Research Article

Topological Soliton Solution and Bifurcation Analysis of the Klein-Gordon-Zakharov Equation in $(1 + 1)$-Dimensions with Power Law Nonlinearity

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This paper addresses the Klein-Gordon-Zakharov equation with power law nonlinearity in $(1 + 1)$-dimensions. The integrability aspect as well as the bifurcation analysis is studied in this paper. The numerical simulations are also given where the finite difference approach was utilized. There are a few constraint conditions that naturally evolve during the course of derivation of the soliton solutions. These constraint conditions must remain valid in order for the soliton solution to exist. For the bifurcation analysis, the phase portraits are also given.

1. Introduction

The theory of nonlinear evolution equations (NLEEs) has come a long way in the past few decades [1–20]. Many of the NLEEs are pretty well known in the area of theoretical physics and applied mathematics. A few of them are the nonlinear Schrödinger's equation, Korteweg-de Vries (KdV) equation, sine-Gordon equation which appear in nonlinear optics, fluid dynamics, and theoretical physics, respectively. It is also very common to come across several combo NLEEs such as the Schrödinger-KdV equation, Klein-Gordon-Zakharov (KGZ) equation, and many others that are also studied in the context of applied mathematics and theoretical physics. This paper is going to focus on the KGZ equation that will be studied with power law nonlinearity in $(1 + 1)$-dimensions.

The integrability aspects and the bifurcation analysis will be the main focus of this paper. The ansatz method will be applied to obtain the topological 1-soliton solution, also known as the shock wave solution, to this equation. The constraint conditions will be naturally formulated in order for the soliton solution to exist. Subsequently, the bifurcation analysis will be carried out for this paper. In this context, the phase portraits will be given. Additionally, other traveling wave solutions will be enumerated. Finally, the numerical simulation to the equation will be given. The finite difference scheme will also be given.

2. Mathematical Analysis

The KGZ equation with power law nonlinearity in $(1 + 1)$-dimensions that are going to be studied in this paper is given by [6]

$$q_{tt} - k^2 q_{xx} + aq + brq + c|q|^{2n}q = 0, \tag{1}$$

$$r_{tt} - k^2 r_{xx} = d\left(|q|^{2n}\right)_{xx}, \tag{2}$$

where a, b, c, d, and k are real valued constants. Additionally $q(x, t)$ is a complex valued dependent variable and $r(x, t)$ is a real valued dependent variable. This section will focus on extracting the shock wave solutions to the KGZ equation (1) and (2) that are also known as topological soliton solution.

Therefore the starting hypothesis will be

$$q(x,t) = A_1 \tanh^{p_1} \tau e^{i\phi}, \tag{3}$$

$$r(x,t) = A_2 \tanh^{p_2} \tau, \tag{4}$$

where

$$\tau = B(x - vt). \tag{5}$$

Here, in (3) and (4) A_1, A_2 and B are free parameters, while v is the velocity of the soliton. The unknown exponents p_1 and p_2 will be determined, in terms of n by the aid of balancing principle. The phase component of (3) is given by

$$\phi = -\kappa x + \omega t + \theta, \tag{6}$$

where κ represents the soliton frequency, ω is the soliton wave number, and θ is the phase constant. Substituting the hypothesis (3) and (4) into (1) and (2) yields

$$
\begin{aligned}
&p_1(p_1-1)(v^2-k^2)B^2\tanh^{p_1-2}\tau \\
&\quad - 2ip_1(v\omega-\kappa^2k^2)B\tanh^{p_1-1}\tau \\
&\quad - \{2p_1^2(v^2-k^2)B^2+\omega^2+K^2\kappa^2\}\tanh^{p_1}\tau \\
&\quad + 2ip_1(v\omega-\kappa^2k^2)B\tanh^{p_1+1}\tau \\
&\quad + p_1(p_1+1)(v^2-k^2)B^2\tanh^{p_1+2}\tau + a\tanh^{p_1}\tau \\
&\quad + bA_2\tanh^{p_1+p_2}\tau + cA_1^{2n}\tanh^{(2n+1)p_1}\tau = 0,
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
&p_2(p_2-1)(v^2-k^2)A_2B^2\tanh^{p_2-2}\tau \\
&\quad - 2p_2^2(v^2-K^2)A_2B^2\tanh^{p_2}\tau \\
&\quad + p_2(p_2+1)(v^2-k^2)A_2B^2\tanh^{p_2+2}\tau \\
&\quad - dA_1^{2n}B^2\{2np_1(2np_1-1)\tanh^{2np_1-2}\tau - 8n^2p_1^2\tanh^{2np_1}\tau \\
&\qquad +2np_1(2np_1+1)\tanh^{2np_1+2}\tau\} = 0,
\end{aligned}
\tag{8}
$$

respectively. Now, splitting (7) into two real and imaginary parts gives

$$
\begin{aligned}
&p_1(p_1-1)(v^2-k^2)B^2\tanh^{p_1-2}\tau \\
&\quad - \{2p_1^2(v^2-k^2)B^2+\omega^2+k^2\kappa^2\}\tanh^{p_1}\tau \\
&\quad + p_1(p_1+1)(v^2-k^2)B^2\tanh^{p_1+2}\tau + a\tanh^{p_1}\tau \\
&\quad + bA_2\tanh^{p_1+p_2}\tau + cA_1^{2n}\tanh^{(2n+1)p_1}\tau = 0,
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
&2ip_1(v\omega-\kappa^2k^2)B\tanh^{p_1-1}\tau \\
&\quad - 2ip_1(v\omega-\kappa^2k^2)B\tanh^{p_1+1}\tau = 0.
\end{aligned}
\tag{10}
$$

From (9), equating the exponents $p_1 + p_2$ and $p_1 + 2$ gives

$$p_2 = 2, \tag{11}$$

and then equating $(2n+1)p_1$ with $p_1 + 2$ gives

$$p_1 = \frac{1}{n}. \tag{12}$$

Finally, equating the exponent pairs $(2n+1)p_1$ and $p_1 + p_2$ gives

$$p_2 = 2np_1. \tag{13}$$

Now the values of p_1 and p_2 from (11) and (12) satisfy (13).

Finally, equating the coefficients of the linearly independent functions $\tanh^{p_1 \pm j}\tau$, $j = \pm 1, 0, \pm 2$ in (9) and (10) to zero gives

$$
\begin{aligned}
&(v\omega - \kappa^2 k^2)B = 0, \\
&2(v^2-k^2)B^2 + n^2(\omega^2+\kappa^2k^2-a) = 0, \\
&(n+1)(v^2-k^2)B^2 + n^2(bA_2+cA_1^{2n}) = 0.
\end{aligned}
\tag{14}
$$

Again, equating the coefficients of the linearly independent functions $\tanh^{p_2 \pm j}\tau$, $j = \pm 1, \pm 2$ in (8) to zero implies

$$(v^2-k^2)A_2 - dA_1^{2n} = 0. \tag{15}$$

Solving (14)-(15) we get

$$\omega = \frac{k^2\kappa^2}{v}, \tag{16}$$

$$B = n\sqrt{\frac{a-k^2\kappa^2-\omega^2}{2(v^2-k^2)}}, \tag{17}$$

$$A_1 = \left[\frac{(n+1)(v^2-k^2)(\omega^2-a-\kappa^2k^2)d}{2\{c(v^2-k^2)+bd\}}\right]^{1/2n}, \tag{18}$$

$$A_2 = \frac{d(n+1)(\omega^2-a-k^2\kappa^2)}{2\{c(v^2-k^2)+bd\}}. \tag{19}$$

The relations (17), (18), and (19) introduce the restrictions given by

$$
\begin{aligned}
&(a-k^2\kappa^2-\omega^2)(v^2-k^2) > 0, \\
&d(v^2-k^2)(\omega^2-a-K^2\kappa^2)\{c(v^2-k^2)+bd\} > 0, \\
&c(v^2-k^2)+bd \neq 0.
\end{aligned}
\tag{20}
$$

Thus the topological solution of the q and r wave functions are given:

$$
\begin{aligned}
&q(x,t) = A_1\tanh^{1/n}[B(x-vt)]e^{i(-\kappa x+\omega t+\theta)}, \\
&r(x,t) = A_2\tanh^2[B(x-vt)].
\end{aligned}
\tag{21}
$$

3. Bifurcation Analysis

This section will carry out the bifurcation analysis of the Klein-Gordon-Zakharov equation with power law nonlinearity. Initially, the phase portraits will be obtained and the corresponding qualitative analysis will be discussed. Several interesting properties of the solution structure will be obtained based on the parameter regimes. Subsequently, the traveling wave solutions will be discussed from the bifurcation analysis.

3.1. Phase Portraits and Qualitative Analysis. We assume that the traveling wave solutions of (1) and (2) are of the form

$$q(x,t) = e^{i\eta}\varphi(\xi), \qquad r(x,t) = \psi(\xi), \qquad (22)$$

$$\eta = mx + lt, \qquad \xi = px - vt, \qquad (23)$$

where $\varphi(\xi)$ and $\psi(\xi)$ are real functions, m, l, p, and v are real constants.

Substituting (22) and (23) into (1) and (2), we find that $p = -vl/mk^2$, φ and ϕ satisfy the following system:

$$\left(v^2 - k^2p^2\right)\varphi'' - \left(l^2 - k^2m^2 - a\right)\varphi + b\varphi\psi + c\varphi^{2n+1} = 0, \qquad (24)$$

$$\left(v^2 + k^2p^2\right)\psi'' - dp^2\left(\varphi^{2n}\right)'' = 0. \qquad (25)$$

Integrating (25) twice and letting the first integral constant be zero, we have

$$\psi = \frac{dp^2\varphi^{2n}}{v^2 - k^2p^2} + g, \quad v^2 - k^2p^2 \neq 0, \qquad (26)$$

where g is the second integral constant.

Substituting (26) into (24), we have

$$\left(v^2 - k^2p^2\right)\varphi'' - \left(l^2 - k^2m^2 - a - bg\right)\varphi + \left(c + \frac{bdp^2}{v^2 - k^2p^2}\right)\varphi^{2n+1} = 0. \qquad (27)$$

To facilitate discussions, we let

$$\delta = \frac{bdp^2 + c\left(v^2 - k^2p^2\right)}{\left(v^2 - k^2p^2\right)^2}, \qquad (28)$$

$$\theta = \frac{l^2 - k^2m^2 - a - bg}{v^2 - k^2p^2}. \qquad (29)$$

Letting $\varphi' = z$, then we get the following planar system:

$$\frac{d\varphi}{d\xi} = z,$$
$$\frac{dz}{d\xi} = -\delta\varphi^{2n+1} + \theta\varphi. \qquad (30)$$

Obviously, the above system (30) is a Hamiltonian system with Hamiltonian function

$$H(\varphi, z) = z^2 + \frac{\delta}{n+1}\varphi^{2n+2} - \theta\varphi^2. \qquad (31)$$

In order to investigate the phase portrait of (30), set

$$f(\varphi) = -\delta\varphi^{2n+1} + \theta\varphi. \qquad (32)$$

Obviously, when $\delta\theta > 0$, $f(\varphi)$ has three zero points, φ_-, φ_0, and φ_+, which are given as follows:

$$\varphi_- = -\left(\frac{\theta}{\delta}\right)^{1/2n}, \qquad \varphi_0 = 0, \qquad \varphi_+ = \left(\frac{\theta}{\delta}\right)^{1/2n}. \qquad (33)$$

When $\delta\theta \leq 0$, $f(\varphi)$ has only one zero point

$$\varphi_0 = 0. \qquad (34)$$

Letting $(\varphi_i, 0)$ be one of the singular points of system (30), then the characteristic values of the linearized system of system (30) at the singular points $(\varphi_i, 0)$ are

$$\lambda_\pm = \pm\sqrt{f'(\varphi_i)}. \qquad (35)$$

From the qualitative theory of dynamical systems, we know the following.

  (I) If $f'(\varphi_i) > 0$, $(\varphi_i, 0)$ is a saddle point.

  (II) If $f'(\varphi_i) < 0$, $(\varphi_i, 0)$ is a center point.

  (III) If $f'(\varphi_i) = 0$, $(\varphi_i, 0)$ is a degenerate saddle point.

Therefore, we obtain the bifurcation phase portraits of system (30) in Figure 1.

Let

$$H(\varphi, z) = h, \qquad (36)$$

where h is Hamiltonian.

Next, we consider the relations between the orbits of (30) and the Hamiltonian h.

Set

$$h^* = \left|H(\varphi_+, 0)\right| = \left|H(\varphi_-, 0)\right|. \qquad (37)$$

According to Figure 1, we get the following propositions.

Proposition 1. *Suppose that $\delta > 0$ and $\theta > 0$, one has the following.*

  (I) *When $h \leq -h^*$, system (30) does not have any closed orbits.*

  (II) *When $-h^* < h < 0$, system (30) has two periodic orbits Γ_1 and Γ_2.*

  (III) *When $h = 0$, system (30) has two homoclinic orbits Γ_3 and Γ_4.*

  (IV) *When $h > 0$, system (30) has a periodic orbit Γ_5.*

Proposition 2. *Suppose that $\delta < 0$ and $\theta < 0$, one has the following.*

  (I) *When $h < 0$ or $h > h^*$, system (30) does not have any closed orbits.*

  (II) *When $0 < h < h^*$, system (30) has three periodic orbits Γ_6, Γ_7, and Γ_8.*
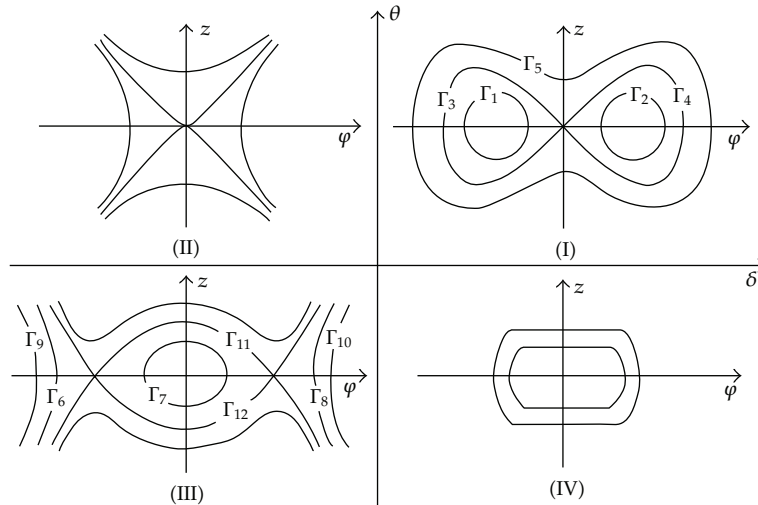
FIGURE 1: The bifurcation phase portraits of system (30). (I) $\delta > 0$, $\theta > 0$, (II) $\delta < 0$, $\theta \geqslant 0$, (III) $\delta < 0$, $\theta < 0$, (IV) $\delta > 0$, $\theta \leqslant 0$.

(III) *When $h = 0$, system (30) has two periodic orbits Γ_9 and Γ_{10}.*

(IV) *When $h = h^*$, system (30) has two heteroclinic orbits Γ_{11} and Γ_{12}.*

Proposition 3. (I) *When $\delta > 0$, $\theta \geqslant 0$ and $h > 0$, system (30) has a periodic orbits.*

(II) *When $\delta < 0$, $\theta \leqslant 0$, system (30) does have not any closed orbits.*

From the qualitative theory of dynamical systems, we know that a smooth solitary wave solution of a partial differential system corresponds to a smooth homoclinic orbit of a traveling wave equation. A smooth kink wave solution or a unbounded wave solution corresponds to a smooth heteroclinic orbit of a traveling wave equation. Similarly, a periodic orbit of a traveling wave equation corresponds to a periodic traveling wave solution of a partial differential system. According to the above analysis, we have the following propositions.

Proposition 4. *If $\delta > 0$ and $\theta > 0$, one has the following.*

(I) *When $-h^* < h < 0$, (1) and (2) have two periodic wave solutions (corresponding to the periodic orbits Γ_1 and Γ_2 in Figure 1).*

(II) *When $h = 0$, (1) and (2) have two solitary wave solutions (corresponding to the homoclinic orbits Γ_3 and Γ_4 in Figure 1).*

(III) *When $h > 0$, (1) and (2) have two periodic wave solutions (corresponding to the periodic orbit Γ_5 in Figure 1).*

Proposition 5. *If $\delta < 0$ and $\theta < 0$, one has the following.*

(I) *When $0 < h < h^*$, (1) and (2) have two periodic wave solutions (corresponding to the periodic orbit Γ_7 in Figure 1) and two periodic blow-up wave solutions*

(corresponding to the periodic orbits Γ_6 and Γ_8 in Figure 1).

(II) *When $h = 0$, (1) and (2) have periodic blow-up wave solutions (corresponding to the periodic orbits Γ_9 and Γ_{10} in Figure 1).*

(III) *When $h = h^*$, (1) and (2) have two kink profile solitary wave solutions. (corresponding to the heteroclinic orbits Γ_{11} and Γ_{12} in Figure 1).*

3.2. Exact Traveling Wave Solutions. Firstly, we will obtain the explicit expressions of traveling wave solutions for (1) and (2) when $\delta > 0$ and $\theta > 0$. From the phase portrait, we see that there are two symmetric homoclinic orbits Γ_3 and Γ_4 connected at the saddle point $(0, 0)$. In (φ, z)-plane the expressions of the homoclinic orbits are given as

$$z = \pm \sqrt{\frac{\delta}{n+1}} \varphi \sqrt{-\varphi^{2n} + \frac{(n+1)\theta}{\delta}}. \tag{38}$$

Substituting (38) into $d\varphi/d\xi = z$ and integrating them along the orbits Γ_3 and Γ_4, we have

$$\pm \int_{\varphi_1}^{\varphi} \frac{1}{s\sqrt{-s^{2n} + (n+1)\theta/\delta}} ds = \sqrt{\frac{\delta}{n+1}} \int_0^{\xi} ds,$$

$$\pm \int_{\varphi_2}^{\varphi} \frac{1}{s\sqrt{-s^{2n} + (n+1)\theta/\delta}} ds = \sqrt{\frac{\delta}{n+1}} \int_0^{\xi} ds, \tag{39}$$

where $\varphi_1 = -((n+1)\theta/\delta)^{1/2n}$ and $\varphi_2 = ((n+1)\theta/\delta)^{1/2n}$.

Completing the above integrals we obtain

$$\varphi = \left(\sqrt{\frac{(n+1)\theta}{\delta}} \operatorname{sech} n\sqrt{\theta}\xi \right)^{1/n}$$

$$\varphi = -\left(\sqrt{\frac{(n+1)\theta}{\delta}} \operatorname{sech} n\sqrt{\theta}\xi \right)^{1/n}. \tag{40}$$

Noting (22), (23), and (26), we get the following solitary wave solutions:

$$q_1(x, y, t) = e^{i\eta}\left(\sqrt{\frac{(n+1)\theta}{\delta}}\operatorname{sech} n\sqrt{\theta}\xi\right)^{1/n},$$

$$r_1(x, y, t) = \frac{-(n+1)\beta\theta\left(\operatorname{sech} n\sqrt{\theta}\xi\right)^2}{\delta(p^2+m^2)} + g,$$

$$q_2(x, y, t) = -e^{i\eta}\left(\sqrt{\frac{(n+1)\theta}{\delta}}\operatorname{sech} n\sqrt{\theta}\xi\right)^{1/n},$$

$$r_2(x, y, t) = \frac{-(n+1)\beta\theta\left(\operatorname{sech} n\sqrt{\theta}\xi\right)^2}{\delta(p^2+m^2)} + g,$$

$$(41)$$

where δ is given by (28), θ is given by (29), $\eta = mx + lt$, and $\xi = px - vt$.

Secondly, we will obtain the explicit expressions of traveling wave solutions for (1) and (2) when $\delta < 0$ and $\theta < 0$. From the phase portrait, we note that there are two special orbits Γ_9 and Γ_{10}, which have the same Hamiltonian with that of the center point $(0, 0)$. In (φ, z)-plane the expressions of the orbits are given as

$$z = \pm\sqrt{-\frac{\delta}{n+1}}\varphi\sqrt{\varphi^{2n} - \frac{(n+1)\theta}{\delta}}. \qquad (42)$$

Substituting (42) into $d\varphi/d\xi = z$ and integrating them along the two orbits Γ_9 and Γ_{10}, it follows that

$$\pm\int_{\varphi}^{+\infty}\frac{1}{s\sqrt{s^{2n} - (n+1)\theta/\delta}}ds = \sqrt{-\frac{\delta}{n+1}}\int_0^{\xi}ds,$$

$$\pm\int_{\varphi_4}^{\varphi}\frac{1}{s\sqrt{s^{2n} - (n+1)\theta/\delta}}ds = \sqrt{-\frac{\delta}{n+1}}\int_0^{\xi}ds,$$

$$(43)$$

where $\varphi_4 = ((n+1)\theta/\delta)^{1/2n}$.

Completing the above integrals we obtain

$$\varphi = \pm\left(\sqrt{\frac{(n+1)\theta}{\delta}}\csc n\sqrt{-\theta}\xi\right)^{1/n},$$

$$\varphi = \pm\left(\sqrt{\frac{(n+1)\theta}{\delta}}\sec n\sqrt{-\theta}\xi\right)^{1/n}. \qquad (44)$$

Noting (22), (23), and (26), we get the following periodic blow-up wave solutions:

$$q_3(x, y, t) = \pm e^{i\eta}\left(\sqrt{\frac{(n+1)\theta}{\delta}}\csc n\sqrt{-\theta}\xi\right)^{1/n},$$

$$r_3(x, y, t) = \frac{-(n+1)\beta\theta\left(\csc n\sqrt{-\theta}\xi\right)^2}{\delta(p^2+m^2)} + g,$$

$$q_4(x, y, t) = \pm e^{i\eta}\left(\sqrt{\frac{(n+1)\theta}{\delta}}\sec n\sqrt{-\theta}\xi\right)^{1/n},$$

$$r_4(x, y, t) = \frac{-(n+1)\beta\theta\left(\sec n\sqrt{-\theta}\xi\right)^2}{\delta(p^2+m^2)} + g,$$

$$(45)$$

where δ is given by (28), θ is given by (29), $\eta = mx + lt$, and $\xi = px - vt$.

4. Numerical Simulation

We decompose the function q in (1) in the form

$$q = u + iv \qquad (46)$$

Substituting in (1) and (2) we have

$$u_{tt} - k^2 u_{xx} + au + brv + c(u^2+v^2)^n u = 0,$$

$$v_{tt} - k^2 v_{xx} + av + brv + c(u^2+v^2)^n v = 0, \qquad (47)$$

$$r_{tt} - k^2 u_{xx} - \alpha(u^2+v^2)^n_{xx} = 0.$$

We assume that u_m^n, v_m^n, r_m^n is the exact solution and U_m^n, V_m^n, R_m^n is the approximate solution at the grid point (x_m, t_n). The proposed scheme can be displayed as

$$\frac{1}{k^2}\delta_t^2 U_m^n - \frac{k^2}{h^2}\delta_x^2 U_m^n + aU_m^n + bR_{m,n}U_m^n$$

$$+ c\left((U_m^n)^2 + (U_m^n)^2\right)^n U_m^n = 0,$$

$$\frac{1}{k^2}\delta_t^2 V_m^n - \frac{k^2}{h^2}\delta_x^2 V_m^n + aV_m^n + bR_{m,n}V_m^n$$

$$+ c\left((U_m^n)^2 + (V_m^n)^2\right)^n V_m^n = 0, \qquad (48)$$

$$\frac{1}{k^2}\delta_t^2 R_m^n - \frac{k^2}{h^2}\delta_x^2 R_m^n + \alpha\left((U_m^n)^2 + (V_m^n)^2\right)^n_{xx} = 0,$$

where

$$\delta_t^2 U_m^n = U_m^{n+1} - 2U_m^n + U_m^{n-1},$$

$$\delta_x^2 U_m^n = U_{m+1}^n - 2U_m^n + U_{m-1}^n. \qquad (49)$$

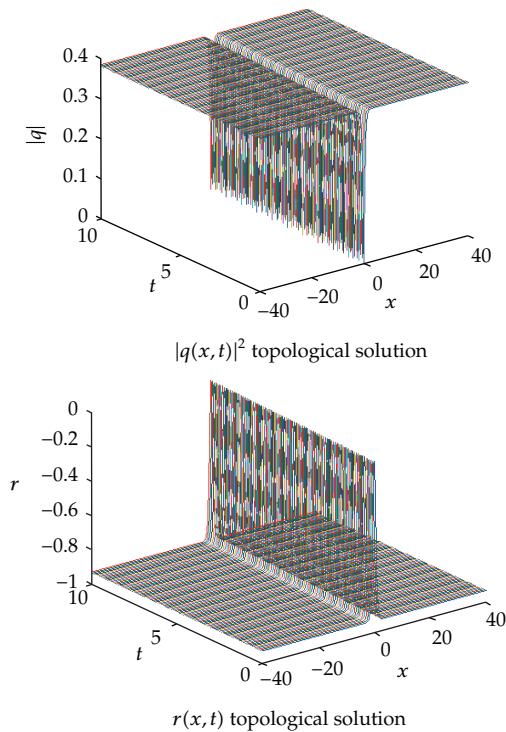$|q(x,t)|^2$ topological solution

$r(x,t)$ topological solution

FIGURE 2: Topological solution for the Klein-Gordon-Zakharov equations.

The similar notation for $\delta_t^2 V_m^n$, $\delta_x^2 V_m^n$ and $\delta_t^2 R_m^n$, $\delta_x^2 R_m^n$ are

$$\delta_t^2 V_m^n = V_m^{n+1} - 2V_m^n + V_m^{n-1},$$

$$\delta_x^2 V_m^n = V_{m+1}^n - 2V_m^n + V_{m-1}^n,$$

$$\delta_t^2 R_m^n = R_m^{n+1} - 2R_m^n + R_m^{n-1},$$

$$\delta_x^2 R_m^n = R_{m+1}^n - 2R_m^n + R_{m-1}^n,$$

(50)

The proposed scheme is implicit and can be easily solved by the fixed point method. The scheme is second order in space and time directions.

To get the numerical solution the initial conditions are taken from the exact solution (21). Figure 2 displays the numerical solutions of $|q(x,t)|$ and $r(x,t)$ at $n = 1$, respectively. We choose the parameter

$$n = 1, \qquad K = 1, \qquad \kappa = 0.5, \qquad a = 0.5, \qquad b = 0.5,$$

$$c = -0.5, \qquad d = 0.5, \qquad c = 0.5, \qquad v = 0.6.$$

(51)

5. Conclusions

This paper studied the KGZ equation in $(1+1)$-D with power law nonlinearity from three different avenues. First, the topological 1-soliton solution to the equation was determined by the aid of ansatz method. The by-product of this solution is a couple of constraint conditions that must remain valid in order for the solitons to exist. Subsequently, the bifurcation

analysis is carried out for this equation that leads to the phase portraits and several other solutions to the equation, using the traveling wave hypothesis. This leads to the solitary waves and periodic singular waves. Finally, the numerical simulation that was conducted using finite difference scheme leads to the simulations for the topological soliton solutions.

These results are pretty complete in analysis. They are going to be extended in the future. An obvious way to expand or generalize these results is going to extend to $(2 + 1)$-D. These results will be reported soon. Another avenue to look into this equation further is to consider the perturbation terms and then obtain exact solution, and additionally study the perturbed KGZ equation using other tools of integrability. These include mapping method, Lie symmetries, expfunction method, and the G'/G-expansion method. These will lead to a further plethora of solutions. Such results will be reported in the future. That is just a foot in the door.

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