in homogenization that we find in Definition 1 and Theorem 4, very weak multiscale convergence provides us with the missing link. As we will see in the homogenization procedure in the next section Theorems 4 and 7 give us the cornerstones for the homogenization procedure that allows us to tackle all appearing passages to limits in a unified way by means of two distinct theorems and without ad hoc constructions. Moreover, Theorem 7 provides us with appropriate upscaling to detect microoscillations in solutions of typical homogenization problems, which are usually of vanishing amplitude, while the global tendency is filtered away as a result of the choice of test functions. See [12].

4. Homogenization

We are now ready to give the main contribution of this paper, the homogenization of the linear parabolic problem (1). The gradient characterization in Theorem 4 and the very weak compactness result from Theorem 7 are crucial for proving the homogenization result, which is presented in Section 4.1. An illustration of how this result can be used in practice is given in Section 4.2.

4.1. The General Case. We study the homogenization of the problem

$$\begin{aligned}
&\partial_t u^\varepsilon(x,t)-\nabla\cdot\left(a\left(\frac{x}{\varepsilon^{q_1}},\dots,\frac{x}{\varepsilon^{q_n}},\frac{t}{\varepsilon^{r_1}},\dots,\frac{t}{\varepsilon^{r_m}}\right)\nabla u^\varepsilon(x,t)\right)\\
&\quad=f(x,t)\quad\text{in }\Omega_T,\\
&\qquad u^\varepsilon(x,t)=0\quad\text{on }\partial\Omega\times(0,T),\\
&\qquad u^\varepsilon(x,0)=u^0(x)\quad\text{in }\Omega,
\end{aligned}\tag{29}$$

where $0 < q_1 < \cdots < q_n$, $0 < r_1 < \cdots < r_m$, $f \in L^2(\Omega_T)$, $u^0 \in L^2(\Omega)$ and where we assume that

(*A1*) $a \in C_\sharp(\mathcal{Y}_{n,m})^{N\times N}$.

(*A2*) $a(y^n, s^m)\xi\cdot\xi \ge \alpha|\xi|^2$ for all $(y^n, s^m) \in \mathbb{R}^{nN}\times\mathbb{R}^m$, all $\xi \in \mathbb{R}^N$ and some $\alpha > 0$.

Under these conditions, (29) allows a unique solution $u^\varepsilon \in W_2^1(0,T;H_0^1(\Omega),L^2(\Omega))$ and for some positive constant C,

$$\|u^\varepsilon\|_{W_2^1(0,T;H_0^1(\Omega),L^2(\Omega))} < C. \tag{30}$$

Given the scale exponents $0 < q_1 < \cdots < q_n$ and $0 < r_1 < \cdots < r_m$, we may define some numbers in order to formulate the theorem below in a convenient way. We define d_i (the number of temporal scales faster than the square of the spatial scale in question) and ρ_i (indicates whether there is nonresonance or resonance), $i = 1, \dots, n$, as follows.

(i) If $2q_i < r_1$, then $d_i = m$, if $r_j \leq 2q_i < r_{j+1}$ for some $j = 1, \ldots, m-1$, then $d_i = m - j$, and if $2q_i \geq r_m$, then $d_i = 0$.

(ii) If $2q_i = r_j$ for some $j = 1, \ldots, m$, that is we have resonance, we let $\rho_i = 1$; otherwise, $\rho_i = 0$.

Note that from the definition of d_i we have in fact in the definition of ρ_i that $j = m - d_i$ in the case of resonance.

Finally, we recall that the lists $\{\varepsilon^{q_1}, \ldots, \varepsilon^{q_n}\}$ and $\{\varepsilon^{r_1}, \ldots, \varepsilon^{r_m}\}$ are jointly well-separated.

Theorem 9. *Let $\{u^\varepsilon\}$ be a sequence of solutions in $W_2^1(0,T; H_0^1(\Omega), L^2(\Omega))$ to (29). Then it holds that*

$$\begin{aligned} u^\varepsilon(x,t) &\longrightarrow u(x,t) \quad \text{in } L^2(\Omega_T), \\ u^\varepsilon(x,t) &\rightharpoonup u(x,t) \quad \text{in } L^2\left(0,T; H_0^1(\Omega)\right), \\ \nabla u^\varepsilon(x,t) &\overset{n+1,m+1}{\rightharpoonup} \nabla u(x,t) + \sum_{j=1}^n \nabla_{y_j} u_j\left(x,t,y^j,s^m\right), \end{aligned} \tag{31}$$

where $u \in W_2^1(0,T; H_0^1(\Omega), L^2(\Omega))$ is the unique solution to

$$\begin{aligned} \partial_t u(x,t) - \nabla \cdot (b(x,t)\nabla u(x,t)) &= f(x,t) \quad \text{in } \Omega_T, \\ u(x,t) &= 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(x,0) &= u^0(x) \quad \text{in } \Omega \end{aligned} \tag{32}$$

with

$$\begin{aligned} &b(x,t)\nabla u(x,t) \\ &= \int_{\mathcal{Y}_{n,m}} a(y^n, s^m) \\ &\quad \times \left(\nabla u(x,t) + \sum_{j=1}^n \nabla_{y_j} u_j\left(x,t,y^j,s^m\right)\right) dy^n\, ds^m. \end{aligned} \tag{33}$$

Here $u_1 \in L^2(\Omega_T \times S^m; H_\sharp^1(Y_1)/\mathbb{R})$ and $u_j \in L^2(\Omega_T \times \mathcal{Y}_{j-1,m}; H_\sharp^1(Y_j)/\mathbb{R})$, $j = 2, \ldots, n$, are the unique solutions to the system of local problems

$$\begin{aligned} &\rho_i \partial_{s_{m-d_i}} u_i\left(x,t,y^i,s^m\right) - \nabla_{y_i} \\ &\quad \cdot \int_{S_{m-d_i+1}} \cdots \int_{S_m} \int_{Y_{i+1}} \cdots \int_{Y_n} a(y^n, s^m) \\ &\quad \times \left(\nabla u(x,t) + \sum_{j=1}^n \nabla_{y_j} u_j\left(x,t,y^j,s^m\right)\right) \\ &\quad \times dy_n \cdots dy_{i+1}\, ds_m \cdots ds_{m-d_i+1} = 0, \end{aligned} \tag{34}$$

for $i = 1, \ldots, n$, where u_i is independent of $s_{m-d_i+1}, \ldots, s_m$.

Remark 10. In the case $d_i = 0$, we naturally interpret the integration in (34) as if there is no local temporal integration involved and that there is no independence of any local temporal variable.

Remark 11. Note that if, for example, u_1 is independent of s_m the function space that u_1 belongs to simplifies to $u_1 \in L^2(\Omega_T \times S^{m-1}; H_\sharp^1(Y_1)/\mathbb{R})$ and when u_1 is also independent of s_{m-1}, we have that $u_1 \in L^2(\Omega_T \times S^{m-2}; H_\sharp^1(Y_1)/\mathbb{R})$ and so on.

Proof of Theorem 9. Since $\{u^\varepsilon\}$ is bounded in $W_2^1(0,T; H_0^1(\Omega), L^2(\Omega))$ and the lists of scales are jointly well-separated, we can apply Theorem 4 and obtain that, up to a subsequence,

$$\begin{aligned} u^\varepsilon(x,t) &\longrightarrow u(x,t) \quad \text{in } L^2(\Omega_T), \\ u^\varepsilon(x,t) &\rightharpoonup u(x,t) \quad \text{in } L^2\left(0,T; H_0^1(\Omega)\right), \\ \nabla u^\varepsilon(x,t) &\overset{n+1,m+1}{\rightharpoonup} \nabla u(x,t) + \sum_{j=1}^n \nabla_{y_j} u_j\left(x,t,y^j,s^m\right), \end{aligned} \tag{35}$$

where $u \in W_2^1(0,T; H_0^1(\Omega), L^2(\Omega))$, $u_1 \in L^2(\Omega_T \times S^m; H_\sharp^1(Y_1)/\mathbb{R})$, and $u_j \in L^2(\Omega_T \times \mathcal{Y}_{j-1,m}; H_\sharp^1(Y_j)/\mathbb{R})$, $j = 2, \ldots, n$.

To obtain the homogenized problem, we introduce the weak form

$$\begin{aligned} &\int_{\Omega_T} -u^\varepsilon(x,t)\, v(x)\, \partial_t c(t) \\ &\quad + a\left(\frac{x}{\varepsilon^{q_1}}, \ldots, \frac{x}{\varepsilon^{q_n}}, \frac{t}{\varepsilon^{r_1}}, \ldots, \frac{t}{\varepsilon^{r_m}}\right) \nabla u^\varepsilon(x,t) \\ &\quad \cdot \nabla v(x)\, c(t)\, dx\, dt = \int_{\Omega_T} f(x,t)\, v(x)\, c(t)\, dx\, dt \end{aligned} \tag{36}$$

of (29) where $v \in H_0^1(\Omega)$ and $c \in D(0,T)$, and letting $\varepsilon \to 0$, we get using Theorem 4

$$\begin{aligned} &\int_{\Omega_T} -u(x,t)\, v(x)\, \partial_t c(t) \\ &\quad + \int_{\mathcal{Y}_{n,m}} a(y^n, s^m) \left(\nabla u(x,t) + \sum_{j=1}^n \nabla_{y_j} u_j\left(x,t,y^j,s^m\right)\right) \\ &\quad \cdot \nabla v(x)\, c(t)\, dy^n\, ds^m\, dx\, dt \\ &= \int_{\Omega_T} f(x,t)\, v(x)\, c(t)\, dx\, dt. \end{aligned} \tag{37}$$

We proceed by deriving the system of local problems (34) and the independencies of the local temporal variables. Fix $i = 1, \ldots, n$ and choose

$$\begin{aligned} v(x) &= \varepsilon^p v_1(x)\, v_2\left(\frac{x}{\varepsilon^{q_1}}\right) \cdots v_{i+1}\left(\frac{x}{\varepsilon^{q_i}}\right), \quad p > 0, \\ c(t) &= c_1(t)\, c_2\left(\frac{t}{\varepsilon^{r_1}}\right) \cdots c_{\lambda+1}\left(\frac{t}{\varepsilon^{r_\lambda}}\right), \quad \lambda = 1, \ldots, m \end{aligned} \tag{38}$$

with $v_1 \in D(\Omega)$, $v_j \in C_\sharp^\infty(Y_{j-1})$ for $j = 2, \ldots, i$, $v_{i+1} \in C_\sharp^\infty(Y_i)/\mathbb{R}$, $c_1 \in D(0,T)$ and $c_l \in C_\sharp^\infty(S_{l-1})$ for $l = 2, \ldots, \lambda + 1$.

Here p and λ will be fixed later. Using this choice of test functions in (36), we have

$$\begin{aligned}
&\int_{\Omega_T} -u^{\varepsilon}(x,t)\,\varepsilon^{p} v_1(x)\, v_2\left(\frac{x}{\varepsilon^{q_1}}\right)\cdots v_{i+1}\left(\frac{x}{\varepsilon^{q_i}}\right)\\
&\quad\times\left(\partial_t c_1(t)\, c_2\left(\frac{t}{\varepsilon^{r_1}}\right)\cdots c_{\lambda+1}\left(\frac{t}{\varepsilon^{r_\lambda}}\right)\right.\\
&\qquad+\sum_{l=2}^{\lambda+1}\varepsilon^{-r_{l-1}} c_1(t)\\
&\qquad\qquad\left.\times c_2\left(\frac{t}{\varepsilon^{r_1}}\right)\cdots \partial_{s_{l-1}} c_l\left(\frac{t}{\varepsilon^{r_{l-1}}}\right)\cdots c_{\lambda+1}\left(\frac{t}{\varepsilon^{r_\lambda}}\right)\right)\\
&\quad + a\left(\frac{x}{\varepsilon^{q_1}},\ldots,\frac{x}{\varepsilon^{q_n}},\frac{t}{\varepsilon^{r_1}},\ldots,\frac{t}{\varepsilon^{r_m}}\right)\nabla u^{\varepsilon}(x,t)\\
&\quad \cdot\left(\varepsilon^{p}\nabla v_1(x)\, v_2\left(\frac{x}{\varepsilon^{q_1}}\right)\cdots v_{i+1}\left(\frac{x}{\varepsilon^{q_i}}\right)\right.\\
&\qquad+\sum_{j=2}^{i+1}\varepsilon^{p-q_{j-1}} v_1(x)\\
&\qquad\qquad\left.\times v_2\left(\frac{x}{\varepsilon^{q_1}}\right)\cdots \nabla_{y_{j-1}} v_j\left(\frac{x}{\varepsilon^{q_{j-1}}}\right)\cdots v_{i+1}\left(\frac{x}{\varepsilon^{q_i}}\right)\right)\\
&\quad\times c_1(t)\, c_2\left(\frac{t}{\varepsilon^{r_1}}\right)\cdots c_{\lambda+1}\left(\frac{t}{\varepsilon^{r_\lambda}}\right) dx\,dt\\
&= \int_{\Omega_T} f(x,t)\,\varepsilon^{p} v_1(x)\, v_2\left(\frac{x}{\varepsilon^{q_1}}\right)\cdots v_{i+1}\left(\frac{x}{\varepsilon^{q_i}}\right)\\
&\qquad\times c_1(t)\, c_2\left(\frac{t}{\varepsilon^{r_1}}\right)\cdots c_{\lambda+1}\left(\frac{t}{\varepsilon^{r_\lambda}}\right) dx\,dt,
\end{aligned}\tag{39}$$

where, for $l = 2$ and $l = \lambda + 1$, the interpretation should be that the partial derivative acts on c_2 and $c_{\lambda+1}$, respectively, and where the $j = 2$ and $j = i + 1$ terms are defined analogously. We let $\varepsilon \to 0$ and using Theorem 4, we obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\Omega_T} -u^{\varepsilon}(x,t)\,\varepsilon^{p} v_1(x)\, v_2\left(\frac{x}{\varepsilon^{q_1}}\right)\cdots v_{i+1}\left(\frac{x}{\varepsilon^{q_i}}\right)\\
&\qquad\times\sum_{l=2}^{\lambda+1}\varepsilon^{-r_{l-1}} c_1(t)\, c_2\left(\frac{t}{\varepsilon^{r_1}}\right)\\
&\qquad\qquad\cdots\partial_{s_{l-1}} c_l\left(\frac{t}{\varepsilon^{r_{l-1}}}\right)\cdots c_{\lambda+1}\left(\frac{t}{\varepsilon^{r_\lambda}}\right)\\
&\qquad + a\left(\frac{x}{\varepsilon^{q_1}},\ldots,\frac{x}{\varepsilon^{q_n}},\frac{t}{\varepsilon^{r_1}},\ldots,\frac{t}{\varepsilon^{r_m}}\right)\nabla u^{\varepsilon}(x,t)\\
&\qquad \cdot\sum_{j=2}^{i+1}\varepsilon^{p-q_{j-1}} v_1(x)\, v_2\left(\frac{x}{\varepsilon^{q_1}}\right)\\
&\qquad\qquad\cdots\nabla_{y_{j-1}} v_j\left(\frac{x}{\varepsilon^{q_{j-1}}}\right)\cdots v_{i+1}\left(\frac{x}{\varepsilon^{q_i}}\right)\\
&\qquad\times c_1(t)\, c_2\left(\frac{t}{\varepsilon^{r_1}}\right)\cdots c_{\lambda+1}\left(\frac{t}{\varepsilon^{r_\lambda}}\right) dx\,dt = 0,
\end{aligned}\tag{40}$$

and extracting a factor ε^{-q_i} in the first term, we get

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\Omega_T} -\varepsilon^{-q_i}u^{\varepsilon}(x,t)\\
&\qquad\times\sum_{l=2}^{\lambda+1}\varepsilon^{p+q_i-r_{l-1}} v_1(x)\, v_2\left(\frac{x}{\varepsilon^{q_1}}\right)\cdots v_{i+1}\left(\frac{x}{\varepsilon^{q_i}}\right)\\
&\qquad\qquad\times c_1(t)\, c_2\left(\frac{t}{\varepsilon^{r_1}}\right)\cdots\partial_{s_{l-1}} c_l\left(\frac{t}{\varepsilon^{r_{l-1}}}\right)\cdots c_{\lambda+1}\left(\frac{t}{\varepsilon^{r_\lambda}}\right)\\
&\qquad + a\left(\frac{x}{\varepsilon^{q_1}},\ldots,\frac{x}{\varepsilon^{q_n}},\frac{t}{\varepsilon^{r_1}},\ldots,\frac{t}{\varepsilon^{r_m}}\right)\nabla u^{\varepsilon}(x,t)\\
&\qquad \cdot\sum_{j=2}^{i+1}\varepsilon^{p-q_{j-1}} v_1(x)\, v_2\left(\frac{x}{\varepsilon^{q_1}}\right)\cdots\nabla_{y_{j-1}}\\
&\qquad\qquad\times v_j\left(\frac{x}{\varepsilon^{q_{j-1}}}\right)\cdots v_{i+1}\left(\frac{x}{\varepsilon^{q_i}}\right)\\
&\qquad\times c_1(t)\, c_2\left(\frac{t}{\varepsilon^{r_1}}\right)\cdots c_{\lambda+1}\left(\frac{t}{\varepsilon^{r_\lambda}}\right) dx\,dt = 0.
\end{aligned}\tag{41}$$

Suppose that $p + q_i - r_\lambda \geq 0$ and $p - q_i \geq 0$ (which also guarantees that $p > 0$ as required above); then, by Theorems 7 and 4, we have left

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\Omega_T} -\varepsilon^{-q_i}u^{\varepsilon}(x,t)\,\varepsilon^{p+q_i-r_\lambda} v_1(x)\, v_2\left(\frac{x}{\varepsilon^{q_1}}\right)\cdots v_{i+1}\left(\frac{x}{\varepsilon^{q_i}}\right)\\
&\qquad\times c_1(t)\, c_2\left(\frac{t}{\varepsilon^{r_1}}\right)\cdots\partial_{s_\lambda} c_{\lambda+1}\left(\frac{t}{\varepsilon^{r_\lambda}}\right)\\
&\qquad + a\left(\frac{x}{\varepsilon^{q_1}},\ldots,\frac{x}{\varepsilon^{q_n}},\frac{t}{\varepsilon^{r_1}},\ldots,\frac{t}{\varepsilon^{r_m}}\right)\nabla u^{\varepsilon}(x,t)\\
&\qquad \cdot\varepsilon^{p-q_i} v_1(x)\, v_2\left(\frac{x}{\varepsilon^{q_1}}\right)\cdots v_i\left(\frac{x}{\varepsilon^{q_{i-1}}}\right)\nabla_{y_i} v_{i+1}\left(\frac{x}{\varepsilon^{q_i}}\right)\\
&\qquad\times c_1(t)\, c_2\left(\frac{t}{\varepsilon^{r_1}}\right)\cdots c_{\lambda+1}\left(\frac{t}{\varepsilon^{r_\lambda}}\right) dx\,dt = 0,
\end{aligned}\tag{42}$$

which is the point of departure for deriving the local problems and the independency.

We distinguish four different cases where ρ_i is either zero (nonresonance) or one (resonance) and d_i is either zero or positive.

Case 1. Consider $\rho_i = 0$ and $d_i = 0$. We choose $\lambda = m$ and $p = q_i$. This means that $p + q_i - r_\lambda = 2q_i - r_m > 0$ since $d_i = \rho_i = 0$ and $p - q_i = q_i - q_i = 0$. This implies that (42) is valid. We get

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\Omega_T} -\varepsilon^{-q_i}u^{\varepsilon}(x,t)\,\varepsilon^{2q_i-r_m} v_1(x)\, v_2\left(\frac{x}{\varepsilon^{q_1}}\right)\cdots v_{i+1}\left(\frac{x}{\varepsilon^{q_i}}\right)\\
&\qquad\times c_1(t)\, c_2\left(\frac{t}{\varepsilon^{r_1}}\right)\cdots\partial_{s_m} c_{m+1}\left(\frac{t}{\varepsilon^{r_m}}\right)\\
&\qquad + a\left(\frac{x}{\varepsilon^{q_1}},\ldots,\frac{x}{\varepsilon^{q_n}},\frac{t}{\varepsilon^{r_1}},\ldots,\frac{t}{\varepsilon^{r_m}}\right)\nabla u^{\varepsilon}(x,t)
\end{aligned}$$

$$\cdot \varepsilon^{0} v_1(x) v_2\left(\frac{x}{\varepsilon^{q_1}}\right)\cdots v_i\left(\frac{x}{\varepsilon^{q_{i-1}}}\right)\nabla_{y_i} v_{i+1}\left(\frac{x}{\varepsilon^{q_i}}\right)$$
$$\times c_1(t) c_2\left(\frac{t}{\varepsilon^{r_1}}\right)\cdots c_{m+1}\left(\frac{t}{\varepsilon^{r_m}}\right) dx\,dt = 0, \tag{43}$$

where we let $\varepsilon \to 0$ and obtain by means of Theorems 7 and 4

$$\int_{\Omega_T}\int_{\mathcal{Y}_{n,m}} a(y^n, s^m)\left(\nabla u(x,t) + \sum_{j=1}^{n}\nabla_{y_j} u_j(x,t,y^j,s^m)\right)$$
$$\cdot v_1(x) v_2(y_1)\cdots v_i(y_{i-1})\nabla_{y_i} v_{i+1}(y_i) c_1(t)$$
$$\times c_2(s_1)\cdots c_{m+1}(s_m)\, dy^n\, ds^m\, dx\, dt = 0. \tag{44}$$

By the Variational Lemma, we have

$$\int_{Y_i}\cdots\int_{Y_n} a(y^n, s^m)\left(\nabla u(x,t) + \sum_{j=1}^{n}\nabla_{y_j} u_j(x,t,y^j,s^m)\right)$$
$$\cdot \nabla_{y_i} v_{i+1}(y_i)\, dy_n\cdots dy_i = 0, \tag{45}$$

a.e. in $\Omega_T \times S^m \times Y_1 \times \cdots \times Y_{i-1}$ for all $v_{i+1} \in C_\sharp^\infty(Y_i)/\mathbb{R}$ and by density for all $v_{i+1} \in H_\sharp^1(Y_i)/\mathbb{R}$. This is the weak form of the local problem in this case. In what follows Theorems 7 and 4, the variational lemma and the density argument are used in a corresponding way.

Case 2. Consider $\rho_i = 1$ and $d_i = 0$. We again choose $\lambda = m$ and $p = q_i$. We then have $p + q_i - r_\lambda = 2q_i - r_m = 0$ since $d_i = 0$ and $\rho_i = 1$ and $p - q_i = q_i - q_i = 0$ which implies that we may again use (42). We get

$$\lim_{\varepsilon\to 0}\int_{\Omega_T} -\varepsilon^{-q_i} u^\varepsilon(x,t)\varepsilon^{0} v_1(x) v_2\left(\frac{x}{\varepsilon^{q_1}}\right)\cdots v_{i+1}\left(\frac{x}{\varepsilon^{q_i}}\right)$$
$$\times c_1(t) c_2\left(\frac{t}{\varepsilon^{r_1}}\right)\cdots \partial_{s_m} c_{m+1}\left(\frac{t}{\varepsilon^{r_m}}\right)$$
$$+ a\left(\frac{x}{\varepsilon^{q_1}},\ldots,\frac{x}{\varepsilon^{q_n}},\frac{t}{\varepsilon^{r_1}},\ldots,\frac{t}{\varepsilon^{r_m}}\right)\nabla u^\varepsilon(x,t) \tag{46}$$
$$\cdot \varepsilon^{0} v_1(x) v_2\left(\frac{x}{\varepsilon^{q_1}}\right)\cdots v_i\left(\frac{x}{\varepsilon^{q_{i-1}}}\right)\nabla_{y_i} v_{i+1}\left(\frac{x}{\varepsilon^{q_i}}\right)$$
$$\times c_1(t) c_2\left(\frac{t}{\varepsilon^{r_1}}\right)\cdots c_{m+1}\left(\frac{t}{\varepsilon^{r_m}}\right) dx\,dt = 0$$

and, passing to the limit,

$$\int_{\Omega_T}\int_{\mathcal{Y}_{n,m}} -u_i(x,t,y^i,s^m) v_1(x) v_2(y_1)\cdots v_{i+1}(y_i)$$
$$\times c_1(t) c_2(s_1)\cdots \partial_{s_m} c_{m+1}(s_m)$$
$$+ a(y^n, s^m)\left(\nabla u(x,t) + \sum_{j=1}^{n}\nabla_{y_j} u_j(x,t,y^j,s^m)\right)$$
$$\cdot v_1(x) v_2(y_1)\cdots v_i(y_{i-1})\nabla_{y_i} v_{i+1}(y_i) c_1(t)$$
$$\times c_2(s_1)\cdots c_{m+1}(s_m)\, dy^n\, ds^m\, dx\, dt = 0. \tag{47}$$

By the variational lemma

$$\int_{S_m}\int_{Y_i}\cdots\int_{Y_n} -u_i(x,t,y^i,s^m) v_{i+1}(y_i)\partial_{s_m} c_{m+1}(s_m)$$
$$+ a(y^n, s^m)\left(\nabla u(x,t) + \sum_{j=1}^{n}\nabla_{y_j} u_j(x,t,y^j,s^m)\right)$$
$$\cdot \nabla_{y_i} v_{i+1}(y_i) c_{m+1}(s_m)\, dy_n\cdots dy_i\, ds_m = 0 \tag{48}$$

a.e. for all $v_{i+1} \in H_\sharp^1(Y_i)/\mathbb{R}$ and $c_{m+1} \in C_\sharp^\infty(S_m)$, which is the weak form of the local problem in this second case.

Case 3. Consider $\rho_i = 0$ and $d_i > 0$. Let λ be fixed and successively be $m, \ldots, m - d_i + 1$. Choose $p = r_\lambda - q_i$ which immediately yields that $p + q_i - r_\lambda = 0$. Furthermore, $p - q_i = r_\lambda - 2q_i > 0$ by the restriction of λ and the definition of d_i.

Thus we have from (42)

$$\lim_{\varepsilon\to 0}\int_{\Omega_T} -\varepsilon^{-q_i} u^\varepsilon(x,t)\varepsilon^{0} v_1(x) v_2\left(\frac{x}{\varepsilon^{q_1}}\right)\cdots v_{i+1}\left(\frac{x}{\varepsilon^{q_i}}\right)$$
$$\times c_1(t) c_2\left(\frac{t}{\varepsilon^{r_1}}\right)\cdots \partial_{s_\lambda} c_{\lambda+1}\left(\frac{t}{\varepsilon^{r_\lambda}}\right)$$
$$+ a\left(\frac{x}{\varepsilon^{q_1}},\ldots,\frac{x}{\varepsilon^{q_n}},\frac{t}{\varepsilon^{r_1}},\ldots,\frac{t}{\varepsilon^{r_m}}\right)\nabla u^\varepsilon(x,t)$$
$$\cdot \varepsilon^{r_\lambda - 2q_i} v_1(x) v_2\left(\frac{x}{\varepsilon^{q_1}}\right)\cdots v_i\left(\frac{x}{\varepsilon^{q_{i-1}}}\right)\nabla_{y_i} v_{i+1}\left(\frac{x}{\varepsilon^{q_i}}\right)$$
$$\times c_1(t) c_2\left(\frac{t}{\varepsilon^{r_1}}\right)\cdots c_{\lambda+1}\left(\frac{t}{\varepsilon^{r_\lambda}}\right) dx\,dt = 0. \tag{49}$$

We let ε tend to zero and obtain

$$\int_{\Omega_T}\int_{\mathcal{Y}_{i,\lambda}} -u_i(x,t,y^i,s^\lambda) v_1(x) v_2(y_1)\cdots v_{i+1}(y_i)$$
$$\times c_1(t) c_2(s_1)\cdots \partial_{s_\lambda} c_{\lambda+1}(s_\lambda)\, dy^i\, ds^\lambda\, dx\, dt = 0 \tag{50}$$

and we have left

$$\int_{S_\lambda} -u_i(x,t,y^i,s^\lambda)\partial_{s_\lambda} c_{\lambda+1}(s_\lambda)\, ds_\lambda = 0, \tag{51}$$

a.e. for all $c_{\lambda+1} \in C_\sharp^\infty(S_\lambda)$. This means that u_i is independent of s_λ; thus, u_i does not depend on $s_{m-d_i+1}, \ldots, s_m$. Next we choose $p = q_i$ and $\lambda = m-d_i$. We have $p+q_i-r_\lambda = 2q_i-r_{m-d_i} > 0$ and $p-q_i = 0$ and we may again use (42). We have

$$\lim_{\varepsilon\to 0}\int_{\Omega_T} -\varepsilon^{-q_i}u^\varepsilon(x,t)\,\varepsilon^{2q_i-r_{m-d_i}}v_1(x)\,v_2\left(\frac{x}{\varepsilon^{q_1}}\right)\cdots v_{i+1}\left(\frac{x}{\varepsilon^{q_i}}\right) \times c_1(t)\,c_2\left(\frac{t}{\varepsilon^{r_1}}\right)\cdots\partial_{s_{m-d_i}}c_{m-d_i+1}\left(\frac{t}{\varepsilon^{r_{m-d_i}}}\right) + a\left(\frac{x}{\varepsilon^{q_1}},\dots,\frac{x}{\varepsilon^{q_n}},\frac{t}{\varepsilon^{r_1}},\dots,\frac{t}{\varepsilon^{r_m}}\right)\nabla u^\varepsilon(x,t) \cdot\varepsilon^0 v_1(x)\,v_2\left(\frac{x}{\varepsilon^{q_1}}\right)\cdots v_i\left(\frac{x}{\varepsilon^{q_{i-1}}}\right)\nabla_{y_i}v_{i+1}\left(\frac{x}{\varepsilon^{q_i}}\right) \times c_1(t)\,c_2\left(\frac{t}{\varepsilon^{r_1}}\right)\cdots c_{m-d_i+1}\left(\frac{t}{\varepsilon^{r_{m-d_i}}}\right)dx\,dt = 0, \tag{52}$$

where a passage to the limit yields

$$\int_{\Omega_T}\int_{\mathcal{Y}_{n,m}} a(y^n,s^m)\left(\nabla u(x,t)+\sum_{j=1}^{n}\nabla_{y_j}u_j\left(x,t,y^j,s^{m-d_i}\right)\right) \cdot v_1(x)\,v_2(y_1)\cdots v_i(y_{i-1})\nabla_{y_i}v_{i+1}(y_i) \times c_1(t)\,c_2(s_1)\cdots c_{m-d_i+1}\left(s_{m-d_i}\right)dy^n\,ds^m\,dx\,dt = 0, \tag{53}$$

and finally

$$\int_{S_{m-d_i+1}}\cdots\int_{S_m}\int_{Y_i}\cdots\int_{Y_n} a(y^n,s^m) \times\left(\nabla u(x,t)+\sum_{j=1}^{n}\nabla_{y_j}u_j\left(x,t,y^j,s^{m-d_i}\right)\right) \cdot\nabla_{y_i}v_{i+1}(y_i)\,dy_n\cdots dy_i\,ds_m\cdots ds_{m-d_i+1} = 0, \tag{54}$$

a.e. for all $v_{i+1}\in H^1_\sharp(Y_i)/\mathbb{R}$, which is the weak form of the local problem.

Case 4. Consider $\rho_i = 1$ and $d_i > 0$. Let λ be fixed and successively be $m,\dots,m-d_i+1$. Choose $p = r_\lambda - q_i$ directly implying that $p+q_i-r_\lambda = 0$. Moreover, $p-q_i = r_\lambda - 2q_i > 0$ by the restriction of λ and the definition of d_i and ρ_i. Hence using (42), we obtain

$$\lim_{\varepsilon\to 0}\int_{\Omega_T} -\varepsilon^{-q_i}u^\varepsilon(x,t)\,\varepsilon^{0}v_1(x)\,v_2\left(\frac{x}{\varepsilon^{q_1}}\right)\cdots v_{i+1}\left(\frac{x}{\varepsilon^{q_i}}\right) \times c_1(t)\,c_2\left(\frac{t}{\varepsilon^{r_1}}\right)\cdots\partial_{s_\lambda}c_{\lambda+1}\left(\frac{t}{\varepsilon^{r_\lambda}}\right) + a\left(\frac{x}{\varepsilon^{q_1}},\dots,\frac{x}{\varepsilon^{q_n}},\frac{t}{\varepsilon^{r_1}},\dots,\frac{t}{\varepsilon^{r_m}}\right)\nabla u^\varepsilon(x,t) \cdot\varepsilon^{r_\lambda-2q_i} v_1(x)\,v_2\left(\frac{x}{\varepsilon^{q_1}}\right)\cdots v_i\left(\frac{x}{\varepsilon^{q_{i-1}}}\right)\nabla_{y_i}v_{i+1}\left(\frac{x}{\varepsilon^{q_i}}\right) \times c_1(t)\,c_2\left(\frac{t}{\varepsilon^{r_1}}\right)\cdots c_{\lambda+1}\left(\frac{t}{\varepsilon^{r_\lambda}}\right)dx\,dt = 0. \tag{55}$$

Passing to the limit, we get

$$\int_{\Omega_T}\int_{\mathcal{Y}_{i,\lambda}} -u_i\left(x,t,y^i,s^\lambda\right)v_1(x)\,v_2(y_1)\cdots v_{i+1}(y_i) \times c_1(t)\,c_2(s_1)\cdots\partial_{s_\lambda}c_{\lambda+1}(s_\lambda)\,dy^i\,ds^\lambda\,dx\,dt = 0. \tag{56}$$

That is,

$$\int_{S_\lambda} -u_i\left(x,t,y^i,s^\lambda\right)\partial_{s_\lambda}c_{\lambda+1}(s_\lambda)\,ds_\lambda = 0 \tag{57}$$

a.e. for all $c_{\lambda+1}\in C^\infty_\sharp(S_\lambda)$, and hence u_i is independent of s_λ. Next we choose $p = q_i$ and $\lambda = m-d_i$ in (42). Thus we have $p+q_i-r_\lambda = 2q_i - r_{m-d_i} = 0$ and $p-q_i = 0$ and we get

$$\lim_{\varepsilon\to 0}\int_{\Omega_T} -\varepsilon^{-q_i}u^\varepsilon(x,t)\,\varepsilon^{0}v_1(x)\,v_2\left(\frac{x}{\varepsilon^{q_1}}\right)\cdots v_{i+1}\left(\frac{x}{\varepsilon^{q_i}}\right) \times c_1(t)\,c_2\left(\frac{t}{\varepsilon^{r_1}}\right)\cdots\partial_{s_{m-d_i}}c_{m-d_i+1}\left(\frac{t}{\varepsilon^{r_{m-d_i}}}\right) + a\left(\frac{x}{\varepsilon^{q_1}},\dots,\frac{x}{\varepsilon^{q_n}},\frac{t}{\varepsilon^{r_1}},\dots,\frac{t}{\varepsilon^{r_m}}\right)\nabla u^\varepsilon(x,t) \cdot\varepsilon^0 v_1(x)\,v_2\left(\frac{x}{\varepsilon^{q_1}}\right)\cdots v_i\left(\frac{x}{\varepsilon^{q_{i-1}}}\right)\nabla_{y_i}v_{i+1}\left(\frac{x}{\varepsilon^{q_i}}\right) \times c_1(t)\,c_2\left(\frac{t}{\varepsilon^{r_1}}\right)\cdots c_{m-d_i+1}\left(\frac{t}{\varepsilon^{r_{m-d_i}}}\right)dx\,dt = 0. \tag{58}$$

We let ε go to zero obtaining

$$\int_{\Omega_T}\int_{\mathcal{Y}_{n,m}} -u_i\left(x,t,y^i,s^{m-d_i}\right)v_1(x)\,v_2(y_1)\cdots v_{i+1}(y_i) \times c_1(t)\,c_2(s_1)\cdots\partial_{s_{m-d_i}}c_{m-d_i+1}\left(s_{m-d_i}\right) + a(y^n,s^m)\left(\nabla u(x,t)+\sum_{j=1}^{n}\nabla_{y_j}u_j\left(x,t,y^j,s^{m-d_i}\right)\right) \cdot v_1(x)\,v_2(y_1)\cdots v_i(y_{i-1})\nabla_{y_i}v_{i+1}(y_i) \times c_1(t)\,c_2(s_1)\cdots c_{m-d_i+1}\left(s_{m-d_i}\right)dy^n\,ds^m\,dx\,dt = 0 \tag{59}$$

and finally we arrive at

$$\int_{S_{m-d_i}}\cdots\int_{S_m}\int_{Y_i}\cdots\int_{Y_n} -u_i\left(x,t,y^i,s^{m-d_i}\right)v_{i+1} \times(y_i)\,\partial_{s_{m-d_i}}c_{m-d_i+1}\left(s_{m-d_i}\right) + a(y^n,s^m)\left(\nabla u(x,t)+\sum_{j=1}^{n}\nabla_{y_j}u_j\left(x,t,y^j,s^{m-d_i}\right)\right) \cdot\nabla_{y_i}v_{i+1}(y_i)\,c_{m-d_i+1}\left(s_{m-d_i}\right)dy_n\cdots dy_i\,ds_m\cdots ds_{m-d_i} = 0 \tag{60}$$

a.e. for all $v_{i+1}\in H^1_\sharp(Y_i)/\mathbb{R}$ and $c_{m-d_i+1}\in C^\infty_\sharp(S_{m-d_i})$, the weak form of the local problem. □

Remark 12. The result above can be extended to any meaningful choice of jointly well-separated scales by means of the general compactness results in Theorems 4 and 7 and are hence not restricted to scales that are powers of ε; see, for example, [11] for the case with an arbitrary number of temporal scales but only one spatial micro scale. To make the exposition clear, we have assumed linearity, but the result can be extended to monotone, not necessarily linear, problems using standard methods.

Remark 13. The wellposedness of the homogenized problem follows from G-convergence; see, for example, Sections 3 and 4 in [20]. See also Theorem 4.19 in [17] for an easily accessible description of the regularity of the G-limit b. The existence of solutions to the local problems follows from the fact that they appear as limits in appropriate convergence processes. Concerning uniqueness, the coercivity of the elliptic part follows along the lines of the proof of Theorem 2.11 in [16] and for those containing a derivative with respect to some local time scale general theory for linear parabolic equations apply, see, for example, Section 23 in [18]. Normally multiscale homogenization results are formulated as in Theorem 9 without separation of variables and if we study slightly more general problems, for example, those with monotone operators where the linearity has been relaxed, such separation is not possible. However, in Corollary 2.12 in [16], a technique similar to separation of variables of the type sometimes used for conventional homogenization problems is developed. Here one scale at the time is removed in an inductive process and the homogenized coefficient is computed. We believe that a similar procedure could be successful also for the type of problem studied here but would be quite technical.

4.2. Illustration of Theorem 9. To illustrate the use of Theorem 9, we apply it to the 3, 3-scaled parabolic homogenization problem

$$\begin{aligned} &\partial_t u^\varepsilon(x,t) - \nabla\cdot\left(a\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},\frac{t}{\varepsilon^{r_1}},\frac{t}{\varepsilon^{r_2}}\right)\nabla u^\varepsilon(x,t)\right) \\ &\quad = f(x,t) \quad \text{in } \Omega_T, \\ &\qquad u^\varepsilon(x,t) = 0 \quad \text{on } \partial\Omega\times(0,T), \\ &\qquad\quad u^\varepsilon(x,0) = u^0(x) \quad \text{in } \Omega, \end{aligned} \tag{61}$$

where $0 < r_1 < r_2$, $f \in L^2(\Omega_T)$, $u^0 \in L^2(\Omega)$, and the structure conditions

(*B1*) $a \in C_\sharp(\mathcal{Y}_{2,2})^{N\times N}$

(*B2*) $a(y^2,s^2)\xi\cdot\xi \geq \alpha|\xi|^2$ for all $(y^2,s^2) \in \mathbb{R}^{2N}\times\mathbb{R}^2$, all $\xi\in\mathbb{R}^N$ and some $\alpha > 0$

are satisfied.

Table 1: d_i and ρ_i for $i = 1$.

r_1 and r_2 relative to $2q_1 = 2$	d_1	ρ_1
$0 < r_1 < r_2 < 2$	0	0
$0 < r_1 < r_2 = 2$	0	1
$0 < r_1 < 2 < r_2$	1	0
$2 = r_1 < r_2$	1	1
$2 < r_1 < r_2$	2	0

Table 2: d_i and ρ_i for $i = 2$.

r_1 and r_2 relative to $2q_2 = 4$	d_2	ρ_2
$0 < r_1 < r_2 < 4$	0	0
$0 < r_1 < r_2 = 4$	0	1
$0 < r_1 < 4 < r_2$	1	0
$4 = r_1 < r_2$	1	1
$4 < r_1 < r_2$	2	0

We note that the assumptions of Theorem 9 are satisfied in this case. Hence the convergence results in (31) hold and, for the homogenized matrix,

$$\begin{aligned} &b(x,t)\nabla u(x,t) \\ &\quad = \int_{\mathcal{Y}_{2,2}} a\left(y^2,s^2\right)\left(\nabla u(x,t) + \nabla_{y_1}u_1\left(x,t,y_1,s^2\right)\right. \\ &\qquad\qquad \left. + \nabla_{y_2}u_2\left(x,t,y^2,s^2\right)\right) dy^2\, ds^2. \end{aligned} \tag{62}$$

Furthermore, $u_1 \in L^2(\Omega_T\times S^2; H^1_\sharp(Y_1)/\mathbb{R})$ and $u_2 \in L^2(\Omega_T\times\mathcal{Y}_{1,2}; H^1_\sharp(Y_2)/\mathbb{R})$ are the unique solutions to the system of local problems

$$\begin{aligned} &\rho_i\partial_{s_{2-d_i}}u_i\left(x,t,y^i,s^2\right) - \nabla_{y_i} \\ &\quad\cdot\int_{S_{2-d_i+1}}\cdots\int_{S_2}\int_{Y_{i+1}}\cdots\int_{Y_2} a\left(y^2,s^2\right) \\ &\quad\times\left(\nabla u(x,t) + \nabla_{y_1}u_1\left(x,t,y_1,s^2\right)\right. \\ &\quad\left. + \nabla_{y_2}u_2\left(x,t,y^2,s^2\right)\right) \\ &\quad\times dy_2\cdots dy_{i+1}\, ds_2\cdots ds_{2-d_i+1} = 0 \end{aligned} \tag{63}$$

for $i = 1, 2$, where u_i is independent of $s_{2-d_i+1},\ldots,s_2$.

To find the local problems and the independencies explicitly, we need to identify which values of d_i, and ρ_i to use. To find d_i, we simply count the number of temporal scales faster than the square of the ith spatial scale for different choices of r_1 and r_2. Moreover, resonance ($\rho_i = 1$) occurs when the square of the ith spatial scale coincides with one of the temporal scales.

First we consider the slowest spatial scale; that is, we let $i = 1$. Note that $2q_1 = 2$. If $2q_1 = 2 < r_1$, then $d_1 = 2$, if $r_1 \leq 2 < r_2$ then $d_1 = 1$ and if $2 \geq r_2$, then $d_1 = 0$. Regarding resonance, if $r_1 = 2$ or $r_2 = 2$; then $\rho_1 = 1$; otherwise, $\rho_1 = 0$. For lucidity, we present which values of r_1 and r_2 that give the different values of d_1 and ρ_1 in Table 1.

In a similar way as above, we get for $i = 2$ Table 2.

We start by sorting out the independencies of the local temporal variables. As noted, for $i = 1, 2$, u_i is independent of $s_{2-d_i+1}, \ldots, s_2$, which means that if $d_i = 1$, then u_i is independent of s_2 and if $d_i = 2$, then u_i is independent of both s_1 and s_2. In terms of r_1 and r_2, we have that for $r_2 > 2$, u_1 is independent of s_2 and for $r_1 > 2$ also independent of s_1, for $r_2 > 4$, u_2 is independent of s_2 and moreover, for $r_1 > 4$ it holds that u_2 is also independent of s_1.

To find the local problems, we examine all possible combinations of (d_1, ρ_1) and (d_2, ρ_2), where 13 are realizable depending on which values r_1 and r_2 may assume. Each row in the tables gives rise to a local problem via (63). This means that each combination gives two local problems. If a row occurs in several combinations, the same local problem reappears. If we start by choosing the first row in the second table, that is $(d_2, \rho_2) = (0, 0)$, this can be combined with all five rows from the first table, which means that the local problem descending from $(d_2, \rho_2) = (0, 0)$ is common to these combinations. By (63), this common local problem is

$$-\nabla_{y_2} \cdot \left(a\left(y^2, s^2\right)\left(\nabla u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2\right)\right) = 0. \qquad (64)$$

If we combine $(d_2, \rho_2) = (0, 0)$ with $(d_1, \rho_1) = (0, 0)$ we have in terms of r_1 and r_2 that $0 < r_1 < r_2 < 2$. The other local problem in this case is

$$-\nabla_{y_1} \cdot \int_{Y_2} a\left(y^2, s^2\right)\left(\nabla u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2\right) dy_2 = 0. \qquad (65)$$

In combination with $(d_1, \rho_1) = (0, 1)$, that is, $0 < r_1 < r_2 = 2$, we obtain instead

$$\partial_{s_2} u_1 - \nabla_{y_1} \cdot \int_{Y_2} a\left(y^2, s^2\right)\left(\nabla u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2\right) dy_2 = 0, \qquad (66)$$

and for $(d_1, \rho_1) = (1, 0)$, which means that $0 < r_1 < 2 < r_2 < 4$, we have

$$-\nabla_{y_1} \cdot \int_{S_2} \int_{Y_2} a\left(y^2, s^2\right)\left(\nabla u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2\right) dy_2\, ds_2 = 0. \qquad (67)$$

The fourth possible combination, that is, with $(d_1, \rho_1) = (1, 1)$, that is $r_1 = 2 < r_2 < 4$, gives

$$\partial_{s_1} u_1 - \nabla_{y_1} \cdot \int_{S_2} \int_{Y_2} a\left(y^2, s^2\right) \times \left(\nabla u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2\right) dy_2\, ds_2 = 0 \qquad (68)$$

and finally for $(d_1, \rho_1) = (2, 0)$, that is $2 < r_1 < r_2 < 4$, the second local problem is

$$-\nabla_{y_1} \cdot \int_{S^2} \int_{Y_2} a\left(y^2, s^2\right)\left(\nabla u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2\right) dy_2\, ds^2 = 0. \qquad (69)$$

Next we consider $(d_2, \rho_2) = (0, 1)$ in Table 2, which corresponds to $0 < r_1 < r_2 = 4$ and gives the local problem

$$\partial_{s_2} u_2 - \nabla_{y_2} \cdot \left(a\left(y^2, s^2\right)\left(\nabla u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2\right)\right) = 0. \qquad (70)$$

Here we have three possible combinations, namely with $(d_1, \rho_1) = (1, 0)$, $(1, 1)$, and $(2, 0)$. We note that we have already derived the local problems corresponding to these rows. Thus, the second local problem for $r_2 = 4$ and $0 < r_1 < 2$ is given by (67) for $r_2 = 4$ and $r_1 = 2$ by (68) and for $2 < r_1 < r_2 = 4$ by (69).

We proceed by choosing $(d_2, \rho_2) = (1, 0)$ in Table 2, yielding

$$-\nabla_{y_2} \cdot \left(\left(\int_{S_2} a\left(y^2, s^2\right) ds_2\right)\left(\nabla u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2\right)\right) = 0. \qquad (71)$$

The choice $(d_2, \rho_2) = (1, 0)$ can be combined with three different rows from Table 1, $(d_1, \rho_1) = (1, 0)$, $(1, 1)$, and $(2, 0)$. In combination with $(d_1, \rho_1) = (1, 0)$, which means that $r_2 > 4$ and $0 < r_1 < 2$, we have

$$-\nabla_{y_1} \cdot \int_{Y_2} \left(\int_{S_2} a\left(y^2, s^2\right) ds_2\right) \times \left(\nabla u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2\right) dy_2 = 0, \qquad (72)$$

which is essentially the same as (67) but with the integration over S_2 directly on $a(y^2, s^2)$ since both u_1 and u_2 are independent of s_2. For $(d_1, \rho_1) = (1, 1)$, that is, $r_2 > 4$ and $r_1 = 2$, we have

$$\partial_{s_1} u_1 - \nabla_{y_1} \cdot \int_{Y_2} \left(\int_{S_2} a\left(y^2, s^2\right) ds_2\right) \times \left(\nabla u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2\right) dy_2 = 0, \qquad (73)$$

which is the same as (68), but where we may integrate directly on $a(y^2, s^2)$ in the same manner as above. For the third possibility, $(d_1, \rho_1) = (2, 0)$, $2 < r_1 < 4 < r_2$, we get

$$-\nabla_{y_1} \cdot \int_{S_1} \int_{Y_2} \left(\int_{S_2} a\left(y^2, s^2\right) ds_2\right) \times \left(\nabla u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2\right) dy_2\, ds_1 = 0, \qquad (74)$$

the same as (69), except for the position of the integration over S_2.

The next row in Table 2 to consider is $(d_2, \rho_2) = (1, 1)$, which can be combined only with $(d_1, \rho_1) = (2, 0)$. This combination corresponds to $4 = r_1 < r_2$ and gives

$$\partial_{s_1} u_2 - \nabla_{y_2} \cdot \left(\left(\int_{S_2} a\left(y^2, s^2\right) ds_2\right)\left(\nabla u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2\right)\right) = 0 \qquad (75)$$

and again (74).

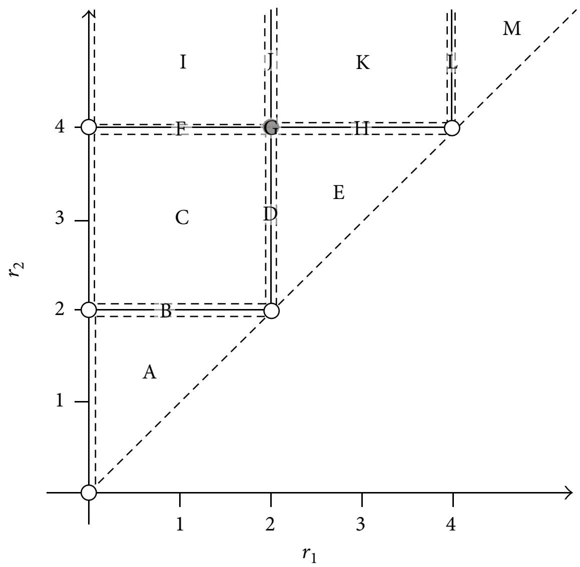

Figure 1: The 13 cases depicted in the r_1r_2 plane in the order of appearance.

Finally, for the row $(d_2, \rho_2) = (2, 0)$ together with $(d_1, \rho_1) = (2, 0)$, that is, $4 < r_1 < r_2$, we get

$$
\begin{aligned}
&-\nabla_{y_2}\cdot\left(\left(\int_{S^2} a\left(y^2, s^2\right) ds^2\right)\left(\nabla u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2\right)\right) = 0,\\
&-\nabla_{y_1}\cdot\int_{Y_2}\left(\int_{S^2} a\left(y^2, s^2\right) ds^2\right)\\
&\qquad\times\left(\nabla u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2\right) dy_2 = 0,
\end{aligned}
\tag{76}
$$

where the latter is essentially the same as (69) and (74).

Thus, having considered all possible combinations of r_1 and r_2, we have obtained 13 different cases, A–M in Figure 1, governed by two local problems each.

In the figure, cases B, D, F, H, J, and L (straight line segments) correspond to single resonance, whereas in the case G (a single point), there is double resonance. In the remaining cases (open two-dimensional regions), there is no resonance.

Remark 14. Note that for a problem with fixed scales the finding of the local problems is very straightforward. For example, if we study (61) with $r_1 = 2$ and $r_2 = 17$, we have $m = 2$, $n = 2$, $d_1 = 1$, $\rho_1 = 1$, $d_2 = 1$, and $\rho_2 = 0$. We obtain that both u_1 and u_2 are independent of s_2. Inserting $d_1 = 1, \rho_1 = 1$ in (34) immediately gives the problem (73) and $d_2 = 1, \rho_2 = 0$ results in (71). The example chosen above with variable time scale exponents reveals more of the applicability and comprehensiveness of the theorem.

Remark 15. The problem (61) was studied already in [17, 19], but using Theorem 9, the process is considerably shortened.

Appendix

Proof of Theorem 4

This section is devoted to the proof of Theorem 4. The theorem was first formulated and proven in a detailed preprint version from 2010 of [11]. It was also given as Theorem 2.74 in [17] together with the proof. We first need the following fundamental compactness result; see also, for example, Theorem 2.66 in [17]. Observe that the concept of jointly separated scales amounts to the obvious modification of jointly well-separated scales.

Theorem A.1. *Let $\{u^\varepsilon\}$ be a bounded sequence in $L^2(\Omega_T)$ and suppose that the lists $\{\varepsilon_1, \ldots, \varepsilon_n\}$ and $\{\varepsilon'_1, \ldots, \varepsilon'_m\}$ are jointly separated. Then there exists u_0 in $L^2(\Omega_T \times \mathcal{Y}_{n,m})$ such that, up to a subsequence,*

$$
u^\varepsilon(x,t) \overset{n+1,m+1}{\rightharpoonup} u_0\left(x,t,y^n,s^m\right). \tag{A.1}
$$

Proof. Introduce the spatiotemporal variable $\tilde{x} = (x, t)$ in $\tilde{\Omega} = \Omega_T$ and let the corresponding local variable $\tilde{y}^{n+m-k} \in \tilde{Y}^{n+m-k} = (Y \times S)^{n+m-k}$, where k is the number of pairs of duplicates, that is, scales which tend to zero equally fast (see Definition 2), be defined in the following manner. Suppose that the resulting combined spatiotemporal list generated from the lists $\{\varepsilon_1, \ldots, \varepsilon_n\}$ and $\{\varepsilon'_1, \ldots, \varepsilon'_m\}$ is $\{\tilde{\varepsilon}_1, \ldots, \tilde{\varepsilon}_{n+m-k}\}$. Fix $l = 1, \ldots, n+m-k$; then, we have three mutually exclusive possibilities for the spatiotemporal scale $\tilde{\varepsilon}_l$. Firstly, if $\tilde{\varepsilon}_l$ tends to zero equally fast as ε_i for some $i = 1, \ldots, n$ but not equally fast as ε'_j for any $j = 1, \ldots, m$, then $\tilde{y}_l = (y_i, s_i^*)$ where $s_i^* \in S_i^* = S$ is a temporal "ghost" variable. Secondly, if $\tilde{\varepsilon}_l$ tends to zero equally fast as ε'_j for some $j = 1, \ldots, m$ but not equally fast as ε_i for any $i = 1, \ldots, n$, then $\tilde{y}_l = (y_j^*, s_j)$ where $y_j^* \in Y_j^* = Y$ is a spatial "ghost" variable. Finally, if $\tilde{\varepsilon}_l$ tends to zero equally fast as both ε_i and ε'_j for some $i = 1, \ldots, n$ and $j = 1, \ldots, m$, then $\tilde{y}_l = (y_i, s_j)$. We collect the introduced $n + m - 2k$ "ghost" variables in the total "ghost" variable $\tilde{y}^* \in \tilde{Y}^*$ where $\tilde{Y}^*$ is a Cartesian product of $n - k$ copies of S and $m - k$ copies of Y.

Within the framework of spatiotemporal quantities as introduced above, let

$$
\begin{aligned}
\tilde{u}^\varepsilon(\tilde{x}) &= u^\varepsilon(x,t),\\
\tilde{v}\left(\tilde{x}, \tilde{y}^{n+m-k}\right) &= v\left(x,t,y^n,s^m\right)
\end{aligned}
\tag{A.2}
$$

for any $v \in L^2(\Omega_T; C_\sharp(\mathcal{Y}_{n,m}))$. Note that the sequence $\{\tilde{u}^\varepsilon\}$ is bounded in $L^2(\tilde{\Omega})$ and that $\tilde{v} \in L^2(\tilde{\Omega}; C_\sharp(\tilde{Y}^{n+m-k}))$ is independent of the local "ghost" variables.

We have by definition

$$
\begin{aligned}
&\int_{\Omega_T} u^\varepsilon(x,t)\, v\left(x, t, \frac{x}{\varepsilon_1}, \ldots, \frac{x}{\varepsilon_n}, \frac{t}{\varepsilon'_1}, \ldots, \frac{t}{\varepsilon'_m}\right) dx\, dt\\
&\quad= \int_{\tilde{\Omega}} \tilde{u}^\varepsilon(\tilde{x})\, \tilde{v}\left(\tilde{x}, \frac{\tilde{x}}{\tilde{\varepsilon}_1}, \ldots, \frac{\tilde{x}}{\tilde{\varepsilon}_{n+m-k}}\right) d\tilde{x}.
\end{aligned}
\tag{A.3}
$$

Since $\{\widetilde{\varepsilon}_1,\ldots,\widetilde{\varepsilon}_{n+m-k}\}$ is separated, Theorem 2.4 in [16] ensures that, up to a subsequence the sequence $\{\widetilde{u}^\varepsilon\}$ does $(n+m-k+1)$-converge to $\widetilde{u}_0 \in L^2(\widetilde{\Omega}\times\widetilde{Y}^{n+m-k})$; that is,

$$\begin{aligned}
&\int_{\widetilde{\Omega}} \widetilde{u}^\varepsilon(\widetilde{x})\,\widetilde{v}\left(\widetilde{x},\frac{\widetilde{x}}{\widetilde{\varepsilon}_1},\ldots,\frac{\widetilde{x}}{\widetilde{\varepsilon}_{n+m-k}}\right)d\widetilde{x}\\
&\longrightarrow \int_{\widetilde{\Omega}}\int_{\widetilde{Y}^{n+m-k}} \widetilde{u}_0\left(\widetilde{x},\widetilde{y}^{n+m-k}\right)\widetilde{v}\left(\widetilde{x},\widetilde{y}^{n+m-k}\right)d\widetilde{y}^{n+m-k}d\widetilde{x}\\
&= \int_{\Omega_T}\int_{\mathcal{Y}_{n,m}}\left(\int_{\widetilde{Y}^*}\widetilde{u}_0\left(\widetilde{x},\widetilde{y}^{n+m-k}\right)d\widetilde{y}^*\right)\\
&\qquad\times v\left(x,t,y^n,s^m\right)dy^n\,ds^m\,dx\,dt\\
&= \int_{\Omega_T}\int_{\mathcal{Y}_{n,m}} u_0\left(x,t,y^n,s^m\right)v\left(x,t,y^n,s^m\right)dy^n\,ds^m\,dx\,dt
\end{aligned}\tag{A.4}$$

as ε tends to zero where

$$u_0\left(x,t,y^n,s^m\right)=\int_{\widetilde{Y}^*}\widetilde{u}_0\left(\widetilde{x},\widetilde{y}^{n+m-k}\right)d\widetilde{y}^*\tag{A.5}$$

belongs to $L^2(\Omega_T\times\mathcal{Y}_{n,m})$ due to the fact that

$$\left\|u_0\right\|_{L^2(\Omega_T\times\mathcal{Y}_{n,m})}\le\left\|\widetilde{u}_0\right\|_{L^2(\widetilde{\Omega}\times\widetilde{Y}^{n+m-k})}<\infty\tag{A.6}$$

by Jensen's inequality. To conclude, we have shown (A.1) and we are done. □

Remark A.2. In the proof above in the case when all spatial and temporal scales can be matched into pairs, we naturally interpret the formal instances of integration over the empty set $\widetilde{Y}^*$ as if there is no local spatiotemporal "ghost" integration involved.

We are now prepared to give the main proof of the appendix.

Proof of Theorem 4. Since $\{u^\varepsilon\}$ is bounded in $W_2^1(0,T;H_0^1(\Omega),L^2(\Omega))$,

$$u^\varepsilon(x,t)\rightharpoonup u(x,t)\quad\text{in } L^2\left(0,T;H_0^1(\Omega)\right)\tag{A.7}$$

for some unique $u\in L^2(0,T;H_0^1(\Omega))$ and, by Lemmas 8.2 and 8.4 in [21],

$$u^\varepsilon(x,t)\longrightarrow u(x,t)\quad\text{in } L^2\left(\Omega_T\right).\tag{A.8}$$

From Theorem A.1 and again using the boundedness of $\{u^\varepsilon\}$ in $W_2^1(0,T;H_0^1(\Omega),L^2(\Omega))$ we have, up to a subsequence,

$$\nabla u^\varepsilon(x,t)\overset{n+1,m+1}{\rightharpoonup} w_0\left(x,t,y^n,s^m\right)\tag{A.9}$$

for some w_0 in $L^2(\Omega_T\times\mathcal{Y}_{n,m})^N$.

We will now characterize w_0 in terms of gradients. Let $v\in D(\Omega;C_\sharp^\infty(Y^n)^N)\cap\mathcal{H}$ and $c\in D(0,T;C_\sharp^\infty(S^m))$ where $\mathcal{H}$ is the subspace of generalized divergence-free functions in $L^2(\Omega\times Y^n)^N$ defined according to

$$\begin{aligned}
\mathcal{H}=\Big\{&\psi\in L^2\left(\Omega;L_\sharp^2(Y^n)^N\right)\\
&:\int_{Y_{k+1}}\cdots\int_{Y_n}\nabla_{y_k}\cdot\psi\left(x,y^n\right)dy_n\cdots dy_{k+1}=0,\\
&k=1,\ldots,n-1\Big\}.
\end{aligned}\tag{A.10}$$

Using vc as a test function in (A.9) we get, up to a subsequence,

$$\begin{aligned}
&\int_{\Omega_T}\nabla u^\varepsilon(x,t)\cdot v\left(x,\frac{x}{\varepsilon_1},\ldots,\frac{x}{\varepsilon_n}\right)c\left(t,\frac{t}{\varepsilon_1'},\ldots,\frac{t}{\varepsilon_m'}\right)dx\,dt\\
&\longrightarrow\int_{\Omega_T}\int_{\mathcal{Y}_{n,m}}w_0\left(x,t,y^n,s^m\right)v\left(x,y^n\right)\\
&\qquad\times c\left(t,s^m\right)dy^n\,ds^m\,dx\,dt.
\end{aligned}\tag{A.11}$$

Using partial integration on Ω, the fact that u^ε and v vanish on $\partial\Omega$ (we only need that one of them does, though) and that $\nabla_{y_n}\cdot v=0$, the left-hand side of (A.11) may be written

$$\begin{aligned}
&-\int_{\Omega_T}u^\varepsilon(x,t)\left(\nabla+\sum_{k=1}^{n-1}\varepsilon_k^{-1}\nabla_{y_k}\right)\\
&\qquad\cdot v\left(x,\frac{x}{\varepsilon_1},\ldots,\frac{x}{\varepsilon_n}\right)c\left(t,\frac{t}{\varepsilon_1'},\ldots,\frac{t}{\varepsilon_m'}\right)dx\,dt.
\end{aligned}\tag{A.12}$$

We claim now that $\nabla_{y_k}\cdot v\in\mathcal{E}_{k+1}$ where

$$\begin{aligned}
\mathcal{E}_r=&\left\{\phi\in D\left(\Omega;C_\sharp^\infty\left(Y^n\right)\right)\right.\\
&\left.:\int_{Y_r}\cdots\int_{Y_n}\phi\left(x,y^n\right)dy_n\cdots dy_r=0\right\},
\end{aligned}\tag{A.13}$$

$r=1,\ldots,n$. Indeed, for any $k=1,\ldots,n-1$, we have $\nabla_{y_k}\cdot v\in D(\Omega;C_\sharp^\infty(Y^n))$ and

$$\int_{Y_{k+1}}\cdots\int_{Y_n}\nabla_{y_k}\cdot v\left(x,y^n\right)dy_n\cdots dy_{k+1}=0,\tag{A.14}$$

where we have simply employed the definition of v being in $\mathcal{H}$ making the multiple integrals to vanish, so $\nabla_{y_k}\cdot v\in\mathcal{E}_{k+1}$. Thus, by Corollary 3.4 in [16], we have that $\{\varepsilon_{k+1}^{-1}\nabla_{y_k}\cdot v(x,x/\varepsilon_1,\ldots,x/\varepsilon_n)\}$ is bounded in $H^{-1}(\Omega)$ for all $k=1,\ldots,n-1$. This boundedness yields an estimation

$$\begin{aligned}
&\left|\int_{\Omega_T}u^\varepsilon(x,t)\sum_{k=1}^{n-1}\varepsilon_k^{-1}\nabla_{y_k}\cdot v\left(x,\frac{x}{\varepsilon_1},\ldots,\frac{x}{\varepsilon_n}\right)\right.\\
&\qquad\left.\times c\left(t,\frac{t}{\varepsilon_1'},\ldots,\frac{t}{\varepsilon_m'}\right)dx\,dt\right|^2
\end{aligned}$$

$$\leq T\int_0^T\left|\int_\Omega u^\varepsilon(x,t)\times\sum_{k=1}^{n-1}\varepsilon_k^{-1}\nabla_{y_k}\cdot v\left(x,\frac{x}{\varepsilon_1},\dots,\frac{x}{\varepsilon_n}\right)c\left(t,\frac{t}{\varepsilon_1'},\dots,\frac{t}{\varepsilon_m'}\right)dx\right|^2dt$$
$$\leq T\int_0^T\left|\left\langle\sum_{k=1}^{n-1}\varepsilon_k^{-1}\nabla_{y_k}\cdot v\left(\cdot,\frac{\cdot}{\varepsilon_1},\dots,\frac{\cdot}{\varepsilon_n}\right),u^\varepsilon(\cdot,t)c\left(t,\frac{t}{\varepsilon_1'},\dots,\frac{t}{\varepsilon_m'}\right)\right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\right|^2dt$$
$$\leq T\int_0^T\left\|\sum_{k=1}^{n-1}\varepsilon_k^{-1}\nabla_{y_k}\cdot v\left(\cdot,\frac{\cdot}{\varepsilon_1},\dots,\frac{\cdot}{\varepsilon_n}\right)\right\|^2_{H^{-1}(\Omega)}\times\left\|u^\varepsilon(\cdot,t)c\left(t,\frac{t}{\varepsilon_1'},\dots,\frac{t}{\varepsilon_m'}\right)\right\|^2_{H_0^1(\Omega)}dt$$
$$\leq C_1\left(\sum_{k=1}^{n-1}\frac{\varepsilon_{k+1}}{\varepsilon_k}\left\|\varepsilon_{k+1}^{-1}\nabla_{y_k}\cdot v\left(\cdot,\frac{\cdot}{\varepsilon_1},\dots,\frac{\cdot}{\varepsilon_n}\right)\right\|_{H^{-1}(\Omega)}\right)^2\times\int_0^T\|u^\varepsilon(\cdot,t)\|^2_{H_0^1(\Omega)}\left|c\left(t,\frac{t}{\varepsilon_1'},\dots,\frac{t}{\varepsilon_m'}\right)\right|dt$$
$$\leq C_2\left(\sum_{k=1}^{n-1}\frac{\varepsilon_{k+1}}{\varepsilon_k}\right)^2\|u^\varepsilon\|^2_{L^2(0,T;H_0^1(\Omega))}\leq C_3\left(\sum_{k=1}^{n-1}\frac{\varepsilon_{k+1}}{\varepsilon_k}\right)^2\longrightarrow 0, \tag{A.15}$$

where we in the first inequality have utilized the Hölder inequality and in the last step have used that the scales are separated. We thus conclude that the left-hand side of (A.9) converges to

$$-\int_{\Omega_T}\int_{\mathcal{Y}_{n,m}}u(x,t)\nabla\cdot v(x,y^n)c(t,s^m)\,dy^n\,ds^m\,dx\,dt=\int_{\Omega_T}\int_{\mathcal{Y}_{n,m}}\nabla u(x,t)\cdot v(x,y^n)c(t,s^m)\,dy^n\,ds^m\,dx\,dt \tag{A.16}$$

for all $v\in D(\Omega;C_\sharp^\infty(Y^n)^N)\cap\mathscr{H}$ and all $c\in D(0,T;C_\sharp^\infty(S^m))$. Hence, from the right-hand side of (A.11), we obtain

$$\int_0^T\int_{S^m}\left(\int_\Omega\int_{Y^n}(w_0(x,t,y^n,s^m)-\nabla u(x,t))\cdot v(x,y^n)\,dy^n\,dx\right)c(t,s^m)\,ds^m\,dt=0. \tag{A.17}$$

By the Variational Lemma and utilizing the density property (i) of Lemma 3.7 in [16] it holds for every $v\in\mathscr{H}$ that

$$\int_\Omega\int_{Y^n}(w_0(x,t,y^n,s^m)-\nabla u(x,t))\cdot v(x,y^n)\,dy^n\,dx=0. \tag{A.18}$$

that is $w_0-\nabla u$ is in the orthogonal of $\mathscr{H}$ almost everywhere in $(0,T)\times S^m$. By property (ii) of Lemma 3.7 in [16], We conclude that

$$w_0(x,t,y^n,s^m)-\nabla u(x,t)=\sum_{k=1}^n\nabla_{y_k}u_k(x,t,y^k,s^m), \tag{A.19}$$

where $u_1\in L^2(\Omega;H_\sharp^1(Y_1)/\mathbb{R})$ and $u_k\in L^2(\Omega\times Y^{k-1};H_\sharp^1(Y_k)/\mathbb{R})$ for $k=2,\dots,n$ almost everywhere in $(0,T)\times S^m$.

What remains is to prove that $u_1\in L^2(\Omega_T\times S^m;H_\sharp^1(Y_1)/\mathbb{R})$ and $u_k\in L^2(\Omega_T\times\mathcal{Y}_{k-1,m};H_\sharp^1(Y_k)/\mathbb{R})$ for $k=2,\dots,n$. We will perform a proof by induction accomplished in two steps: the Base Case followed by the Inductive Step.

Base Case. We show that $u_1\in L^2(\Omega_T\times S^m;H_\sharp^1(Y_1)/\mathbb{R})$. We have, almost everywhere in $\Omega_T\times\mathcal{Y}_{1,m}$,

$$\begin{aligned}\nabla_{y_1}u_1(x,t,y_1,s^m)&=\int_{Y_2}\cdots\int_{Y_n}\nabla_{y_1}u_1(x,t,y_1,s^m)\,dy_n\cdots dy_2\\&=\int_{Y_2}\cdots\int_{Y_n}\sum_{i=1}^n\nabla_{y_i}u_i(x,t,y^i,s^m)\,dy_n\cdots dy_2\\&=\int_{Y_2}\cdots\int_{Y_n}(w_0(x,t,y^n,s^m)-\nabla u(x,t))\,dy_n\cdots dy_2\\&=\int_{Y_2}\cdots\int_{Y_n}w_0(x,t,y^n,s^m)\,dy_n\cdots dy_2-\nabla u(x,t),\end{aligned} \tag{A.20}$$

where the second equality follows from the fact that u_i is Y_i-periodic. Thus,

$$\begin{aligned}\|u_1\|_{L^2(\Omega_T\times S^m;H_\sharp^1(Y_1)/\mathbb{R})}&=\|\nabla_{y_1}u_1\|_{L^2(\Omega_T\times\mathcal{Y}_{1,m})^N}\\&=\left\|\int_{Y_2}\cdots\int_{Y_n}w_0(x,t,y^n,s^m)\,dy_n\cdots dy_2-\nabla u\right\|_{L^2(\Omega_T\times\mathcal{Y}_{1,m})^N}\\&\leq\left\|\int_{Y_2}\cdots\int_{Y_n}w_0(x,t,y^n,s^m)\,dy_n\cdots dy_2\right\|_{L^2(\Omega_T\times\mathcal{Y}_{1,m})^N}+\|\nabla u\|_{L^2(\Omega_T\times\mathcal{Y}_{1,m})^N}.\end{aligned} \tag{A.21}$$

Clearly, $\nabla u\in L^2(\Omega_T)^N$, and since $w_0\in L^2(\Omega_T\times\mathcal{Y}_{n,m})^N$, we have that

$$\int_{Y_2}\cdots\int_{Y_n}w_0(x,t,y^n,s^m)\,dy_n\cdots dy_2 \tag{A.22}$$

belongs to $L^2(\Omega_T \times \mathscr{Y}_{1,m})^N$ by the Hölder inequality. We get

$$\|u_1\|_{L^2(\Omega_T \times S^m; H^1_\sharp(Y_1)/\mathbb{R})} < \infty, \tag{A.23}$$

and the Base Case is verified.

Inductive Step. Fix $r = 1, \ldots, n-1$ where $n > 1$. Assume that $u_1 \in L^2(\Omega_T \times S^m; H^1_\sharp(Y_1)/\mathbb{R})$ and, provided that $r > 1$, that $u_j \in L^2(\Omega_T \times \mathscr{Y}_{j-1,m}; H^1_\sharp(Y_j)/\mathbb{R})$ for all $j = 2, \ldots, r$. We must show that this assumption implies that $u_{r+1} \in L^2(\Omega_T \times \mathscr{Y}_{r,m}; H^1_\sharp(Y_{r+1})/\mathbb{R})$. We have, almost everywhere in $\Omega_T \times \mathscr{Y}_{r+1,m}$,

$$\begin{aligned}
&\nabla_{y_{r+1}} u_{r+1}\left(x,t,y^{r+1},s^m\right) \\
&\quad = \int_{Y_{r+2}} \cdots \int_{Y_n} \nabla_{y_{r+1}} u_{r+1}\left(x,t,y^{r+1},s^m\right) dy_n \cdots dy_{r+2} \\
&\quad = \int_{Y_{r+2}} \cdots \int_{Y_n} \sum_{i=1}^{n} \nabla_{y_i} u_i\left(x,t,y^i,s^m\right) dy_n \cdots dy_{r+2} \\
&\qquad - \int_{Y_{r+2}} \cdots \int_{Y_n} \sum_{i=1}^{r} \nabla_{y_i} u_i\left(x,t,y^i,s^m\right) dy_n \cdots dy_{r+2} \\
&\quad = \int_{Y_{r+2}} \cdots \int_{Y_n} \left(w_0\left(x,t,y^n,s^m\right) - \nabla u(x,t)\right) dy_n \cdots dy_{r+2} \\
&\qquad - \sum_{i=1}^{r} \nabla_{y_i} u_i\left(x,t,y^i,s^m\right) \\
&\quad = \int_{Y_{r+2}} \cdots \int_{Y_n} w_0\left(x,t,y^n,s^m\right) dy_n \cdots dy_{r+2} \\
&\qquad - \nabla u(x,t) - \sum_{i=1}^{r} \nabla_{y_i} u_i\left(x,t,y^i,s^m\right),
\end{aligned} \tag{A.24}$$

where the second equality follows from the fact that u_i is Y_i-periodic. We get the estimation

$$\begin{aligned}
&\|u_{r+1}\|_{L^2(\Omega_T \times \mathscr{Y}_{r,m}; H^1_\sharp(Y_{r+1})/\mathbb{R})} \\
&\quad = \left\|\nabla_{y_{r+1}} u_{r+1}\right\|_{L^2(\Omega_T \times \mathscr{Y}_{r+1,m})^N} \\
&\quad = \left\| \int_{Y_{r+2}} \cdots \int_{Y_n} w_0\left(x,t,y^n,s^m\right) dy_n \cdots dy_{r+2} \right. \\
&\qquad \left. - \nabla u - \sum_{i=1}^{r} \nabla_{y_i} u_i \right\|_{L^2(\Omega_T \times \mathscr{Y}_{r+1,m})^N} \\
&\quad \leq \left\| \int_{Y_{r+2}} \cdots \int_{Y_n} w_0\left(x,t,y^n,s^m\right) dy_n \cdots dy_{r+2} \right\|_{L^2(\Omega_T \times \mathscr{Y}_{r+1,m})^N} \\
&\qquad + \|\nabla u\|_{L^2(\Omega_T \times \mathscr{Y}_{r+1,m})^N} + \sum_{i=1}^{r} \left\|\nabla_{y_i} u_i\right\|_{L^2(\Omega_T \times \mathscr{Y}_{r+1,m})^N} \\
&\quad \leq \left\| \int_{Y_{r+2}} \cdots \int_{Y_n} w_0\left(x,t,y^n,s^m\right) dy_n \cdots dy_{r+2} \right\|_{L^2(\Omega_T \times \mathscr{Y}_{r+1,m})^N} \\
&\qquad + \|\nabla u\|_{L^2(\Omega_T \times \mathscr{Y}_{r+1,m})^N} + \sum_{i=1}^{r} \|u_i\|_{L^2(\Omega_T \times \mathscr{Y}_{i-1,m}; H^1_\sharp(Y_i)/\mathbb{R})}.
\end{aligned} \tag{A.25}$$

Using the same arguments as in the Base Case, $\nabla u \in L^2(\Omega_T)^N$ and

$$\int_{Y_{r+2}} \cdots \int_{Y_n} w_0\left(x,t,y^n,s^m\right) dy_n \cdots dy_{r+2} \tag{A.26}$$

belongs to $L^2(\Omega_T \times \mathscr{Y}_{1,m})^N$. By the inductive assumption, we have that $u_1 \in L^2(\Omega_T \times S^m; H^1_\sharp(Y_1)/\mathbb{R})$ and $u_j \in L^2(\Omega_T \times \mathscr{Y}_{j-1,m}; H^1_\sharp(Y_j)/\mathbb{R})$ for all $j = 2, \ldots, r$. Thus

$$\|u_{r+1}\|_{L^2(\Omega_T \times \mathscr{Y}_{r,m}; H^1_\sharp(Y_{r+1})/\mathbb{R})} < \infty, \tag{A.27}$$

and the Inductive Step is complete and we are done. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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