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Research Article

Three-Dimensional Biorthogonal Divergence-Free and Curl-Free Wavelets with Free-Slip Boundary

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This paper deals with the construction of divergence-free and curl-free wavelets on the unit cube, which satisfies the free-slip boundary conditions. First, interval wavelets adapted to our construction are introduced. Then, we provide the biorthogonal divergence-free and curl-free wavelets with free-slip boundary and simple structure, based on the characterization of corresponding spaces. Moreover, the bases are also stable.

1. Introduction

In recent years, divergence-free and curl-free wavelets are generally studied, due to their potential use in many physical problems [1–5]. Anisotropic divergence-free and curl-free wavelets on the hypercube are firstly constructed in [6, 7], but all these functions only satisfy slip boundary conditions. However, the free-slip boundary is important in many cases, such as the solution of partial differential equations in incompressible fluids and electromagnetism. Inspired by this fact, [8, 9] give the construction of anisotropic divergence-free and curl-free wavelets with free-slip boundary, but the structure is very complicated and the basis functions are not explicit. Recently, based on a simple characterization of 2D divergence-free space, Harouna and Perrier proposed an alternative construction to [8] for divergence-free wavelets in two-dimensional case [10]. Following the similar but non-trivial line, we mainly study the anisotropic 3D divergence-free and curl-free wavelet bases with free-slip boundary in this paper. The traditional understanding that 3D curl-free wavelets are more difficult to construct than divergence-free wavelets is not always right, due to our procedure.

In Section 2, interval wavelets that we will use are introduced. Based on the spaces characterization, 3D biorthogonal divergence-free and curl-free wavelet bases are given in Sections 3 and 4, respectively.

2. Interval Wavelets on $[0, 1]$

In this part, we will introduce the interval wavelets used in the subsequent construction.

The existence of divergence-free and curl-free wavelets on R^d follows from the following fundamental proposition [11].

Proposition 1. *Let $(V_j^1(R), \widetilde{V}_j^1(R))$ be a biorthogonal MRA of $L^2(R)$, with compactly supported scaling functions $(\varphi^1, \widetilde{\varphi}^1)$ and wavelets $(\psi^1, \widetilde{\psi}^1)$, such that $\varphi^1, \psi^1 \in C^{1+\varepsilon}$ for $\varepsilon > 0$. Then there exists a biorthogonal MRA $(V_j^0(R), \widetilde{V}_j^0(R))$, with associated scaling functions $(\varphi^0, \widetilde{\varphi}^0)$ and wavelets $(\psi^0, \widetilde{\psi}^0)$, such that*

$$\left(\varphi^1\right)'(x) = \varphi^0(x) - \varphi^0(x-1), \qquad \left(\psi^1\right)' = 4\psi^0. \quad (1)$$

The dual functions verify $\int_x^{x+1} \widetilde{\varphi}^1(t)dt = \widetilde{\varphi}^0(x)$ and $(\widetilde{\psi}^0)' = -4\widetilde{\psi}^1$.

Based on the above proposition, Jouini and Lemarié-Rieusset [12] proved the existence of two one dimensional MRAs of $L^2(0,1)$ linked by

$$\frac{d}{dx}V_j^1 = V_j^0,$$
$$\widetilde{V}_j^0 = H_0^1(0,1) \bigcap \int_0^x \widetilde{V}_j^1 = \left\{f : f' \in \widetilde{V}_j^1, f(0) = f(1) = 0\right\}. \quad (2)$$

In the following, we simply introduce the construction of these spaces. Suppose that φ^1 in Proposition 1 is supported on $[n_{\min}, n_{\max}]$($n_{\min}, n_{\max}$ integers) and reproduces polynomials up to degree $r-1$:

$$0 \le \ell \le r-1, \quad \frac{x^\ell}{\ell!} = \sum_{k=-\infty}^{+\infty} \widetilde{p}_\ell^1(k)\varphi^1(x-k), \quad x \in R, \tag{3}$$

with $\widetilde{p}_\ell^1(k) = \langle x^\ell/\ell!, \widetilde{\varphi}^1(x-k)\rangle$. Similarly, $\widetilde{\varphi}^1$ is supported on $[\widetilde{n}_{\min}, \widetilde{n}_{\max}]$ and reproduces polynomials up to degree $\widetilde{r}-1$.

For j being sufficiently large, the spaces V_j^1 have the structure

$$\begin{aligned} V_j^1 &= \operatorname{span}\left\{\Phi_{j,\ell}^{1,\flat} = 2^{j/2}\Phi_\ell^{1,\flat}\left(2^j x\right)\right\}_{\ell=0,\dots,r-1} \oplus V_j^{1,\text{int}} \\ &\quad \oplus \operatorname{span}\left\{\Phi_{j,\ell}^{1,\sharp} = 2^{j/2}\Phi_\ell^{1,\sharp}\left(2^j x\right)\right\}_{\ell=0,\dots,r-1}, \end{aligned} \tag{4}$$

where $V_j^{1,\text{int}} = \operatorname{span}\{\varphi_{j,k}^1 = 2^{j/2}\varphi^1(2^j x - k) : k = k_\flat, \dots, 2^j - k_\sharp\}$ is the space whose supports are included into $[\delta_\flat/2^j, 1-\delta_\sharp/2^j] \subset [0,1]$ ($\delta_\flat, \delta_\sharp \in N$ be two fixed parameters), and $k_\flat = \delta_\flat - n_{\min}$, $k_\sharp = \delta_\sharp + n_{\max}$. Moreover, $\Phi_\ell^{1,\flat}$ are the edge scaling functions at the edge 0 being defined by

$$\Phi_\ell^{1,\flat}(x) = \sum_{k=1-n_{\max}}^{k_\flat-1} \widetilde{p}_\ell^1(k)\varphi^1(x-k)\chi_{[0,+\infty)}. \tag{5}$$

At the edge 1, $\Phi_\ell^{1,\flat}$ are defined by symmetry using $Tf(x) = f(1-x)$.

Similarly, the biorthogonal spaces $\widetilde{V}_j^1$ are defined with the same structure as

$$\widetilde{V}_j^1 = \operatorname{span}\left\{\widetilde{\Phi}_{j,\ell}^{1,\flat}\right\}_{\ell=0,\dots,\widetilde{r}-1} \oplus \widetilde{V}_j^{1,\text{int}} \oplus \operatorname{span}\left\{\widetilde{\Phi}_{j,\ell}^{1,\sharp}\right\}_{\ell=0,\dots,\widetilde{r}-1}. \tag{6}$$

Adjusting the parameters such that

$$\begin{aligned} \Delta_j &= \dim\left(V_j^1\right) = \dim\left(\widetilde{V}_j^1\right) \\ &= 2^j - \left(\delta_{\flat+\delta_\sharp}\right) - \left(n_{\max} - n_{\min}\right) + 2r + 1. \end{aligned} \tag{7}$$

The last step of the construction is the biorthogonalization process, since the edge scaling functions of V_j^1 and $\widetilde{V}_j^1$ are no more biorthogonal. Finally, $(V_j^1, \widetilde{V}_j^1)$ form a biorthogonal MRA of $L^2(0,1)$.

As described in [13], removing the edge scaling functions $\Phi_0^{1,\flat}$ and $\Phi_0^{1,\sharp}$ leads to

$$\begin{aligned} V_j^D &= \operatorname{span}\left\{\Phi_{j,\ell}^{1,\flat}\right\}_{\ell=1,\dots,r-1} \oplus V_j^{1,\text{int}} \oplus \operatorname{span}\left\{\Phi_{j,\ell}^{1,\sharp}\right\}_{\ell=1,\dots,r-1} \\ &=: \operatorname{span}\left\{\varphi_{j,k}^D : k = 1, \dots, \Delta_j - 2\right\}. \end{aligned} \tag{8}$$

Similarly, define $\widetilde{V}_j^D = \operatorname{span}\{\widetilde{\Phi}_{j,\ell}^{1,\flat}\}_{\ell=1,\dots,\widetilde{r}-1} \oplus \widetilde{V}_j^{1,\text{int}} \oplus \operatorname{span}\{\widetilde{\Phi}_{j,\ell}^{1,\sharp}\}_{\ell=1,\dots,\widetilde{r}-1}$. After a biorthogonalization process, we finally note that

$$\widetilde{V}_j^D = \operatorname{span}\left\{\widetilde{\varphi}_{j,k}^D : k = 1, \dots, \Delta_j - 2\right\}, \tag{9}$$

and the spaces $(V_j^D, \widetilde{V}_j^D)$ form a biorthogonal MRA of $H_0^1(0,1)$.

The construction of $(V_j^0, \widetilde{V}_j^0)$ follows the same structure. Since $(\varphi^1)'(x) = \varphi^0(x) - \varphi^0(x-1)$, φ^0 has compact support $[n_{\min}, n_{\max}-1]$ and reproduces polynomials up to degree $r-2$:

$$0 \le \ell \le r-2, \quad \frac{x^\ell}{\ell!} = \sum_{k=-\infty}^{+\infty} \widetilde{p}_\ell^0(k)\varphi^0(x-k), \tag{10}$$

with $\widetilde{p}_\ell^0(k) = \langle x^\ell/\ell!, \widetilde{\varphi}^0(x-k)\rangle$. The scaling function $\widetilde{\varphi}^0(x) = \int_x^{x+1}\widetilde{\varphi}^1(t)dt$ has support $[\widetilde{n}_{\min}-1, \widetilde{n}_{\max}]$ and reproduces polynomials up to degree $\widetilde{r}$. Consider

$$\begin{aligned} V_j^0 &= \operatorname{span}\left\{\Phi_{j,\ell}^{0,\flat} = 2^{j/2}\Phi_\ell^{0,\flat}\left(2^j x\right)\right\}_{\ell=0,\dots,r-2} \oplus V_j^{0,\text{int}} \\ &\quad \oplus \operatorname{span}\left\{\Phi_{j,\ell}^{0,\sharp} = 2^{j/2}\Phi_\ell^{0,\sharp}\left(2^j x\right)\right\}_{\ell=0,\dots,r-2}, \end{aligned} \tag{11}$$

where $V_j^{0,\text{int}} = \operatorname{span}\{\varphi_{j,k}^0 = 2^{j/2}\varphi^0(2^j x - k) : k = k_\flat, \dots, 2^j - k_\sharp + 1\}$ and supports are included into $[\delta_\flat/2^j, 1-\delta_\sharp/2^j] \subset [0,1]$. The left edge scaling functions are

$$\Phi_\ell^{0,\flat}(x) = \sum_{k=2-n_{\max}}^{k_\flat-1} \widetilde{p}_\ell^0(k)\varphi^0(x-k)\chi_{[0,+\infty)}. \tag{12}$$

Biorthogonal spaces $\widetilde{V}_j^0$ are similarly defined, but by satisfying vanishing boundary conditions at 0 and 1, then

$$\widetilde{V}_j^0 = \operatorname{span}\left\{\widetilde{\Phi}_{j,\ell}^{0,\flat}\right\}_{\ell=1,\dots,\widetilde{r}} \oplus \widetilde{V}_j^{0,\text{int}} \oplus \operatorname{span}\left\{\widetilde{\Phi}_{j,\ell}^{0,\sharp}\right\}_{\ell=1,\dots,\widetilde{r}}, \tag{13}$$

with $\widetilde{V}_j^{0,\text{int}} = \operatorname{span}\{\widetilde{\varphi}_{j,k}^0 : k = \widetilde{k}_\flat + 1, \dots, 2^j - \widetilde{k}_\sharp\}$ and $\widetilde{\Phi}_\ell^{0,\flat} = \sum_{k=1-\widetilde{n}_{\max}}^{\widetilde{k}_\flat} p_\ell^0(k)\widetilde{\varphi}^0(x-k)\chi_{[0,+\infty)}$ for $\ell = 1, \dots, \widetilde{r}$. It is easy to know $\dim(V_j^0) = \dim(\widetilde{V}_j^0) = \Delta_j - 1$.

In practice, we choose $j \ge j_{\min}$ with

$$\begin{aligned} j_{\min} > \max\Big\{&\log_2\left[n_{\max} - n_{\min} + \delta_\sharp + \delta_\flat + 1\right], \\ &\log_2\left[\widetilde{n}_{\max} - \widetilde{n}_{\min} + \widetilde{\delta}_\sharp + \widetilde{\delta}_\flat + 1\right]\Big\} \end{aligned} \tag{14}$$

to ensure that the supports of edge scaling functions at 0 do not intersect the supports of edge scaling functions at 1.

The construction of wavelet spaces $(W_j^1, \widetilde{W}_j^1)$ can be seen from [13]. Moreover, they satisfy the following result.

Proposition 2 (see [12]). *Let $(V_j^1, \widetilde{V}_j^1)$ and $(V_j^0, \widetilde{V}_j^0)$ be MRAs satisfying $(d/dx)V_j^1 = V_j^0$ and $\widetilde{V}_j^0 = H_0^1 \bigcap \int_0^x \widetilde{V}_j^1$; then the wavelet spaces W_j^0 and $\widetilde{W}_j^0$ are linked to the biorthogonal wavelet spaces associated to $(V_j^1, \widetilde{V}_j^1)$ by*

$$W_j^0 = \frac{d}{dx}W_j^1, \qquad \widetilde{W}_j^0 = \int_0^x \widetilde{W}_j^1. \tag{15}$$

Moreover, let $\{\psi_{j,k}^1\}_{k=1,\dots,2^j}$ and $\{\widetilde{\psi}_{j,k}^1\}_{k=1,\dots,2^j}$ be two biorthogonal wavelet bases of W_j^1 and $\widetilde{W}_j^1$. Biorthogonal wavelet bases of W_j^0 and $\widetilde{W}_j^0$ are directly defined by

$$\psi_{j,k}^0 = 2^{-j}\left(\psi_{j,k}^1\right)', \qquad \widetilde{\psi}_{j,k}^0 = -2^j\int_0^x \widetilde{\psi}_{j,k}^1. \tag{16}$$

3. Divergence-Free Wavelets on $[0,1]^3$

Let $\Omega = [0,1]^3$ and let $\vec{n}$ be the normal vector; the boundary condition considered in [6] is $\vec{u}\cdot\vec{n} = 0$ on $\Gamma = \bigcup_{k=1}^{3}\Gamma_k$ with

$$\Gamma_k = [0,1]^{k-1}\times\{0\}\times[0,1]^{3-k}, \quad 1 \le k \le 3. \tag{17}$$

It holds that $\vec{u}\cdot\vec{n} = 0$ on Γ if and only if $u_k = 0$ on Γ_k $(1 \le k \le 3)$. We call it a slip boundary, which is shown in Figure 1.

In this section, we mainly consider the following space with free-slip boundary as Figure 2

$$\mathscr{H}_{\mathrm{div}}(\Omega) = \left\{\vec{u} \in \left(L^2(\Omega)\right)^3 : \operatorname{div}\vec{u} = 0, \vec{u}\cdot\vec{n} = 0 \ on\ \partial\Omega\right\}. \tag{18}$$

For $\vec{u}(x,y,z) = (u_1,u_2,u_3)^T$, the 3D curl-operator is defined as

$$\operatorname{curl}\vec{u} = \left(\partial_2 u_3 - \partial_3 u_2, \partial_3 u_1 - \partial_1 u_3, \partial_1 u_2 - \partial_2 u_1\right)^T. \tag{19}$$

Remark 3. Taking Fourier transform on the both sides of $\operatorname{div}\vec{u} = 0$ leads to the equation

$$\xi_1\hat{u}_1(\xi) + \xi_2\hat{u}_2(\xi) + \xi_3\hat{u}_3(\xi) = 0, \quad \xi = (\xi_1,\xi_2,\xi_3). \tag{20}$$

In $L^2(R^3)$, define the following functions

$$\hat{\varphi}_1(\xi) = \frac{\xi_3\hat{u}_2 - \xi_2\hat{u}_3}{i(\xi_1^2+\xi_2^2+\xi_3^2)}, \qquad \hat{\varphi}_2(\xi) = \frac{\xi_1\hat{u}_3 - \xi_3\hat{u}_1}{i(\xi_1^2+\xi_2^2+\xi_3^2)},$$
$$\hat{\varphi}_3(\xi) = \frac{\xi_2\hat{u}_1 - \xi_1\hat{u}_2}{i(\xi_1^2+\xi_2^2+\xi_3^2)}. \tag{21}$$

Then, according to (20), it is easy to verify that

$$\begin{aligned}\hat{u}_1(\xi) &= i\left(\xi_2\hat{\varphi}_3(\xi) - \xi_3\hat{\varphi}_2(\xi)\right),\\ \hat{u}_2(\xi) &= i\left(\xi_3\hat{\varphi}_1(\xi) - \xi_1\hat{\varphi}_3(\xi)\right),\\ \hat{u}_3(\xi) &= i\left(\xi_1\hat{\varphi}_2(\xi) - \xi_2\hat{\varphi}_1(\xi)\right),\end{aligned} \tag{22}$$

which is equivalent to $\vec{u} = \operatorname{curl}\vec{\varphi}$. Therefore, any function $\vec{u} \in (L^2(R^3))^3$ which satisfies $\operatorname{div}\vec{u} = 0$ can be characterized by curl operator as $\vec{u} = \operatorname{curl}\vec{\varphi}$ with $\vec{\varphi} \in (H^1(R^3))^3$. In fact, a similar result holds in 3D nonsmooth domains.

Proposition 4 (see [14]). *There is a characterization*

$$\mathscr{H}_{\mathrm{div}}(\Omega) = \left\{\vec{u} = \operatorname{curl}\vec{\varphi} : \vec{\varphi} \in \left(H^1(\Omega)\right)^3, \vec{\varphi}\times\vec{n} = \vec{0}\ on\ \partial\Omega\right\}. \tag{23}$$

Based on Proposition 4, we give the following definition of divergence-free scaling function spaces.

Definition 5. For $j \ge j_{\min}$, the divergence-free scaling function spaces $\vec{V}_j^{\mathrm{div}}$ are defined by

$$\begin{aligned}\vec{V}_j^{\mathrm{div}} &= \operatorname{curl}\left\{\left(V_j^0\otimes V_j^D\otimes V_j^D\right)\times\left(V_j^D\otimes V_j^0\otimes V_j^D\right)\right.\\ &\quad\left.\times\left(V_j^D\otimes V_j^D\otimes V_j^0\right)\right\}\\ &= \operatorname{span}\left\{\Phi_{j,\mathbf{k}}^{\mathrm{div},1}, \Phi_{j,\mathbf{k}}^{\mathrm{div},2}, \Phi_{j,\mathbf{k}}^{\mathrm{div},3}\right\},\end{aligned} \tag{24}$$

where the divergence-free scaling functions are given by

$$\begin{aligned}\Phi_{j,\mathbf{k}}^{\mathrm{div},1} &=: \frac{1}{\sqrt{2}}\operatorname{curl}\left[\left(\varphi_{j,k_1}^0\cdot\varphi_{j,k_2}^D\cdot\varphi_{j,k_3}^D, 0, 0\right)^T\right]\\ &= \frac{1}{\sqrt{2}}\left[\varphi_{j,k_1}^0\cdot\varphi_{j,k_2}^D\cdot\left(\varphi_{j,k_3}^D\right)'\delta_2 - \varphi_{j,k_1}^0\cdot\left(\varphi_{j,k_2}^D\right)'\cdot\varphi_{j,k_3}^D\delta_3\right],\\ \Phi_{j,\mathbf{k}}^{\mathrm{div},2} &=: \frac{1}{\sqrt{2}}\operatorname{curl}\left[\left(0, \varphi_{j,k_1}^D\cdot\varphi_{j,k_2}^0\cdot\varphi_{j,k_3}^D, 0\right)^T\right]\\ &= \frac{1}{\sqrt{2}}\left[\left(\varphi_{j,k_1}^D\right)'\cdot\varphi_{j,k_2}^0\cdot\varphi_{j,k_3}^D\delta_3 - \varphi_{j,k_1}^D\cdot\varphi_{j,k_2}^0\cdot\left(\varphi_{j,k_3}^D\right)'\delta_1\right],\\ \Phi_{j,\mathbf{k}}^{\mathrm{div},3} &=: \frac{1}{\sqrt{2}}\operatorname{curl}\left[\left(0, 0, \varphi_{j,k_1}^D\cdot\varphi_{j,k_2}^D\cdot\varphi_{j,k_3}^0\right)^T\right]\\ &= \frac{1}{\sqrt{2}}\left[\varphi_{j,k_1}^D\cdot\left(\varphi_{j,k_2}^D\right)'\cdot\varphi_{j,k_3}^0\delta_1 - \left(\varphi_{j,k_1}^D\right)'\cdot\varphi_{j,k_2}^D\cdot\varphi_{j,k_3}^0\delta_2\right].\end{aligned} \tag{25}$$

For proving the consequent main result, we also consider the standard MRA $\vec{V}_j$ of $(L^2(\Omega))^3$:

$$\begin{aligned}\vec{V}_j &= \left(V_j^1\otimes V_j^0\otimes V_j^0\right)\times\left(V_j^0\otimes V_j^1\otimes V_j^0\right)\\ &\quad\times\left(V_j^0\otimes V_j^0\otimes V_j^1\right).\end{aligned} \tag{26}$$

The following conclusion shows that the space $\vec{V}_j$ preserves the divergence-free condition.

Proposition 6. *If* $\vec{u} \in (L^2(\Omega))^3$ *and* $\operatorname{div}\vec{u} = 0$, *then* $\operatorname{div}[\vec{P}_j\vec{u}] = 0$, *where* $\vec{P}_j = (p_j^1\otimes p_j^0\otimes p_j^0, p_j^0\otimes p_j^1\otimes p_j^0, p_j^0\otimes p_j^0\otimes p_j^1)$ *is the biorthogonal projector on* $\vec{V}_j$.

Proof. Let $\vec{u} = (u_1,u_2,u_3)^T$; then

$$\vec{P}_j\vec{u} = \left(p_j^1\otimes p_j^0\otimes p_j^0 u_1, p_j^0\otimes p_j^1\otimes p_j^0 u_2, p_j^0\otimes p_j^0\otimes p_j^1 u_3\right)^T. \tag{27}$$

Therefore, by the fact $d/dx\circ p_j^1 f = p_j^0\circ(d/dx)f$ in [10], we obtain

$$\begin{aligned}\operatorname{div}\left[\vec{P}_j\vec{u}\right] &= \frac{\partial}{\partial x}p_j^1\otimes p_j^0\otimes p_j^0 u_1 + \frac{\partial}{\partial y}p_j^0\otimes p_j^1\otimes p_j^0 u_2\\ &\quad + \frac{\partial}{\partial z}p_j^0\otimes p_j^0\otimes p_j^1 u_3\\ &= p_j^0\otimes p_j^0\otimes p_j^0\left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}\right)\\ &= p_j^0\otimes p_j^0\otimes p_j^0(\operatorname{div}\vec{u}) = 0.\end{aligned} \tag{28}$$

□

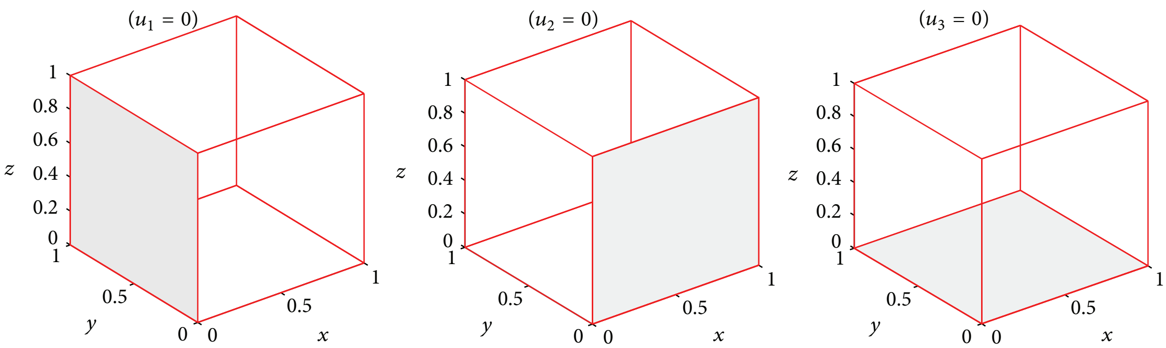

Figure 1: Slip boundary condition (divergence).

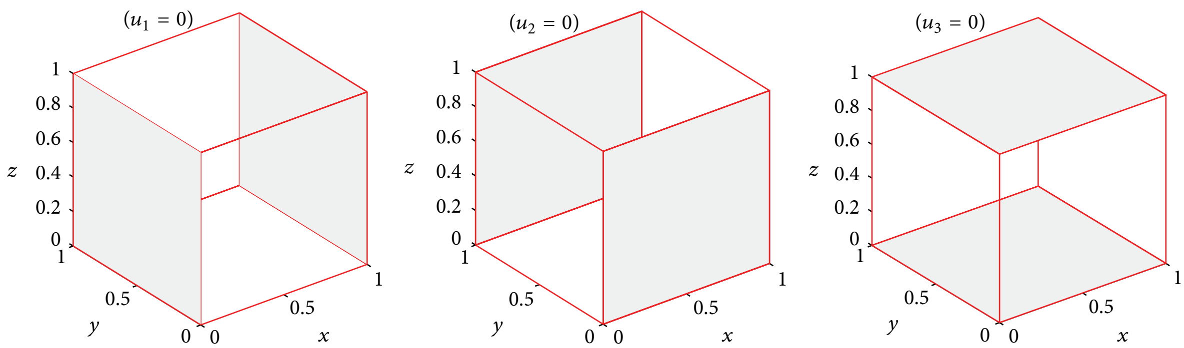

Figure 2: Free-slip boundary condition (divergence).

Theorem 7. *The divergence-free scaling function spaces $\{\vec{V}_j^{\mathrm{div}}\}_{j \ge j_{\min}}$ is a multiresolution analysis of $\mathscr{H}_{\mathrm{div}}(\Omega)$.*

Proof. Since $\mathscr{H}_{\mathrm{div}}(\Omega) \cap \vec{V}_j$ are a multiresolution analysis of $\mathscr{H}_{\mathrm{div}}(\Omega)$, it is reduced to prove

$$\vec{V}_j^{\mathrm{div}} = \mathscr{H}_{\mathrm{div}}(\Omega) \cap \vec{V}_j. \tag{29}$$

Noting that $(d/dx)V_j^1 = V_j^0$ and $V_j^D \subseteq V_j^1$ from (2) and (8), we know $\vec{V}_j^{\mathrm{div}} \subset \vec{V}_j$. Furthermore, $\vec{V}_j^{\mathrm{div}} \subset \mathscr{H}_{\mathrm{div}}(\Omega)$ by construction. Therefore, $\vec{V}_j^{\mathrm{div}} \subset \mathscr{H}_{\mathrm{div}}(\Omega) \cap \vec{V}_j$.

Conversely, letting $\vec{u} \in \mathscr{H}_{\mathrm{div}}(\Omega) \cap \vec{V}_j$, we are going to prove $\vec{u} \in \vec{V}_j^{\mathrm{div}}$. On the one hand, since $\vec{u} \in \vec{V}_j$, we have $\vec{u} = \vec{P}_j \vec{u}$. On the other hand, since $\vec{u} \in \mathscr{H}_{\mathrm{div}}(\Omega)$, there exists a $\vec{\varphi} = (\varphi_1, \varphi_2, \varphi_3)^T \in (H^1(\Omega))^3$ such that $\vec{u} = \mathrm{curl}(\vec{\varphi})$. Moreover, $\vec{\varphi} \times \vec{n} = \vec{0}$. Thus, $\vec{u} = \vec{P}_j[\mathrm{curl}(\vec{\varphi})]$. Furthermore, we can decompose $\vec{\varphi}$ by isotropic vector wavelets as

$$\vec{\varphi} = \vec{P}_j^D(\vec{\varphi}) + \sum_{j' \ge j} \vec{Q}_{j'}^D(\vec{\varphi}), \tag{30}$$

where $\vec{P}_j^D(\vec{\varphi}) = \sum_k c_{1,k}\varphi_{j,k_1}^0 \varphi_{j,k_2}^D \varphi_{j,k_3}^D \delta_1 + \sum_k c_{2,k}\varphi_{j,k_1}^D \varphi_{j,k_2}^0 \varphi_{j,k_3}^D \delta_2 + \sum_k c_{3,k}\varphi_{j,k_1}^D \varphi_{j,k_2}^D \varphi_{j,k_3}^0 \delta_3$

$$\begin{aligned}
&\vec{Q}_{j'}^D(\vec{\varphi}) \\
&= \sum_k \Big(d_{j',k}^{1,1}\varphi_{j',k_1}^0 \varphi_{j',k_2}^D \psi_{j',k_3}^D + d_{j',k}^{1,2}\varphi_{j',k_1}^0 \psi_{j',k_2}^D \varphi_{j',k_3}^D \\
&\quad + d_{j',k}^{1,3}\psi_{j',k_1}^0 \varphi_{j',k_2}^D \varphi_{j',k_3}^D + d_{j',k}^{1,4}\varphi_{j',k_1}^0 \psi_{j',k_2}^D \psi_{j',k_3}^D \\
&\quad + d_{j',k}^{1,5}\psi_{j',k_1}^0 \varphi_{j',k_2}^D \psi_{j',k_3}^D + d_{j',k}^{1,6}\psi_{j',k_1}^0 \psi_{j',k_2}^D \varphi_{j',k_3}^D \\
&\quad + d_{j',k}^{1,7}\psi_{j',k_1}^0 \psi_{j',k_2}^D \psi_{j',k_3}^D \Big)\delta_1 \\
&+ \sum_k \Big(d_{j',k}^{2,1}\varphi_{j',k_1}^D \varphi_{j',k_2}^0 \psi_{j',k_3}^D + d_{j',k}^{2,2}\varphi_{j',k_1}^D \psi_{j',k_2}^0 \varphi_{j',k_3}^D \\
&\quad + d_{j',k}^{2,3}\psi_{j',k_1}^D \varphi_{j',k_2}^0 \varphi_{j',k_3}^D + d_{j',k}^{2,4}\varphi_{j',k_1}^D \psi_{j',k_2}^0 \psi_{j',k_3}^D \\
&\quad + d_{j',k}^{2,5}\psi_{j',k_1}^D \varphi_{j',k_2}^0 \psi_{j',k_3}^D + d_{j',k}^{2,6}\psi_{j',k_1}^D \psi_{j',k_2}^0 \varphi_{j',k_3}^D \\
&\quad + d_{j',k}^{2,7}\psi_{j',k_1}^D \psi_{j',k_2}^0 \psi_{j',k_3}^D \Big)\delta_2 \\
&+ \sum_k \Big(d_{j',k}^{3,1}\varphi_{j',k_1}^D \varphi_{j',k_2}^D \psi_{j',k_3}^0 + d_{j',k}^{3,2}\varphi_{j',k_1}^D \psi_{j',k_2}^D \varphi_{j',k_3}^0 \\
&\quad + d_{j',k}^{3,3}\psi_{j',k_1}^D \varphi_{j',k_2}^D \varphi_{j',k_3}^0 + d_{j',k}^{3,4}\varphi_{j',k_1}^D \psi_{j',k_2}^D \psi_{j',k_3}^0 \\
&\quad + d_{j',k}^{3,5}\psi_{j',k_1}^D \varphi_{j',k_2}^D \psi_{j',k_3}^0 + d_{j',k}^{3,6}\psi_{j',k_1}^D \psi_{j',k_2}^D \varphi_{j',k_3}^0 \\
&\quad + d_{j',k}^{3,7}\psi_{j',k_1}^D \psi_{j',k_2}^D \psi_{j',k_3}^0 \Big)\delta_3.
\end{aligned} \tag{31}$$

Since $\operatorname{curl}[\varphi^0_{j',k_1}\varphi^D_{j',k_2}\psi^D_{j',k_3}\delta_1] \in (V^D_{j'}\otimes V^0_{j'}\otimes W^0_{j'})\times(V^0_{j'}\otimes V^D_{j'}\otimes W^0_{j'})\times(V^0_{j'}\otimes V^0_{j'}\otimes W^D_{j'})$, then

$$\vec{P}_j\left[\operatorname{curl}\left(\varphi^0_{j',k_1}\varphi^D_{j',k_2}\psi^D_{j',k_3}\delta_1\right)\right] = \vec{0}. \tag{32}$$

Similarly, every term in the right sides of (31) satisfies (32). Finally,

$$\vec{P}_j\left[\operatorname{curl}\left(\vec{Q}^D_{j'}(\vec{\varphi})\right)\right] = \vec{0}. \tag{33}$$

Furthermore, we can obtain

$$\vec{u} = \vec{P}_j\left[\operatorname{curl}(\vec{\varphi})\right] = \vec{P}_j\left[\operatorname{curl}\left(\vec{P}^D_j\vec{\varphi}\right)\right] = \operatorname{curl}\left(\vec{P}^D_j\vec{\varphi}\right). \tag{34}$$

Here, we have used the fact $\operatorname{curl}(\vec{P}^D_j\vec{\varphi}) \in \vec{V}_j$ in the last step of (34). By construction, we have $\operatorname{curl}(\vec{P}^D_j\vec{\varphi}) \in \vec{V}^{\mathrm{div}}_j$, which means $\vec{u} \in \vec{V}^{\mathrm{div}}_j$ and the proof is completed. □

Based on the constructive method of vector wavelets and the following decompositions:

$$\begin{aligned}
V^0_j\otimes V^D_j\otimes V^D_j &= \left(V^0_{j_{\min}}\oplus^{j-1}_{j_1=j_{\min}}W^0_{j_1}\right)\\
&\quad\otimes\left(V^D_{j_{\min}}\oplus^{j-1}_{j_2=j_{\min}}W^D_{j_2}\right)\\
&\quad\otimes\left(V^D_{j_{\min}}\oplus^{j-1}_{j_3=j_{\min}}W^D_{j_3}\right)\\
&= \left(V^0_{j_{\min}}\otimes V^D_{j_{\min}}\otimes V^D_{j_{\min}}\right)\\
&\quad\oplus^{j-1}_{j_3=j_{\min}}\left(V^0_{j_{\min}}\otimes V^D_{j_{\min}}\otimes W^D_{j_3}\right)\\
&\quad\oplus^{j-1}_{j_2=j_{\min}}\left(V^0_{j_{\min}}\otimes W^D_{j_2}\otimes V^D_{j_{\min}}\right)\\
&\quad\oplus^{j-1}_{j_1=j_{\min}}\left(W^0_{j_1}\otimes V^D_{j_{\min}}\otimes V^D_{j_{\min}}\right)\\
&\quad\oplus^{j-1}_{j_2,j_3=j_{\min}}\left(V^0_{j_{\min}}\otimes W^D_{j_2}\otimes W^D_{j_3}\right)\\
&\quad\oplus^{j-1}_{j_1,j_3=j_{\min}}\left(W^0_{j_1}\otimes V^D_{j_{\min}}\otimes W^D_{j_3}\right)\\
&\quad\oplus^{j-1}_{j_1,j_2=j_{\min}}\left(W^0_{j_1}\otimes W^D_{j_2}\otimes V^D_{j_{\min}}\right)\\
&\quad\oplus^{j-1}_{j_1,j_2,j_3=j_{\min}}\left(W^0_{j_1}\otimes W^D_{j_2}\otimes W^D_{j_3}\right),\\
V^D_j\otimes V^0_j\otimes V^D_j &= \left(V^D_{j_{\min}}\otimes V^0_{j_{\min}}\otimes V^D_{j_{\min}}\right)\\
&\quad\oplus^{j-1}_{j_3=j_{\min}}\left(V^D_{j_{\min}}\otimes V^0_{j_{\min}}\otimes W^D_{j_3}\right)\\
&\quad\oplus^{j-1}_{j_2=j_{\min}}\left(V^D_{j_{\min}}\otimes W^0_{j_2}\otimes V^D_{j_{\min}}\right)\\
&\quad\oplus^{j-1}_{j_1=j_{\min}}\left(W^D_{j_1}\otimes V^0_{j_{\min}}\otimes V^D_{j_{\min}}\right)\\
&\quad\oplus^{j-1}_{j_2,j_3=j_{\min}}\left(V^D_{j_{\min}}\otimes W^0_{j_2}\otimes W^D_{j_3}\right)\\
&\quad\oplus^{j-1}_{j_1,j_3=j_{\min}}\left(W^D_{j_1}\otimes V^0_{j_{\min}}\otimes W^D_{j_3}\right)\\
&\quad\oplus^{j-1}_{j_1,j_2=j_{\min}}\left(W^D_{j_1}\otimes W^0_{j_2}\otimes V^D_{j_{\min}}\right)\\
&\quad\oplus^{j-1}_{j_1,j_2,j_3=j_{\min}}\left(W^D_{j_1}\otimes W^0_{j_2}\otimes W^D_{j_3}\right),\\
V^D_j\otimes V^D_j\otimes V^0_j &= \left(V^D_{j_{\min}}\otimes V^D_{j_{\min}}\otimes V^0_{j_{\min}}\right)\\
&\quad\oplus^{j-1}_{j_3=j_{\min}}\left(V^D_{j_{\min}}\otimes V^D_{j_{\min}}\otimes W^0_{j_3}\right)\\
&\quad\oplus^{j-1}_{j_2=j_{\min}}\left(V^D_{j_{\min}}\otimes W^D_{j_2}\otimes V^0_{j_{\min}}\right)\\
&\quad\oplus^{j-1}_{j_1=j_{\min}}\left(W^D_{j_1}\otimes V^D_{j_{\min}}\otimes V^0_{j_{\min}}\right)\\
&\quad\oplus^{j-1}_{j_2,j_3=j_{\min}}\left(V^D_{j_{\min}}\otimes W^D_{j_2}\otimes W^0_{j_3}\right)\\
&\quad\oplus^{j-1}_{j_1,j_3=j_{\min}}\left(W^D_{j_1}\otimes V^D_{j_{\min}}\otimes W^0_{j_3}\right)\\
&\quad\oplus^{j-1}_{j_1,j_2=j_{\min}}\left(W^D_{j_1}\otimes W^D_{j_2}\otimes V^0_{j_{\min}}\right)\\
&\quad\oplus^{j-1}_{j_1,j_2,j_3=j_{\min}}\left(W^D_{j_1}\otimes W^D_{j_2}\otimes W^0_{j_3}\right),
\end{aligned} \tag{35}$$

we can give the definition of anisotropic divergence-free wavelets as follows.

Definition 8. For j_1, j_2, and $j_3 \geq j_{\min}$, the anisotropic divergence-free wavelets are defined by

$$\begin{aligned}
\Psi^{\mathrm{div},(1,1)}_{\mathbf{j},\mathbf{k}} &= \frac{1}{\sqrt{4^{j_3}+1}}\operatorname{curl}\left[\varphi^0_{j_{\min},k_1}\cdot\varphi^D_{j_{\min},k_2}\cdot\psi^D_{j_3,k_3}\delta_1\right]\\
\Psi^{\mathrm{div},(1,2)}_{\mathbf{j},\mathbf{k}} &= \frac{1}{\sqrt{4^{j_2}+1}}\operatorname{curl}\left[\varphi^0_{j_{\min},k_1}\cdot\psi^D_{j_2,k_2}\cdot\varphi^D_{j_{\min},k_3}\delta_1\right]\\
\Psi^{\mathrm{div},(1,3)}_{\mathbf{j},\mathbf{k}} &= \frac{1}{\sqrt{2}}\operatorname{curl}\left[\psi^0_{j_1,k_1}\cdot\varphi^D_{j_{\min},k_2}\cdot\varphi^D_{j_{\min},k_3}\delta_1\right]\\
\Psi^{\mathrm{div},(1,4)}_{\mathbf{j},\mathbf{k}} &= \frac{1}{\sqrt{4^{j_2}+4^{j_3}}}\operatorname{curl}\left[\varphi^0_{j_{\min},k_1}\cdot\psi^D_{j_2,k_2}\cdot\psi^D_{j_3,k_3}\delta_1\right]\\
\Psi^{\mathrm{div},(1,5)}_{\mathbf{j},\mathbf{k}} &= \frac{1}{\sqrt{4^{j_3}+1}}\operatorname{curl}\left[\psi^0_{j_1,k_1}\cdot\varphi^D_{j_{\min},k_2}\cdot\psi^D_{j_3,k_3}\delta_1\right]\\
\Psi^{\mathrm{div},(1,6)}_{\mathbf{j},\mathbf{k}} &= \frac{1}{\sqrt{4^{j_2}+1}}\operatorname{curl}\left[\psi^0_{j_1,k_1}\cdot\psi^D_{j_2,k_2}\cdot\varphi^D_{j_{\min},k_3}\delta_1\right]\\
\Psi^{\mathrm{div},(1,7)}_{\mathbf{j},\mathbf{k}} &= \frac{1}{\sqrt{4^{j_2}+4^{j_3}}}\operatorname{curl}\left[\psi^0_{j_1,k_1}\cdot\psi^D_{j_2,k_2}\cdot\psi^D_{j_3,k_3}\delta_1\right],\\
\Psi^{\mathrm{div},(2,1)}_{\mathbf{j},\mathbf{k}} &= \frac{1}{\sqrt{4^{j_3}+1}}\operatorname{curl}\left[\varphi^D_{j_{\min},k_1}\cdot\varphi^0_{j_{\min},k_2}\cdot\psi^D_{j_3,k_3}\delta_2\right]\\
\Psi^{\mathrm{div},(2,2)}_{\mathbf{j},\mathbf{k}} &= \frac{1}{\sqrt{2}}\operatorname{curl}\left[\varphi^D_{j_{\min},k_1}\cdot\psi^0_{j_2,k_2}\cdot\varphi^D_{j_{\min},k_3}\delta_2\right]\\
\Psi^{\mathrm{div},(2,3)}_{\mathbf{j},\mathbf{k}} &= \frac{1}{\sqrt{4^{j_1}+1}}\operatorname{curl}\left[\psi^D_{j_1,k_1}\cdot\varphi^0_{j_{\min},k_2}\cdot\varphi^D_{j_{\min},k_3}\delta_2\right]\\
\Psi^{\mathrm{div},(2,4)}_{\mathbf{j},\mathbf{k}} &= \frac{1}{\sqrt{4^{j_3}+1}}\operatorname{curl}\left[\varphi^D_{j_{\min},k_1}\cdot\psi^0_{j_2,k_2}\cdot\psi^D_{j_3,k_3}\delta_2\right]\\
\Psi^{\mathrm{div},(2,5)}_{\mathbf{j},\mathbf{k}} &= \frac{1}{\sqrt{4^{j_1}+4^{j_3}}}\operatorname{curl}\left[\psi^D_{j_1,k_1}\cdot\varphi^0_{j_{\min},k_2}\cdot\psi^D_{j_3,k_3}\delta_2\right]\\
\Psi^{\mathrm{div},(2,6)}_{\mathbf{j},\mathbf{k}} &= \frac{1}{\sqrt{4^{j_1}+1}}\operatorname{curl}\left[\psi^D_{j_1,k_1}\cdot\psi^0_{j_2,k_2}\cdot\varphi^D_{j_{\min},k_3}\delta_2\right]
\end{aligned}$$

$$\Psi_{\mathbf{j,k}}^{\mathrm{div},(2,7)} = \frac{1}{\sqrt{4^{j_1}+4^{j_3}}}\operatorname{curl}\left[\psi_{j_1,k_1}^{D}\cdot\psi_{j_2,k_2}^{0}\cdot\psi_{j_3,k_3}^{D}\delta_2\right],$$

$$\Psi_{\mathbf{j,k}}^{\mathrm{div},(3,1)} = \frac{1}{\sqrt{2}}\operatorname{curl}\left[\varphi_{j_{\min},k_1}^{D}\cdot\varphi_{j_{\min},k_2}^{D}\cdot\psi_{j_3,k_3}^{0}\delta_3\right]$$

$$\Psi_{\mathbf{j,k}}^{\mathrm{div},(3,2)} = \frac{1}{\sqrt{4^{j_2}+1}}\operatorname{curl}\left[\varphi_{j_{\min},k_1}^{D}\cdot\psi_{j_2,k_2}^{D}\cdot\varphi_{j_{\min},k_3}^{0}\delta_3\right]$$

$$\Psi_{\mathbf{j,k}}^{\mathrm{div},(3,3)} = \frac{1}{\sqrt{4^{j_1}+1}}\operatorname{curl}\left[\psi_{j_1,k_1}^{D}\cdot\varphi_{j_{\min},k_2}^{D}\cdot\varphi_{j_{\min},k_3}^{0}\delta_3\right]$$

$$\Psi_{\mathbf{j,k}}^{\mathrm{div},(3,4)} = \frac{1}{\sqrt{4^{j_2}+1}}\operatorname{curl}\left[\varphi_{j_{\min},k_1}^{D}\cdot\psi_{j_2,k_2}^{D}\cdot\psi_{j_3,k_3}^{0}\delta_3\right]$$

$$\Psi_{\mathbf{j,k}}^{\mathrm{div},(3,5)} = \frac{1}{\sqrt{4^{j_1}+1}}\operatorname{curl}\left[\psi_{j_1,k_1}^{D}\cdot\varphi_{j_{\min},k_2}^{D}\cdot\psi_{j_3,k_3}^{0}\delta_3\right]$$

$$\Psi_{\mathbf{j,k}}^{\mathrm{div},(3,6)} = \frac{1}{\sqrt{4^{j_1}+4^{j_2}}}\operatorname{curl}\left[\psi_{j_1,k_1}^{D}\cdot\psi_{j_2,k_2}^{D}\cdot\varphi_{j_{\min},k_3}^{0}\delta_3\right]$$

$$\Psi_{\mathbf{j,k}}^{\mathrm{div},(3,7)} = \frac{1}{\sqrt{4^{j_1}+4^{j_2}}}\operatorname{curl}\left[\psi_{j_1,k_1}^{D}\cdot\psi_{j_2,k_2}^{D}\cdot\psi_{j_3,k_3}^{0}\delta_3\right]. \tag{36}$$

Remark 9. The coefficients before the operator "curl" are used to guarantee the biorthogonality in the following construction of dual wavelets.

Proposition 10. *Defining the wavelet spaces*

$$\vec{W}_{\mathbf{j}}^{\mathrm{div},(\varepsilon,n)} = \operatorname{span}\left\{\Psi_{\mathbf{j,k}}^{\mathrm{div},(\varepsilon,n)}\right\},\qquad \varepsilon=1,2,3,\ n=1,2,\ldots,7; \tag{37}$$

then $\vec{V}_j^{\mathrm{div}} = \vec{V}_{j_{\min}}^{\mathrm{div}}\oplus_{j_{\min}\le j_1,j_2,j_3\le j-1}(\oplus_{\varepsilon=1,2,3,n=1,2,\ldots,7}\vec{W}_{\mathbf{j}}^{\mathrm{div},(\varepsilon,n)})$.

Proof. It can be easily obtained from (35) and Definition 8. □

Definition 11. Biorthogonal divergence-free scaling functions and wavelets are defined by

$$\widetilde{\Phi}_{j,\mathbf{k}}^{\mathrm{div},1} = \frac{1}{\sqrt{2}}\left[\widetilde{\varphi}_{j,k_1}^{0}\cdot\widetilde{\varphi}_{j,k_2}^{D}\cdot\widetilde{\gamma}_{j,k_3}\delta_2 - \widetilde{\varphi}_{j,k_1}^{0}\cdot\widetilde{\gamma}_{j,k_2}\cdot\widetilde{\varphi}_{j,k_3}^{D}\delta_3\right]$$

$$\widetilde{\Phi}_{j,\mathbf{k}}^{\mathrm{div},2} = \frac{1}{\sqrt{2}}\left[\widetilde{\gamma}_{j,k_1}\cdot\widetilde{\varphi}_{j,k_2}^{0}\cdot\widetilde{\varphi}_{j,k_3}^{D}\delta_3 - \widetilde{\varphi}_{j,k_1}^{D}\cdot\widetilde{\varphi}_{j,k_2}^{0}\cdot\widetilde{\gamma}_{j,k_3}\delta_1\right]$$

$$\widetilde{\Phi}_{j,\mathbf{k}}^{\mathrm{div},3} = \frac{1}{\sqrt{2}}\left[\widetilde{\varphi}_{j,k_1}^{D}\cdot\widetilde{\gamma}_{j,k_2}\cdot\widetilde{\varphi}_{j,k_3}^{0}\delta_1 - \widetilde{\gamma}_{j,k_1}\cdot\widetilde{\varphi}_{j,k_2}^{D}\cdot\widetilde{\varphi}_{j,k_3}^{0}\delta_2\right],$$

$$\widetilde{\Psi}_{\mathbf{j,k}}^{\mathrm{div},(1,1)} = \frac{1}{\sqrt{4^{j_3}+1}}\left[2^{j_3}\widetilde{\varphi}_{j_{\min},k_1}^{0}\cdot\widetilde{\varphi}_{j_{\min},k_2}^{D}\cdot\widetilde{\psi}_{j_3,k_3}^{0}\delta_2 - \widetilde{\varphi}_{j_{\min},k_1}^{0}\cdot\widetilde{\gamma}_{j_{\min},k_2}\cdot\widetilde{\psi}_{j_3,k_3}^{D}\delta_3\right]$$

$$\widetilde{\Psi}_{\mathbf{j,k}}^{\mathrm{div},(1,2)} = \frac{1}{\sqrt{4^{j_2}+1}}\left[\widetilde{\varphi}_{j_{\min},k_1}^{0}\cdot\widetilde{\psi}_{j_2,k_2}^{D}\cdot\widetilde{\gamma}_{j_{\min},k_3}\delta_2 - 2^{j_2}\widetilde{\varphi}_{j_{\min},k_1}^{0}\cdot\widetilde{\psi}_{j_2,k_2}^{0}\cdot\widetilde{\varphi}_{j_{\min},k_3}^{D}\delta_3\right]$$

$$\widetilde{\Psi}_{\mathbf{j,k}}^{\mathrm{div},(1,3)} = \frac{1}{\sqrt{2}}\left[\widetilde{\psi}_{j_1,k_1}^{0}\cdot\widetilde{\varphi}_{j_{\min},k_2}^{D}\cdot\widetilde{\gamma}_{j_{\min},k_3}\delta_2 - \widetilde{\psi}_{j_1,k_1}^{0}\cdot\widetilde{\gamma}_{j_{\min},k_2}\cdot\widetilde{\varphi}_{j_{\min},k_3}^{D}\delta_3\right]$$

$$\widetilde{\Psi}_{\mathbf{j,k}}^{\mathrm{div},(1,4)} = \frac{1}{\sqrt{4^{j_2}+4^{j_3}}}\left[2^{j_3}\widetilde{\varphi}_{j_{\min},k_1}^{0}\cdot\widetilde{\psi}_{j_2,k_2}^{D}\cdot\widetilde{\psi}_{j_3,k_3}^{0}\delta_2 - 2^{j_2}\widetilde{\varphi}_{j_{\min},k_1}^{0}\cdot\widetilde{\psi}_{j_2,k_2}^{0}\cdot\widetilde{\psi}_{j_3,k_3}^{D}\delta_3\right]$$

$$\widetilde{\Psi}_{\mathbf{j,k}}^{\mathrm{div},(1,5)} = \frac{1}{\sqrt{4^{j_3}+1}}\left[2^{j_3}\widetilde{\psi}_{j_1,k_1}^{0}\cdot\widetilde{\varphi}_{j_{\min},k_2}^{D}\cdot\widetilde{\psi}_{j_3,k_3}^{0}\delta_2 - \widetilde{\psi}_{j_1,k_1}^{0}\cdot\widetilde{\gamma}_{j_{\min},k_2}\cdot\widetilde{\psi}_{j_3,k_3}^{D}\delta_3\right]$$

$$\widetilde{\Psi}_{\mathbf{j,k}}^{\mathrm{div},(1,6)} = \frac{1}{\sqrt{4^{j_2}+1}}\left[\widetilde{\psi}_{j_1,k_1}^{0}\cdot\widetilde{\psi}_{j_2,k_2}^{D}\cdot\widetilde{\gamma}_{j_{\min},k_3}\delta_2 - 2^{j_2}\widetilde{\psi}_{j_1,k_1}^{0}\cdot\widetilde{\psi}_{j_2,k_2}^{0}\cdot\widetilde{\varphi}_{j_{\min},k_3}^{D}\delta_3\right]$$

$$\widetilde{\Psi}_{\mathbf{j,k}}^{\mathrm{div},(1,7)} = \frac{1}{\sqrt{4^{j_2}+4^{j_3}}}\left[2^{j_3}\widetilde{\psi}_{j_1,k_1}^{0}\cdot\widetilde{\psi}_{j_2,k_2}^{D}\cdot\widetilde{\psi}_{j_3,k_3}^{0}\delta_2 - 2^{j_2}\widetilde{\psi}_{j_1,k_1}^{0}\cdot\widetilde{\psi}_{j_2,k_2}^{0}\cdot\widetilde{\psi}_{j_3,k_3}^{D}\delta_3\right],$$

$$\widetilde{\Psi}_{\mathbf{j,k}}^{\mathrm{div},(2,1)} = \frac{1}{\sqrt{4^{j_3}+1}}\left[\widetilde{\gamma}_{j_{\min},k_1}\cdot\widetilde{\varphi}_{j_{\min},k_2}^{0}\cdot\widetilde{\psi}_{j_3,k_3}^{D}\delta_3 - 2^{j_3}\widetilde{\varphi}_{j_{\min},k_1}^{D}\cdot\widetilde{\varphi}_{j_{\min},k_2}^{0}\cdot\widetilde{\psi}_{j_3,k_3}^{0}\delta_1\right]$$

$$\widetilde{\Psi}_{\mathbf{j,k}}^{\mathrm{div},(2,2)} = \frac{1}{\sqrt{2}}\left[\widetilde{\gamma}_{j_{\min},k_1}\cdot\widetilde{\psi}_{j_2,k_2}^{0}\cdot\widetilde{\varphi}_{j_{\min},k_3}^{D}\delta_3 - \widetilde{\varphi}_{j_{\min},k_1}^{D}\cdot\widetilde{\psi}_{j_2,k_2}^{0}\cdot\widetilde{\gamma}_{j_{\min},k_3}\delta_1\right]$$

$$\widetilde{\Psi}_{\mathbf{j,k}}^{\mathrm{div},(2,3)} = \frac{1}{\sqrt{4^{j_1}+1}}\left[2^{j_1}\widetilde{\psi}_{j_1,k_1}^{0}\cdot\widetilde{\varphi}_{j_{\min},k_2}^{0}\cdot\widetilde{\varphi}_{j_{\min},k_3}^{D}\delta_3 - \widetilde{\psi}_{j_1,k_1}^{D}\cdot\widetilde{\varphi}_{j_{\min},k_2}^{0}\cdot\widetilde{\gamma}_{j_{\min},k_3}\delta_1\right]$$

$$\widetilde{\Psi}_{\mathbf{j,k}}^{\mathrm{div},(2,4)} = \frac{1}{\sqrt{4^{j_3}+1}}\left[\widetilde{\gamma}_{j_{\min},k_1}\cdot\widetilde{\psi}_{j_2,k_2}^{0}\cdot\widetilde{\psi}_{j_3,k_3}^{D}\delta_3 - 2^{j_3}\widetilde{\varphi}_{j_{\min},k_1}^{D}\cdot\widetilde{\psi}_{j_2,k_2}^{0}\cdot\widetilde{\psi}_{j_3,k_3}^{0}\delta_1\right]$$

$$\widetilde{\Psi}_{\mathbf{j,k}}^{\mathrm{div},(2,5)} = \frac{1}{\sqrt{4^{j_1}+4^{j_3}}}\left[2^{j_1}\widetilde{\psi}_{j_1,k_1}^{0}\cdot\widetilde{\varphi}_{j_{min},k_2}^{0}\cdot\widetilde{\psi}_{j_3,k_3}^{D}\delta_3 - 2^{j_3}\widetilde{\psi}_{j_1,k_1}^{D}\cdot\widetilde{\varphi}_{j_{\min},k_2}^{0}\cdot\widetilde{\psi}_{j_3,k_3}^{0}\delta_1\right]$$

$$\widetilde{\Psi}_{\mathbf{j,k}}^{\mathrm{div},(2,6)} = \frac{1}{\sqrt{4^{j_1}+1}}\left[2^{j_1}\widetilde{\psi}_{j_1,k_1}^{0}\cdot\widetilde{\psi}_{j_2,k_2}^{0}\cdot\widetilde{\varphi}_{j_{\min},k_3}^{D}\delta_3 - \widetilde{\psi}_{j_1,k_1}^{D}\cdot\widetilde{\psi}_{j_2,k_2}^{0}\cdot\widetilde{\gamma}_{j_{\min},k_3}\delta_1\right]$$

$$\widetilde{\Psi}_{\mathbf{j,k}}^{\mathrm{div},(2,7)} = \frac{1}{\sqrt{4^{j_1}+4^{j_3}}}\left[2^{j_1}\widetilde{\psi}_{j_1,k_1}^{0}\cdot\widetilde{\psi}_{j_2,k_2}^{0}\cdot\widetilde{\psi}_{j_3,k_3}^{D}\delta_3 - 2^{j_3}\widetilde{\psi}_{j_1,k_1}^{D}\cdot\widetilde{\psi}_{j_2,k_2}^{0}\cdot\widetilde{\psi}_{j_3,k_3}^{0}\delta_1\right],$$

$$\widetilde{\Psi}_{\mathbf{j,k}}^{\mathrm{div},(3,1)} = \frac{1}{\sqrt{2}}\left[\widetilde{\varphi}_{j_{\min},k_1}^{D}\cdot\widetilde{\gamma}_{j_{\min},k_2}\cdot\widetilde{\psi}_{j_3,k_3}^{0}\delta_1 - \widetilde{\gamma}_{j_{\min},k_1}\cdot\widetilde{\varphi}_{j_{\min},k_2}^{D}\cdot\widetilde{\psi}_{j_3,k_3}^{0}\delta_2\right]$$

$$\widetilde{\Psi}^{\text{div},(3,2)}_{\mathbf{j},\mathbf{k}} = \frac{1}{\sqrt{4^{j_2}+1}}\left[2^{j_2}\widetilde{\varphi}^{D}_{j_{\min},k_1}\cdot\widetilde{\psi}^{0}_{j_2,k_2}\cdot\widetilde{\varphi}^{0}_{j_{\min},k_3}\delta_1 - \widetilde{\gamma}_{j_{\min},k_1}\cdot\widetilde{\psi}^{D}_{j_2,k_2}\cdot\widetilde{\varphi}^{0}_{j_{\min},k_3}\delta_2\right]$$

$$\widetilde{\Psi}^{\text{div},(3,3)}_{\mathbf{j},\mathbf{k}} = \frac{1}{\sqrt{4^{j_1}+1}}\left[\widetilde{\psi}^{D}_{j_1,k_1}\cdot\widetilde{\gamma}_{j_{\min},k_2}\cdot\widetilde{\varphi}^{0}_{j_{\min},k_3}\delta_1 - 2^{j_1}\widetilde{\psi}^{0}_{j_1,k_1}\cdot\widetilde{\varphi}^{D}_{j_{\min},k_2}\cdot\widetilde{\varphi}^{0}_{j_{\min},k_3}\delta_2\right]$$

$$\widetilde{\Psi}^{\text{div},(3,4)}_{\mathbf{j},\mathbf{k}} = \frac{1}{\sqrt{4^{j_2}+1}}\left[2^{j_2}\widetilde{\varphi}^{D}_{j_{\min},k_1}\cdot\widetilde{\psi}^{0}_{j_2,k_2}\cdot\widetilde{\psi}^{0}_{j_3,k_3}\delta_1 - \widetilde{\gamma}_{j_{\min},k_1}\cdot\widetilde{\psi}^{D}_{j_2,k_2}\cdot\widetilde{\psi}^{0}_{j_3,k_3}\delta_2\right]$$

$$\widetilde{\Psi}^{\text{div},(3,5)}_{\mathbf{j},\mathbf{k}} = \frac{1}{\sqrt{4^{j_1}+1}}\left[\widetilde{\psi}^{D}_{j_1,k_1}\cdot\widetilde{\gamma}_{j_{\min},k_2}\cdot\widetilde{\psi}^{0}_{j_3,k_3}\delta_1 - 2^{j_1}\widetilde{\psi}^{0}_{j_1,k_1}\cdot\widetilde{\varphi}^{D}_{j_{\min},k_2}\cdot\widetilde{\psi}^{0}_{j_3,k_3}\delta_2\right]$$

$$\widetilde{\Psi}^{\text{div},(3,6)}_{\mathbf{j},\mathbf{k}} = \frac{1}{\sqrt{4^{j_1}+4^{j_2}}}\left[2^{j_2}\widetilde{\psi}^{D}_{j_1,k_1}\cdot\widetilde{\psi}^{0}_{j_2,k_2}\cdot\widetilde{\varphi}^{0}_{j_{\min},k_3}\delta_1 - 2^{j_1}\widetilde{\psi}^{0}_{j_1,k_1}\cdot\widetilde{\psi}^{D}_{j_2,k_2}\cdot\widetilde{\varphi}^{0}_{j_{\min},k_3}\delta_2\right]$$

$$\widetilde{\Psi}^{\text{div},(3,7)}_{\mathbf{j},\mathbf{k}} = \frac{1}{\sqrt{4^{j_1}+4^{j_2}}}\left[2^{j_2}\widetilde{\psi}^{D}_{j_1,k_1}\cdot\widetilde{\psi}^{0}_{j_2,k_2}\cdot\widetilde{\psi}^{0}_{j_3,k_3}\delta_1 - 2^{j_1}\widetilde{\psi}^{0}_{j_1,k_1}\cdot\widetilde{\psi}^{D}_{j_2,k_2}\cdot\widetilde{\psi}^{0}_{j_3,k_3}\delta_2\right]. \quad (38)$$

Here, $\widetilde{\gamma}_{j,k} = -\int_0^x \widetilde{\varphi}^{D}_{j,k}(t)dt$.

Proposition 12. *The families* $\{\Phi^{\text{div},\varepsilon}_{j,\mathbf{k}}, \Psi^{\text{div},(\varepsilon,n)}_{\mathbf{j},\mathbf{k}} : j_1, j_2, j_3 \geq j,\ \varepsilon = 1,2,3,\ n = 1,2,\ldots,7\}$ *and* $\{\widetilde{\Phi}^{\text{div},\varepsilon}_{j,\mathbf{k}}, \widetilde{\Psi}^{\text{div},(\varepsilon,n)}_{\mathbf{j},\mathbf{k}} : j_1, j_2, j_3 \geq j,\ \varepsilon = 1,2,3,\ n = 1,2,\ldots,7\}$ *are biorthogonal in* $(L^2(\Omega))^3$.

Proof. It is easily proved by the fact that $\widetilde{\psi}^{0}_{j,k} = -2^j\int_0^x \widetilde{\psi}^{D}_{j,k}(t)dt$, which is shown in (16). □

Theorem 13. *The set* $\{\Phi^{\text{div},\varepsilon}_{j_{\min},\mathbf{k}}, \Psi^{\text{div},(\varepsilon,n)}_{\mathbf{j},\mathbf{k}} : j_1, j_2, j_3 \geq j_{\min},\ \varepsilon = 1,2,3,\ n = 1,2,\ldots,7\}$ *is a Riesz basis of* $\mathscr{H}_{\text{div}}(\Omega)$.

Proof. The completeness is ensured by Theorem 7 and Proposition 10. Now, it remains to prove the L^2-stability of the basis. By assumption of 1D scaling and wavelet functions, the divergence-free wavelets $\Psi^{\text{div},(\varepsilon,n)}_{\mathbf{j},\mathbf{k}}$ are compactly supported, have zero mean value, and belong to the spaces C^{ε} for some $\varepsilon > 0$; then they constitute a vaguelette-family ([12]). Furthermore, the Riesz stability follows from the existence of a biorthogonal wavelet family given by Proposition 12. □

4. Curl-Free Wavelets on $[0,1]^3$

The boundary condition considered in [7] is $\vec{u}\times\vec{n} = \vec{0}$ on $\Gamma = \bigcup_{k=1}^{3}\Gamma_k$ with

$$\Gamma_k = \bigcup_{m=1,m\neq k}^{3}[0,1]^{m-1}\times\{0\}\times[0,1]^{3-m}, \quad 1\leq k\leq 3. \quad (39)$$

It holds that $\vec{u}\times\vec{n} = \vec{0}$ on Γ if and only if $u_k = 0$ on Γ_k $(1\leq k\leq 3)$, which is shown in Figure 3.

In this section, we mainly consider the following space:

$$\mathscr{H}_{\text{curl}}(\Omega) = \left\{\vec{u}\in\left(L^2(\Omega)\right)^3 =: \operatorname{curl}\vec{u} = \vec{0},\ \vec{u}\times\vec{n} = \vec{0} \text{ on } \partial\Omega\right\} \quad (40)$$

with free-slip boundary as Figure 4.

An equivalent characterization is firstly given for $\mathscr{H}_{\text{curl}}(\Omega)$; and then we will give the MRA and wavelets for it.

Proposition 14. *There is the characterization* $\mathscr{H}_{\text{curl}}(\Omega) = \{\vec{u} = \operatorname{grad}\varphi : \varphi\in H_0^1(\Omega)\}$.

Proof. Suppose $\varphi\in H_0^1(\Omega)$; then $\vec{u} = \operatorname{grad}\varphi = (\partial_1\varphi,\partial_2\varphi,\partial_3\varphi)^T\in(L^2(\Omega))^3$. Moreover,

$$\begin{aligned}\operatorname{curl}\vec{u} &= \operatorname{curl}\cdot\operatorname{grad}\varphi\\ &= \left(\partial_2\partial_3\varphi-\partial_3\partial_2\varphi,\partial_3\partial_1\varphi-\partial_1\partial_3\varphi,\partial_1\partial_2\varphi-\partial_2\partial_1\varphi\right)^T\\ &= \vec{0}.\end{aligned} \quad (41)$$

Note that

$$\begin{aligned}\partial_1\varphi(x,y,0) &= \lim_{\Delta x\to 0}\frac{\varphi(x+\Delta x,y,0)-\varphi(x,y,0)}{\Delta x} = 0,\\ \partial_1\varphi(x,y,1) &= \lim_{\Delta x\to 0}\frac{\varphi(x+\Delta x,y,1)-\varphi(x,y,1)}{\Delta y} = 0,\\ \partial_1\varphi(x,0,z) &= \lim_{\Delta x\to 0}\frac{\varphi(x+\Delta x,0,z)-\varphi(x,0,z)}{\Delta x} = 0,\\ \partial_1\varphi(x,1,z) &= \lim_{\Delta x\to 0}\frac{\varphi(x+\Delta x,1,z)-\varphi(x,1,z)}{\Delta y} = 0;\end{aligned} \quad (42)$$

therefore,

$$\begin{aligned}u_1(x,y,0) = u_1(x,y,1) = 0, &\quad \forall 0\leq x,\ y\leq 1.\\ u_1(x,0,z) = u_1(x,1,z) = 0, &\quad \forall 0\leq x,\ z\leq 1.\end{aligned} \quad (43)$$

In the same way, one can obtain

$$\begin{aligned}u_2(x,y,0) = u_2(x,y,1) = 0, &\quad \forall 0\leq x,\ y\leq 1,\\ u_2(0,y,z) = u_2(1,y,z) = 0, &\quad \forall 0\leq y,\ z\leq 1,\\ u_3(0,y,z) = u_3(1,y,z) = 0, &\quad \forall 0\leq y,\ z\leq 1,\\ u_3(x,0,z) = u_3(x,1,z) = 0, &\quad \forall 0\leq x,\ z\leq 1.\end{aligned} \quad (44)$$

This is equivalent to $\vec{u}\times\vec{n} = \vec{0}$. Therefore, $\vec{u} = \operatorname{grad}\varphi\in\mathscr{H}_{\text{curl}}(\Omega)$.

On the other hand, suppose $\vec{u}\in\mathscr{H}_{\text{curl}}(\Omega)$; then we will prove that there exists a function $\varphi\in H_0^1(\Omega)$, such that $\vec{u} = \operatorname{grad}\varphi$. Since $\operatorname{curl}\vec{u} = \vec{0}$, then

$$\partial_2 u_3 = \partial_3 u_2, \qquad \partial_3 u_1 = \partial_1 u_3, \qquad \partial_1 u_2 = \partial_2 u_1. \quad (45)$$

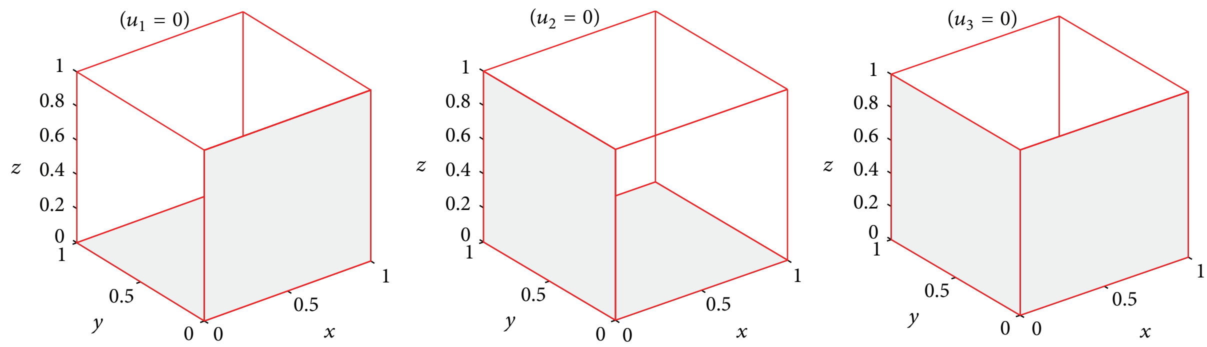

Figure 3: Slip boundary condition (curl).

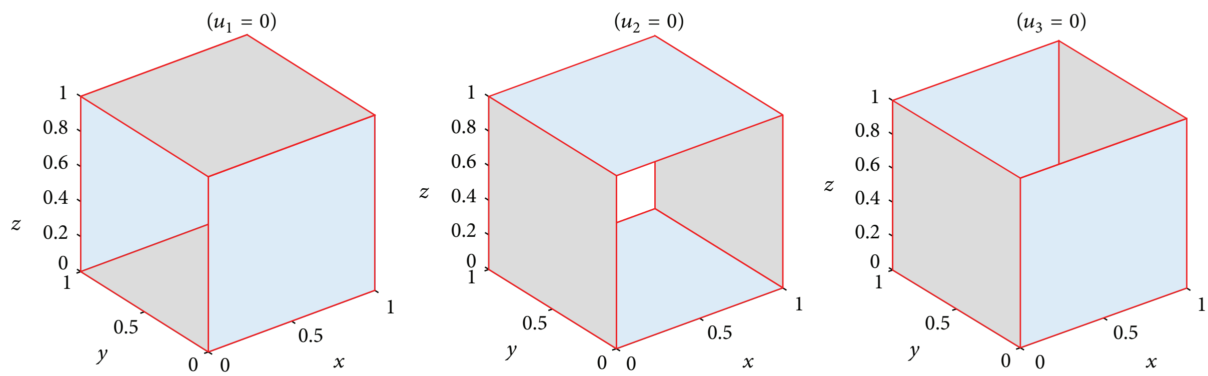

Figure 4: Free-slip boundary condition (curl).

By Stokes formula, there exists a primitive function $\varphi \in H^1(\Omega)$ such that

$$\begin{aligned} d\varphi(x,y,z) &\\ &= u_1(x,y,z)\,dx + u_2(x,y,z)\,dy + u_3(x,y,z)\,dz. \end{aligned} \tag{46}$$

Therefore, $\partial\varphi/\partial x = u_1$, $\partial\varphi/\partial y = u_2$, and $\partial\varphi/\partial z = u_3$; that is $\vec{u} = \operatorname{grad}\varphi$. Furthermore, $\vec{u}\times\vec{n} = \vec{0}$ means that

$$\begin{aligned} u_1(x,y,0) &= u_1(x,y,1) = 0, \quad \forall 0 \le x,\ y \le 1, \\ u_2(0,y,z) &= u_2(1,y,z) = 0, \quad \forall 0 \le y,\ z \le 1, \\ u_3(x,0,z) &= u_3(x,1,z) = 0, \quad \forall 0 \le x,\ z \le 1. \end{aligned} \tag{47}$$

Noting that

$$\begin{aligned} \varphi(x,y,z) &= \int_{x_0}^{x} u_1(r,y,z)\,dr = \int_{y_0}^{y} u_2(x,s,z)\,ds \\ &= \int_{z_0}^{z} u_3(x,y,t)\,dt, \end{aligned} \tag{48}$$

we obtain $\varphi(x,y,0) = \varphi(x,y,1) = 0$, $\varphi(0,y,z) = \varphi(1,y,z) = 0$, and $\varphi(x,0,z) = \varphi(x,1,z) = 0$. Therefore, $\varphi \in H_0^1(\Omega)$. □

Noting that $V_j^D \otimes V_j^D \otimes V_j^D$ is an MRA of $(H_0^1(\Omega))^3$, we give the following definition.

Definition 15. For $j \ge j_{\min}$, the curl-free scaling function spaces $\vec{V}_j^{\text{curl}}$ are defined by

$$\vec{V}_j^{\text{curl}} = \operatorname{grad}\left(V_j^D \otimes V_j^D \otimes V_j^D\right) = \operatorname{span}\left\{\Phi_{j,\mathbf{k}}^{\text{curl}}\right\}, \tag{49}$$

where the curl-free scaling functions are given by

$$\begin{aligned} \Phi_{j,\mathbf{k}}^{\text{curl}} &=: \frac{1}{\sqrt{3}}\operatorname{grad}\left(\varphi_{j,k_1}^D \cdot \varphi_{j,k_2}^D \cdot \varphi_{j,k_3}^D\right) \\ &= \frac{1}{\sqrt{3}}\left(\left(\varphi_{j,k_1}^D\right)' \cdot \varphi_{j,k_2}^D \cdot \varphi_{j,k_3}^D, \varphi_{j,k_1}^D \cdot \left(\varphi_{j,k_2}^D\right)' \right. \\ &\quad \left. \cdot \varphi_{j,k_3}^D, \varphi_{j,k_1}^D \cdot \varphi_{j,k_2}^D \cdot \left(\varphi_{j,k_3}^D\right)'\right)^T. \end{aligned} \tag{50}$$

For convenience, we also consider the standard MRA $\vec{V}_j$ of $(L^2(\Omega))^3$:

$$\vec{V}_j = \left(V_j^0 \otimes V_j^1 \otimes V_j^1\right) \times \left(V_j^1 \otimes V_j^0 \otimes V_1^1\right) \times \left(V_j^1 \otimes V_j^1 \otimes V_j^0\right). \tag{51}$$

Theorem 16. *The curl-free scaling function spaces* $\{\vec{V}_j^{\text{curl}}\}_{j \ge j_{\min}}$ *are a multiresolution analysis of* $\mathscr{H}_{\text{curl}}(\Omega)$.

Proof. Since $\mathscr{H}_{\text{curl}}(\Omega) \cap \vec{V}_j$ is a multiresolution analysis of $\mathscr{H}_{\text{curl}}(\Omega)$, it is reduced to prove

$$\vec{V}_j^{\text{curl}} = \mathscr{H}_{\text{curl}}(\Omega) \cap \vec{V}_j. \tag{52}$$

Noting that $(d/dx)V_j^1 = V_j^0$ and $V_j^D \subseteq V_j^1$, we know $\vec{V}_j^{\text{curl}} \subset \vec{V}_j$. Furthermore, $\vec{V}_j^{\text{curl}} \subset \mathscr{H}_{\text{curl}}(\Omega)$ by construction. Therefore, $\vec{V}_j^{\text{curl}} \subset \mathscr{H}_{\text{curl}}(\Omega) \cap \vec{V}_j$.

Conversely, let $\vec{u} \in \mathscr{H}_{\text{curl}}(\Omega) \cap \vec{V}_j$, we are going to prove $\vec{u} \in \vec{V}_j^{\text{curl}}$. Let $\vec{P}_j$ be the biorthogonal projector on $\vec{V}_j$. On the one hand, since $\vec{u} \in \vec{V}_j$, we have $\vec{u} = \vec{P}_j\vec{u}$. On the other hand, since $\vec{u} \in \mathscr{H}_{\text{curl}}(\Omega)$, there exists a $\varphi \in H_0^1(\Omega)$ such that $\vec{u} = \text{grad}\varphi$. Thus,

$$\vec{u} = \vec{P}_j \left[\text{grad}\varphi\right]. \tag{53}$$

Since $(V_j^D \otimes V_j^D \otimes V_j^D)_{j \ge j_{\min}}$ forms an MRA of $H_0^1(\Omega)$, we can decompose φ as

$$\varphi = P_j^D(\varphi) + \sum_{j_1, j_2, j_3 \ge j} \sum_{n=1}^{7} Q_{n,J}^D(\varphi), \quad J = (j_1, j_2, j_3), \tag{54}$$

where

$$\begin{aligned}
P_j^D(\varphi) &= \sum_{\mathbf{k}} c_{\mathbf{k}} \varphi_{j,k_1}^D \varphi_{j,k_2}^D \varphi_{j,k_3}^D, \\
Q_{1,J}^D(\varphi) &= \sum_{j_3 \ge j} \sum_{\mathbf{k}} d_{j_3,\mathbf{k}}^1 \varphi_{j,k_1}^D \varphi_{j,k_2}^D \psi_{j_3,k_3}^D, \\
Q_{2,J}^D(\varphi) &= \sum_{j_2 \ge j} \sum_{\mathbf{k}} d_{j_2,\mathbf{k}}^2 \varphi_{j,k_1}^D \psi_{j_2,k_2}^D \varphi_{j,k_3}^D, \\
Q_{3,J}^D(\varphi) &= \sum_{j_1 \ge j} \sum_{\mathbf{k}} d_{j_1,\mathbf{k}}^3 \psi_{j_1,k_1}^D \varphi_{j,k_2}^D \varphi_{j,k_3}^D, \\
Q_{4,J}^D(\varphi) &= \sum_{j_2, j_3 \ge j} \sum_{\mathbf{k}} d_{j_2,j_3,\mathbf{k}}^4 \varphi_{j,k_1}^D \psi_{j_2,k_2}^D \psi_{j_3,k_3}^D, \\
Q_{5,J}^D(\varphi) &= \sum_{j_1, j_3 \ge j} \sum_{\mathbf{k}} d_{j_1,j_3,\mathbf{k}}^5 \psi_{j_1,k_1}^D \varphi_{j,k_2}^D \psi_{j_3,k_3}^D, \\
Q_{6,J}^D(\varphi) &= \sum_{j_1, j_2 \ge j} \sum_{\mathbf{k}} d_{j_1,j_2,\mathbf{k}}^6 \psi_{j_1,k_1}^D \psi_{j_2,k_2}^D \varphi_{j,k_3}^D, \\
Q_{7,J}^D(\varphi) &= \sum_{j_1, j_2, j_3 \ge j} \sum_{\mathbf{k}} d_{j_1,j_2,j_3,\mathbf{k}}^7 \psi_{j_1,k_1}^D \psi_{j_2,k_2}^D \psi_{j_3,k_3}^D
\end{aligned} \tag{55}$$

are the biorthogonal projectors on, respectively, $V_j^D \otimes V_j^D \otimes V_j^D$, $V_j^D \otimes V_j^D \otimes W_{j_3}^D$, $V_j^D \otimes W_{j_2}^D \otimes V_j^D$, $W_{j_1}^D \otimes V_j^D \otimes V_j^D$, $V_j^D \otimes W_{j_2}^D \otimes W_{j_3}^D$, $W_{j_1}^D \otimes V_j^D \otimes W_{j_3}^D$, $W_{j_1}^D \otimes W_{j_2}^D \otimes V_j^D$, and $W_{j_1}^D \otimes W_{j_2}^D \otimes W_{j_3}^D$.

Noting that

$$\begin{aligned}
\text{grad}\left(\varphi_{j,k_1}^D \varphi_{j,k_2}^D \psi_{j_3,k_3}^D\right) &\in \left(V_j^0 \otimes V_j^D \otimes W_{j_3}^D\right) \\
&\quad \times \left(V_j^D \otimes V_j^0 \otimes W_{j_3}^D\right) \times \left(V_j^D \otimes V_j^D \otimes W_{j_3}^0\right),
\end{aligned} \tag{56}$$

then $\vec{P}_j[\text{grad}\, Q_{1,J}^D(\varphi)] = \vec{0}$. Similarly, $\vec{P}_j[\text{grad}\, Q_{n,J}^D(\varphi)] = \vec{0}$ for $2 \le n \le 7$. Therefore,

$$\vec{u} = \vec{P}_j\left[\text{grad}\varphi\right] = \vec{P}_j\left[\text{grad}\, P_j^D(\varphi)\right]. \tag{57}$$

Since $\text{grad}\, P_j^D(\varphi) \in \vec{V}_j^{\text{curl}} \subset \vec{V}_j$, then we obtain

$$\vec{u} = \text{grad}\, P_j^D(\varphi) \in \vec{V}_j^{\text{curl}}. \tag{58}$$

□

Definition 17. For j_1, j_2, and $j_3 \ge j_{\min}$, the anisotropic curl-free wavelets and wavelet spaces are defined by

$$\begin{aligned}
\Psi_{\mathbf{j},\mathbf{k}}^{\text{curl},1} &= \frac{1}{\sqrt{4^{j_3}+2}} \text{grad}\left[\varphi_{j_{\min},k_1}^D \cdot \varphi_{j_{\min},k_2}^D \cdot \psi_{j_3,k_3}^D\right] \\
\Psi_{\mathbf{j},\mathbf{k}}^{\text{curl},2} &= \frac{1}{\sqrt{4^{j_2}+2}} \text{grad}\left[\varphi_{j_{\min},k_1}^D \cdot \psi_{j_2,k_2}^D \cdot \varphi_{j_{\min},k_3}^D\right] \\
\Psi_{\mathbf{j},\mathbf{k}}^{\text{curl},3} &= \frac{1}{\sqrt{4^{j_1}+2}} \text{grad}\left[\psi_{j_1,k_1}^D \cdot \varphi_{j_{\min},k_2}^D \cdot \varphi_{j_{\min},k_3}^D\right] \\
\Psi_{\mathbf{j},\mathbf{k}}^{\text{curl},4} &= \frac{1}{\sqrt{4^{j_2}+4^{j_3}+1}} \text{grad}\left[\varphi_{j_{\min},k_1}^D \cdot \psi_{j_2,k_2}^D \cdot \psi_{j_3,k_3}^D\right] \\
\Psi_{\mathbf{j},\mathbf{k}}^{\text{curl},5} &= \frac{1}{\sqrt{4^{j_1}+4^{j_3}+1}} \text{grad}\left[\psi_{j_1,k_1}^D \cdot \varphi_{j_{\min},k_2}^D \cdot \psi_{j_3,k_3}^D\right] \\
\Psi_{\mathbf{j},\mathbf{k}}^{\text{curl},6} &= \frac{1}{\sqrt{4^{j_1}+4^{j_2}+1}} \text{grad}\left[\psi_{j_1,k_1}^D \cdot \psi_{j_2,k_2}^D \cdot \varphi_{j_{\min},k_3}^D\right] \\
\Psi_{\mathbf{j},\mathbf{k}}^{\text{curl},7} &= \frac{1}{\sqrt{4^{j_1}+4^{j_2}+4^{j_3}}} \text{grad}\left[\psi_{j_1,k_1}^D \cdot \psi_{j_2,k_2}^D \cdot \psi_{j_3,k_3}^D\right].
\end{aligned} \tag{59}$$

Proposition 18. *Defining the wavelet spaces* $\vec{W}_{\mathbf{j}}^{\text{curl},n} = \text{span}\{\Psi_{\mathbf{j},\mathbf{k}}^{\text{curl},n}\}$ *for* $n = 1, 2, \ldots, 7$, *then*

$$\vec{V}_j^{\text{curl}} = \vec{V}_{j_{\min}}^{\text{curl}} \oplus_{j_{\min} \le j_1, j_2, j_3 \le j-1} \left(\oplus_{\varepsilon=1,2,\ldots,7} \vec{W}_{\mathbf{j}}^{\text{curl},\varepsilon}\right). \tag{60}$$

Proof. The result follows from the following fact:

$$\begin{aligned}
V_j^D \otimes V_j^D \otimes V_j^D &= \left(V_{j_{\min}}^D \oplus_{j_1=j_{\min}}^{j-1} W_{j_1}^D\right) \\
&\quad \otimes \left(V_{j_{\min}}^D \oplus_{j_2=j_{\min}}^{j-1} W_{j_2}^D\right) \\
&\quad \otimes \left(V_{j_{\min}}^D \oplus_{j_3=j_{\min}}^{j-1} W_{j_3}^D\right) \\
&= \left(V_{j_{\min}}^D \otimes V_{j_{\min}}^D \otimes V_{j_{\min}}^D\right) \\
&\quad \oplus_{j_3=j_{\min}}^{j-1} \left(V_{j_{\min}}^D \otimes V_{j_{\min}}^D \otimes W_{j_3}^D\right) \\
&\quad \oplus_{j_2=j_{\min}}^{j-1} \left(V_{j_{\min}}^D \otimes W_{j_2}^D \otimes V_{j_{\min}}^D\right) \\
&\quad \oplus_{j_1=j_{\min}}^{j-1} \left(W_{j_1}^D \otimes V_{j_{\min}}^D \otimes V_{j_{\min}}^D\right) \\
&\quad \oplus_{j_2,j_3=j_{\min}}^{j-1} \left(V_{j_{\min}}^D \otimes W_{j_2}^D \otimes W_{j_3}^D\right) \\
&\quad \oplus_{j_1,j_3=j_{\min}}^{j-1} \left(W_{j_1}^D \otimes V_{j_{\min}}^D \otimes W_{j_3}^D\right) \\
&\quad \oplus_{j_1,j_2=j_{\min}}^{j-1} \left(W_{j_1}^D \otimes W_{j_2}^D \otimes V_{j_{\min}}^D\right) \\
&\quad \oplus_{j_1,j_2,j_3=j_{\min}}^{j-1} \left(W_{j_1}^D \otimes W_{j_2}^D \otimes W_{j_3}^D\right).
\end{aligned} \tag{61}$$

□

Definition 19. Biorthogonal curl-free scaling functions and wavelets are defined by

$$\begin{aligned}
\widetilde{\Phi}_{j,\mathbf{k}}^{\text{curl}} &= \frac{1}{\sqrt{3}} \left[\widetilde{\gamma}_{j,k_1} \cdot \widetilde{\varphi}_{j,k_2}^D \cdot \widetilde{\varphi}_{j,k_3}^D \delta_1 + \widetilde{\varphi}_{j,k_1}^D \cdot \widetilde{\gamma}_{j,k_2} \cdot \widetilde{\varphi}_{j,k_3}^D \delta_2 \right. \\
&\quad \left. + \widetilde{\varphi}_{j,k_1}^D \cdot \widetilde{\varphi}_{j,k_2}^D \cdot \widetilde{\gamma}_{j,k_3} \delta_3\right],
\end{aligned}$$

$$
\begin{aligned}
\widetilde{\Psi}^{\mathrm{curl},1}_{\mathbf{j},\mathbf{k}} &= \frac{1}{\sqrt{4^{j_3}+2}} \Big[\widetilde{\gamma}_{j_{\min},k_1}\widetilde{\varphi}^{D}_{j_{\min},k_2}\widetilde{\psi}^{D}_{j_3,k_3}\delta_1 \\
&\quad + \widetilde{\varphi}^{D}_{j_{\min},k_1}\widetilde{\gamma}_{j_{\min},k_2}\widetilde{\psi}^{D}_{j_3,k_3}\delta_2 \\
&\quad + 2^{j_3}\widetilde{\varphi}^{D}_{j_{\min},k_1}\widetilde{\varphi}^{D}_{j_{\min},k_2}\widetilde{\psi}^{0}_{j_3,k_3}\delta_3\Big], \\
\widetilde{\Psi}^{\mathrm{curl},2}_{\mathbf{j},\mathbf{k}} &= \frac{1}{\sqrt{4^{j_2}+2}} \Big[\widetilde{\gamma}_{j_{\min},k_1}\widetilde{\psi}^{D}_{j_2,k_2}\widetilde{\varphi}^{D}_{j_{\min},k_3}\delta_1 \\
&\quad + 2^{j_2}\widetilde{\varphi}^{D}_{j_{\min},k_1}\widetilde{\psi}^{0}_{j_2,k_2}\widetilde{\varphi}^{D}_{j_{\min},k_3}\delta_2 \\
&\quad + \widetilde{\varphi}^{D}_{j_{\min},k_1}\widetilde{\psi}^{D}_{j_2,k_2}\widetilde{\gamma}_{j_{\min},k_3}\delta_3\Big], \\
\widetilde{\Psi}^{\mathrm{curl},3}_{\mathbf{j},\mathbf{k}} &= \frac{1}{\sqrt{4^{j_1}+2}} \Big[2^{j_1}\widetilde{\psi}^{0}_{j_1,k_1}\widetilde{\varphi}^{D}_{j_{\min},k_2}\widetilde{\varphi}^{D}_{j_{\min},k_3}\delta_1 \\
&\quad + \widetilde{\psi}^{D}_{j_1,k_1}\widetilde{\gamma}_{j_{\min},k_2}\widetilde{\varphi}^{D}_{j_{\min},k_3}\delta_2 \\
&\quad + \widetilde{\psi}^{D}_{j_1,k_1}\widetilde{\varphi}^{D}_{j_{\min},k_2}\widetilde{\gamma}_{j_{\min},k_3}\delta_3\Big], \\
\widetilde{\Psi}^{\mathrm{curl},4}_{\mathbf{j},\mathbf{k}} &= \frac{1}{\sqrt{4^{j_2}+4^{j_3}+1}} \Big[\widetilde{\gamma}_{j_{\min},k_1}\widetilde{\psi}^{D}_{j_2,k_2}\widetilde{\psi}^{D}_{j_3,k_3}\delta_1 \\
&\quad + 2^{j_2}\widetilde{\varphi}^{D}_{j_{\min},k_1}\widetilde{\psi}^{0}_{j_2,k_2}\widetilde{\psi}^{D}_{j_3,k_3}\delta_2 \\
&\quad + 2^{j_3}\widetilde{\varphi}^{D}_{j_{\min},k_1}\widetilde{\psi}^{D}_{j_2,k_2}\widetilde{\psi}^{0}_{j_3,k_3}\delta_3\Big], \\
\widetilde{\Psi}^{\mathrm{curl},5}_{\mathbf{j},\mathbf{k}} &= \frac{1}{\sqrt{4^{j_1}+4^{j_3}+1}} \Big[2^{j_1}\widetilde{\psi}^{0}_{j_1,k_1}\widetilde{\varphi}^{D}_{j_{\min},k_2}\widetilde{\psi}^{D}_{j_3,k_3}\delta_1 \\
&\quad + \widetilde{\psi}^{D}_{j_1,k_1}\widetilde{\gamma}_{j_{\min},k_2}\widetilde{\psi}^{D}_{j_3,k_3}\delta_2 \\
&\quad + 2^{j_3}\widetilde{\psi}^{D}_{j_1,k_1}\widetilde{\varphi}^{D}_{j_{\min},k_2}\widetilde{\psi}^{0}_{j_3,k_3}\delta_3\Big], \\
\widetilde{\Psi}^{\mathrm{curl},6}_{\mathbf{j},\mathbf{k}} &= \frac{1}{\sqrt{4^{j_1}+4^{j_2}+1}} \Big[2^{j_1}\widetilde{\psi}^{0}_{j_1,k_1}\widetilde{\psi}^{D}_{j_2,k_2}\widetilde{\varphi}^{D}_{j_{\min},k_3}\delta_1 \\
&\quad + 2^{j_2}\widetilde{\psi}^{D}_{j_1,k_1}\widetilde{\psi}^{0}_{j_2,k_2}\widetilde{\varphi}^{D}_{j_{\min},k_3}\delta_2 \\
&\quad + \widetilde{\psi}^{D}_{j_1,k_1}\widetilde{\psi}^{D}_{j_2,k_2}\widetilde{\gamma}_{j_{\min},k_3}\delta_3\Big], \\
\widetilde{\Psi}^{\mathrm{curl},7}_{\mathbf{j},\mathbf{k}} &= \frac{1}{\sqrt{4^{j_1}+4^{j_2}+4^{j_3}}} \Big[2^{j_1}\widetilde{\psi}^{0}_{j_1,k_1}\widetilde{\psi}^{D}_{j_2,k_2}\widetilde{\psi}^{D}_{j_3,k_3}\delta_1 \\
&\quad + 2^{j_2}\widetilde{\psi}^{D}_{j_1,k_1}\widetilde{\psi}^{0}_{j_2,k_2}\widetilde{\psi}^{D}_{j_3,k_3}\delta_2 \\
&\quad + 2^{j_3}\widetilde{\psi}^{D}_{j_1,k_1}\widetilde{\psi}^{D}_{j_2,k_2}\widetilde{\psi}^{0}_{j_3,k_3}\delta_3\Big].
\end{aligned}
\tag{62}
$$

Here, $\widetilde{\gamma}_{j,k}$ is defined as in Definition 11.

Proposition 20. *The families* $\{\Phi^{\mathrm{curl}}_{j,\mathbf{k}}, \Psi^{\mathrm{curl},\varepsilon}_{\mathbf{j},\mathbf{k}} : j_1, j_2, j_3 \geq j, \varepsilon = 1, 2, \ldots, 7\}$ *and* $\{\widetilde{\Phi}^{\mathrm{curl}}_{j,\mathbf{k}}, \widetilde{\Psi}^{\mathrm{curl},\varepsilon}_{\mathbf{j},\mathbf{k}} : j_1, j_2, j_3 \geq j, \varepsilon = 1, 2, \ldots, 7\}$ *are biorthogonal in* $(L^2(\Omega))^3$.

Theorem 21. *The set* $\{\Phi^{\mathrm{curl}}_{j_{\min},\mathbf{k}}, \Psi^{\mathrm{curl},\varepsilon}_{\mathbf{j},\mathbf{k}} : j_1, j_2, j_3 \geq j_{\min}, \varepsilon = 1, 2, \ldots, 7\}$ *is a Riesz basis of* $\mathscr{H}_{\mathrm{curl}}(\Omega)$.

Proof. It can be proved by the same method as Theorem 13. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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