

Research Article

Homotopy Perturbation Method for Fractional Gas Dynamics Equation Using Sumudu Transform

Jagdev Singh,¹ Devendra Kumar,² and A. Kılıçman³

¹ Department of Mathematics, Jagannath University, Rampura, Chaksu, Jaipur, Rajasthan 303901, India

² Department of Mathematics, Jagannath Gupta Institute of Engineering & Technology, Jaipur, Rajasthan 302022, India

³ Department of Mathematics and Institute for Mathematical Research, University Putra Malaysia (UPM), 43400 Serdang, Selangor, Malaysia

Correspondence should be addressed to A. Kılıçman; akilicman@putra.upm.edu.my

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A user friendly algorithm based on new homotopy perturbation Sumudu transform method (HPSTM) is proposed to solve nonlinear fractional gas dynamics equation. The fractional derivative is considered in the Caputo sense. Further, the same problem is solved by Adomian decomposition method (ADM). The results obtained by the two methods are in agreement and hence this technique may be considered an alternative and efficient method for finding approximate solutions of both linear and nonlinear fractional differential equations. The HPSTM is a combined form of Sumudu transform, homotopy perturbation method, and He's polynomials. The nonlinear terms can be easily handled by the use of He's polynomials. The numerical solutions obtained by the proposed method show that the approach is easy to implement and computationally very attractive.

1. Introduction

Fractional calculus is a field of applied mathematics that deals with derivatives and integrals of arbitrary orders. During the last decade, fractional calculus has found applications in numerous seemingly diverse fields of science and engineering. Fractional differential equations are increasingly used to model problems in fluid mechanics, acoustics, biology, electromagnetism, diffusion, signal processing, and many other physical processes [1–19].

There exists a wide class of literature dealing with the problems of approximate solutions to fractional differential equations with various different methodologies, called perturbation methods. The perturbation methods have some limitations; for example, the approximate solution involves series of small parameters which poses difficulty since the majority of nonlinear problems have no small parameters at all. Although appropriate choices of small parameters sometimes lead to ideal solution, in most of the cases unsuitable choices lead to serious effects in the solutions. Therefore, an analytical method is welcome which does not require a small parameter in the equation modeling the phenomenon.

Recently, there is a very comprehensive literature review in some new asymptotic methods for the search for the solitary solutions of nonlinear differential equations, nonlinear differential-difference equations, and nonlinear fractional differential equations; see [20]. The homotopy perturbation method (HPM) was first introduced by He [21]. The HPM was also studied by many authors to handle linear and nonlinear equations arising in various scientific and technological fields [22–32]. The Adomian decomposition method (ADM) [33] and variational iteration method (VIM) [34] have also been applied to study the various physical problems.

In a recent paper, Singh et al. [35] have paid attention to study the solutions of linear and nonlinear partial differential equations by using the homotopy perturbation Sumudu transform method (HPSTM). The HPSTM is a combination of Sumudu transform, HPM, and He's polynomials and is mainly due to Ghorbani and Saberi-Nadjafi [36] and Ghorbani [37].

In this paper, we consider the following nonlinear time-fractional gas dynamics equation of the form

$$D_t^\alpha U + \frac{1}{2} (U^2)_x - U(1-U) = 0, \quad t > 0, \quad 0 < \alpha \leq 1, \quad (1)$$

with the initial condition

$$U(x, 0) = e^{-x}, \tag{2}$$

where α is a parameter describing the order of the fractional derivative. The function $U(x, t)$ is the probability density function, t is the time, and x is the spatial coordinate. The derivative is understood in the Caputo sense. The general response expression contains a parameter describing the order of the fractional derivative that can be varied to obtain various responses. In the case of $\alpha = 1$ the fractional gas dynamics equation reduces to the classical gas dynamics equation. The gas dynamics equations are based on the physical laws of conservation, namely, the laws of conservation of mass, conservation of momentum, conservation of energy, and so forth. The nonlinear fractional gas dynamics has been studied previously by Das and Kumar [38].

Further, we apply the HPSTM and ADM to solve the nonlinear time-fractional gas dynamics equation. The objective of the present paper is to extend the application of the HPSTM to obtain analytic and approximate solutions to the time-fractional gas dynamics equation. The advantage of the HPSTM is its capability of combining two powerful methods for obtaining exact and approximate analytical solutions for nonlinear equations. It provides the solutions in terms of convergent series with easily computable components in a direct way without using linearization, perturbation, or restrictive assumptions. It is worth mentioning that the HPSTM is capable of reducing the volume of the computational work as compared to the classical methods while still maintaining the high accuracy of the numerical result; the size reduction amounts to an improvement of the performance of the approach.

2. Sumudu Transform

In the early 90's, Watugala [39] introduced a new integral transform, named the Sumudu transform and applied it to the solution of ordinary differential equation in control engineering problems. The Sumudu transform, is defined over the set of functions

$$A = \{f(t) \mid \exists M, \tau_1, \tau_2 > 0, |f(t)| < Me^{t/\tau_j}, \text{ if } t \in (-1)^j \times [0, \infty)\} \tag{3}$$

by the following formula:

$$\bar{f}(u) = S[f(t)] = \int_0^\infty f(ut) e^{-t} dt, \quad u \in (-\tau_1, \tau_2). \tag{4}$$

Some of the properties were established by Weerakoon in [40, 41]. In [42], by Aşiru, further fundamental properties of this transform were also established. Similarly, this transform was applied to the one-dimensional neutron transport equation in [43] by Kadem. In fact it was shown that there is a strong relationship between Sumudu and other integral transforms; see Kılıçman et al. [44]. In particular the relation

between Sumudu transform and Laplace transforms was proved in Kılıçman and Gadain [45].

Further, in Eltayeb et al. [46], the Sumudu transform was extended to the distributions and some of their properties were also studied in Kılıçman and Eltayeb [47]. Recently, this transform is applied to solve the system of differential equations; see Kılıçman et al. in [48].

Note that a very interesting fact about Sumudu transform is that the original function and its Sumudu transform have the same Taylor coefficients except the factor n ; see Zhang [49]. Thus if $f(t) = \sum_{n=0}^\infty a_n t^n$ then $F(u) = \sum_{n=0}^\infty n! a_n u^n$; see Kılıçman et al. [44]. Similarly, the Sumudu transform sends combinations, $C(m, n)$, into permutations, $P(m, n)$, and hence it will be useful in the discrete systems.

3. Basic Definitions of Fractional Calculus

In this section, we mention the following basic definitions of fractional calculus which are used further in the present paper.

Definition 1. The Riemann-Liouville fractional integral operator of order $\alpha > 0$, of a function $f(t) \in C_\mu$, and $\mu \geq -1$ is defined as [5]

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau) d\tau, \quad (\alpha > 0), \tag{5}$$

$$J^0 f(t) = f(t). \tag{6}$$

For the Riemann-Liouville fractional integral, we have

$$J^\alpha t^\gamma = \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + \alpha + 1)} t^{\alpha+\gamma}. \tag{7}$$

Definition 2. The fractional derivative of $f(t)$ in the Caputo sense is defined as [10]

$$D_t^\alpha f(t) = J^{m-\alpha} D^m f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t - \tau)^{m-\alpha-1} f^{(m)}(\tau) d\tau, \tag{8}$$

for $m - 1 < \alpha \leq m, m \in N, t > 0$.

For the Riemann-Liouville fractional integral and the Caputo fractional derivative, we have the following relation:

$$J_t^\alpha D_t^\alpha f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0+) \frac{t^k}{k!}. \tag{9}$$

Definition 3. The Sumudu transform of the Caputo fractional derivative is defined as follows [50]:

$$S[D_t^\alpha f(t)] = u^{-\alpha} S[f(t)] - \sum_{k=0}^{m-1} u^{-\alpha+k} f^{(k)}(0+), \quad (m - 1 < \alpha \leq m). \tag{10}$$

4. Solution by Homotopy Perturbation Sumudu Transform Method (HPSTM)

4.1. *Basic Idea of HPSTM.* To illustrate the basic idea of this method, we consider a general fractional nonlinear nonhomogeneous partial differential equation with the initial condition of the form

$$D_t^\alpha U(x, t) + RU(x, t) + NU(x, t) = g(x, t), \tag{11}$$

$$U(x, 0) = f(x), \tag{12}$$

where $D_t^\alpha U(x, t)$ is the Caputo fractional derivative of the function $U(x, t)$, R is the linear differential operator, N represents the general nonlinear differential operator, and $g(x, t)$ is the source term.

Applying the Sumudu transform (denoted in this paper by S) on both sides of (11), we get

$$S[D_t^\alpha U(x, t)] + S[RU(x, t)] + S[NU(x, t)] = S[g(x, t)]. \tag{13}$$

Using the property of the Sumudu transform, we have

$$S[U(x, t)] = f(x) + u^\alpha S[g(x, t)] - u^\alpha S[RU(x, t) + NU(x, t)]. \tag{14}$$

Operating with the Sumudu inverse on both sides of (14) gives

$$U(x, t) = G(x, t) - S^{-1}[u^\alpha S[RU(x, t) + NU(x, t)]], \tag{15}$$

where $G(x, t)$ represents the term arising from the source term and the prescribed initial conditions. Now we apply the HPM:

$$U(x, t) = \sum_{n=0}^{\infty} p^n U_n(x, t), \tag{16}$$

and the nonlinear term can be decomposed as

$$NU(x, t) = \sum_{n=0}^{\infty} p^n H_n(U), \tag{17}$$

for some He's polynomials $H_n(U)$ [37] that are given by

$$H_n(U_0, U_1, \dots, U_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left[N \left(\sum_{i=0}^{\infty} p^i U_i \right) \right]_{p=0}, \quad n = 0, 1, 2, \dots \tag{18}$$

Substituting (16) and (17) in (15), we get

$$\sum_{n=0}^{\infty} p^n U_n(x, t) = G(x, t) - p \left(S^{-1} \left[u^\alpha S \left[R \sum_{n=0}^{\infty} p^n U_n(x, t) + \sum_{n=0}^{\infty} p^n H_n(U) \right] \right] \right), \tag{19}$$

which is the coupling of the Sumudu transform and the HPM using He's polynomials. Comparing the coefficients of like powers of p , the following approximations are obtained:

$$p^0 : U_0(x, t) = G(x, t),$$

$$p^1 : U_1(x, t) = -S^{-1} [u^\alpha S [RU_0(x, t) + H_0(U)]],$$

$$p^2 : U_2(x, t) = -S^{-1} [u^\alpha S [RU_1(x, t) + H_1(U)]], \tag{20}$$

$$p^3 : U_3(x, t) = -S^{-1} [u^\alpha S [RU_2(x, t) + H_2(U)]],$$

⋮

Proceeding in this same manner, the rest of the components $U_n(x, t)$ can be completely obtained and the series solution is thus entirely determined. Finally, we approximate the analytical solution $U(x, t)$ by truncated series:

$$U(x, t) = \lim_{N \rightarrow \infty} \sum_{n=0}^N U_n(x, t). \tag{21}$$

The above series solutions generally converge very rapidly.

4.2. *Solution of the Problem.* Consider the following nonlinear time-fractional gas dynamics equation:

$$D_t^\alpha U + \frac{1}{2} (U^2)_x - U(1 - U) = 0, \quad 0 < \alpha \leq 1, \tag{22}$$

with the initial condition

$$U(x, 0) = e^{-x}. \tag{23}$$

Applying the Sumudu transform on both sides of (22), subject to the initial condition (23), we have

$$S[U(x, t)] = e^{-x} - u^\alpha S \left[\frac{1}{2} (U^2)_x - U(1 - U) \right]. \tag{24}$$

The inverse Sumudu transform implies that

$$U(x, t) = e^{-x} - S^{-1} \left[u^\alpha S \left[\frac{1}{2} (U^2)_x - U(1 - U) \right] \right]. \tag{25}$$

Now applying the HPM, we get

$$\begin{aligned} & \sum_{n=0}^{\infty} p^n U_n(x, t) \\ &= e^{-x} - p \left(S^{-1} \left[u^\alpha S \left[\frac{1}{2} \left(\sum_{n=0}^{\infty} p^n H_n(U) \right) - \left(\sum_{n=0}^{\infty} p^n U_n(x, t) \right) + \left(\sum_{n=0}^{\infty} p^n H'_n(U) \right) \right] \right] \right), \end{aligned} \tag{26}$$

where $H_n(U)$ and $H'_n(U)$ are He's polynomials [37] that represent the nonlinear terms. So, the He's polynomials are given by

$$\sum_{n=0}^{\infty} p^n H_n(U) = (U^2)_x. \tag{27}$$

The first few components of He's polynomials are given by

$$\begin{aligned} H_0(U) &= (U_0^2)_x, \\ H_1(U) &= 2(U_0U_1)_x, \\ H_2(U) &= (U_1^2 + 2U_0U_2)_x, \\ &\vdots \end{aligned} \tag{28}$$

and for $H'_n(U)$, we find that

$$\begin{aligned} \sum_{n=0}^{\infty} p^n H'_n(U) &= U^2, \\ H'_0(U) &= U_0^2, \\ H'_1(U) &= 2U_0U_1, \\ H'_2(U) &= U_1^2 + 2U_0U_2, \\ &\vdots \end{aligned} \tag{29}$$

Comparing the coefficients of like powers of p , we have

$$\begin{aligned} p^0 : U_0(x, t) &= e^{-x}, \\ p^1 : U_1(x, t) &= -S^{-1} \left[u^\alpha S \left[\frac{1}{2} H_0(U) - U_0 + H'_0(U) \right] \right] \\ &= e^{-x} \frac{t^\alpha}{\Gamma(\alpha + 1)}, \\ p^2 : U_2(x, t) &= -S^{-1} \left[u^\alpha S \left[\frac{1}{2} H_1(U) - U_1 + H'_1(U) \right] \right] \\ &= e^{-x} \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)}, \\ p^3 : U_3(x, t) &= -S^{-1} \left[u^\alpha S \left[\frac{1}{2} H_2(U) - U_2 + H'_2(U) \right] \right] \\ &= e^{-x} \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)}, \\ &\vdots \end{aligned} \tag{30}$$

Therefore, the series solution is

$$U(x, t) = e^{-x} \left[1 + \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)} + \dots \right]. \tag{31}$$

Setting $\alpha = 1$ in (31), we reproduce the solution of the problem as follows:

$$U(x, t) = e^{-x} \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots \right). \tag{32}$$

This solution is equivalent to the exact solution in closed form:

$$U(x, t) = e^{t-x}. \tag{33}$$

Now, we calculate numerical results of the probability density function $U(x, t)$ for different time-fractional Brownian motions $\alpha = 1/3, 2/3, 1$ and for various values of t and x . The numerical results for the approximate solution (31) obtained by using HPSTM and the exact solution (33) for various values of t, x , and α are shown in Figures 1(a)–1(d) and those for different values of t and α at $x = 1$ are depicted in Figure 2.

It is observed from Figures 1 and 2 that $U(x, t)$ increases with the increase in t and decreases with the increase in α . Figures 1(c) and 1(d) clearly show that, when $\alpha = 1$, the approximate solution (31) obtained by the present method is very near to the exact solution. It is to be noted that only the third-order term of the HPSTM was used in evaluating the approximate solutions for Figures 1 and 2. It is evident that the efficiency of the present method can be dramatically enhanced by computing further terms of $U(x, t)$ when the HPSTM is used.

5. Solution by Adomian Decomposition Method (ADM)

5.1. *Basic Idea of ADM.* To illustrate the basic idea of ADM [51, 52], we consider a general fractional nonlinear nonhomogeneous partial differential equation with the initial condition of the form

$$D_t^\alpha U(x, t) + RU(x, t) + NU(x, t) = g(x, t), \tag{34}$$

where $D_t^\alpha U(x, t)$ is the Caputo fractional derivative of the function $U(x, t)$, R is the linear differential operator, N represents the general nonlinear differential operator, and $g(x, t)$ is the source term.

Applying the operator J_t^α on both sides of (34) and using result (9), we have

$$\begin{aligned} U(x, t) &= \sum_{k=0}^{m-1} \left(\frac{\partial^k U}{\partial t^k} \right)_{t=0} \frac{t^k}{k!} \\ &\quad + J_t^\alpha g(x, t) - J_t^\alpha [RU(x, t) + NU(x, t)]. \end{aligned} \tag{35}$$

Next, we decompose the unknown function $U(x, t)$ into sum of an infinite number of components given by the decomposition series

$$U = \sum_{n=0}^{\infty} U_n, \tag{36}$$

and the nonlinear term can be decomposed as

$$NU = \sum_{n=0}^{\infty} A_n, \tag{37}$$

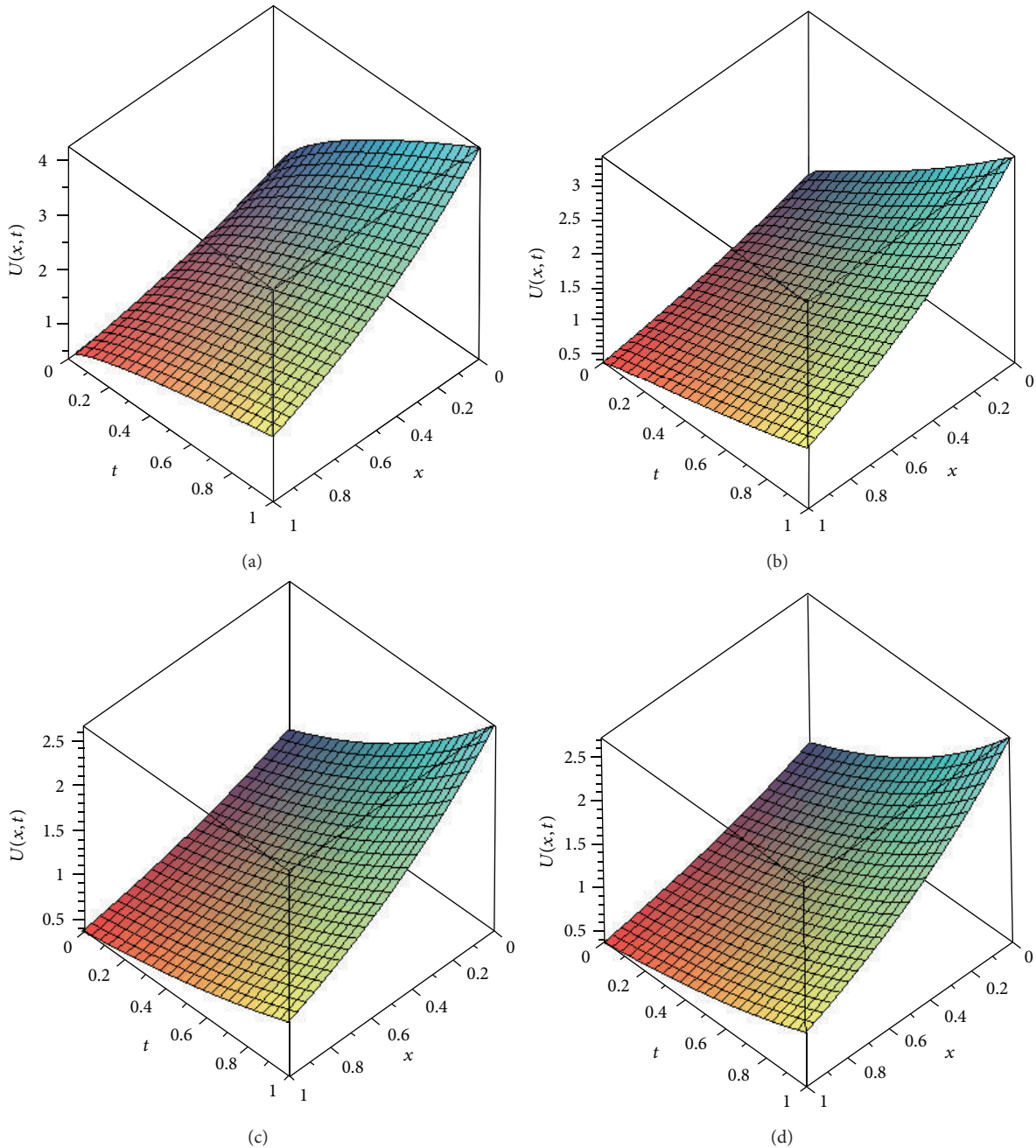


FIGURE 1: The behaviour of the $U(x, t)$ w.r.t. x and t are obtained when (a) $\alpha = 1/3$, (b) $\alpha = 2/3$, (c) $\alpha = 1$, and (d) exact solution.

where A_n are Adomian polynomials that are given by

$$A_n = \frac{1}{n!} \left[\frac{d^n}{d\lambda^n} N \left(\sum_{i=0}^n \lambda^i U_i \right) \right]_{\lambda=0}, \quad n = 0, 1, 2, \dots \quad (38)$$

The components U_0, U_1, U_2, \dots are determined recursively by substituting (36) and (37) into (34) leading to

$$\sum_{n=0}^{\infty} U_n = \sum_{k=0}^{m-1} \left(\frac{\partial^k U}{\partial t^k} \right)_{t=0} \frac{t^k}{k!} + J_t^\alpha g(x, t) - J_t^\alpha \left[R \left(\sum_{n=0}^{\infty} U_n \right) + \sum_{n=0}^{\infty} A_n \right]. \quad (39)$$

This can be written as

$$U_0 + U_1 + U_2 + \dots = \sum_{k=0}^{m-1} \left(\frac{\partial^k U}{\partial t^k} \right)_{t=0} \frac{t^k}{k!} + J_t^\alpha g(x, t) - J_t^\alpha [R(U_0 + U_1 + U_2 + \dots) + (A_0 + A_1 + A_2 + \dots)]. \quad (40)$$

Adomian method uses the formal recursive relations as

$$U_0 = \sum_{k=0}^{m-1} \left(\frac{\partial^k U}{\partial t^k} \right)_{t=0} \frac{t^k}{k!} + J_t^\alpha g(x, t), \quad (41)$$

$$U_{n+1} = -J_t^\alpha [R(U_n) + A_n], \quad n \geq 0.$$

TABLE 1: Comparison study between HPSTM, ADM, and the exact solution when $\alpha = 1$.

x	t	HPSTM	ADM	Exact solution
0	0.1	1.221333333	1.221333333	1.221402758
0.2	0.1	0.9999431595	0.9999431595	1.000000000
0.4	0.1	0.8186842160	0.8186842160	0.8187307531
0.6	0.1	0.6702819447	0.6702819447	0.6703200460
0.8	0.1	0.5487804413	0.5487804413	0.5488116361
1.0	0.1	0.4493037263	0.4493037263	0.4493289641

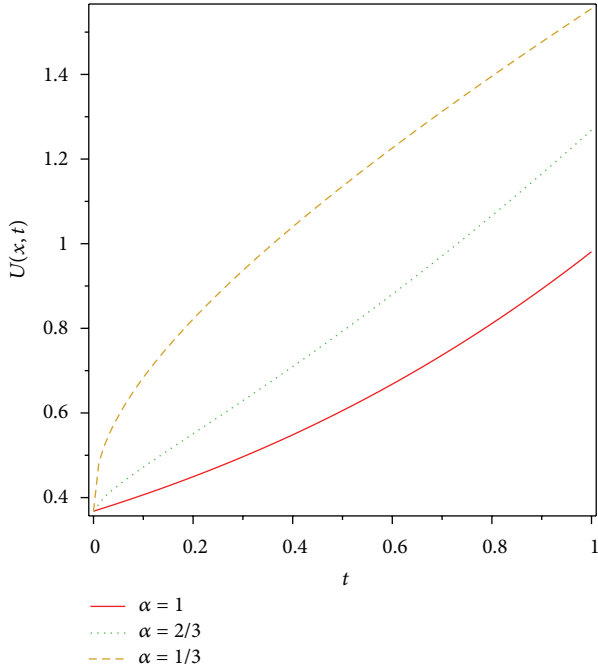


FIGURE 2: Plots of $U(x, t)$ versus t at $x = 1$ for different values of α .

5.2. *Solution of the Problem.* Consider the following nonlinear time-fractional gas dynamics equation:

$$D_t^\alpha U + \frac{1}{2} (U^2)_x - U(1 - U) = 0, \quad 0 < \alpha \leq 1, \quad (42)$$

with the initial condition

$$U(x, 0) = e^{-x}. \quad (43)$$

Applying the operator J_t^α on both sides of (42) and using result (9), we have

$$U = \sum_{k=0}^{1-1} \frac{t^k}{k!} [D_t^k U]_{t=0} - J_t^\alpha \left[\frac{1}{2} (U^2)_x - U + U^2 \right]. \quad (44)$$

This gives the following recursive relations using (41):

$$U_0 = \sum_{k=0}^0 \frac{t^k}{k!} [D_t^k U]_{t=0}, \quad (45)$$

$$U_{n+1} = -J_t^\alpha [A_n - U_n], \quad n = 0, 1, 2, \dots,$$

where

$$A_n = \frac{1}{n!} \left[\left(\frac{1}{2} \frac{\partial}{\partial x} + 1 \right) \frac{d^n}{d\lambda^n} \left(\sum_{i=0}^n \lambda^i U_i \right)^2 \right]_{\lambda=0}, \quad (46)$$

$$n = 0, 1, 2, \dots$$

which using the results (7), (5), and (43) gives

$$U_0(x, t) = e^{-x},$$

$$A_0 = 0,$$

$$U_1(x, t) = e^{-x} \frac{t^\alpha}{\Gamma(\alpha + 1)},$$

$$A_1 = 0,$$

$$U_2(x, t) = e^{-x} \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)}, \quad (47)$$

$$A_2 = 0,$$

$$U_3(x, t) = e^{-x} \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)},$$

\vdots

Therefore, the decomposition series solution is

$$U(x, t) = e^{-x} \left[1 + \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)} + \dots \right], \quad (48)$$

which is the same solution as obtained by using HPSTM.

From Table 1, it is observed that the values of the approximate solution at different grid points obtained by the HPSTM and ADM are close to the values of the exact solution with high accuracy at the third-term approximation. It can also be noted that the accuracy increases as the order of approximation increases.

The comparison between the third iteration solution of the HPSTM and the second iteration solution of the ADM is given in Figure 3.

It is observed that for $x = 1$ and $\alpha = 1$, there is a good agreement between the two methods.

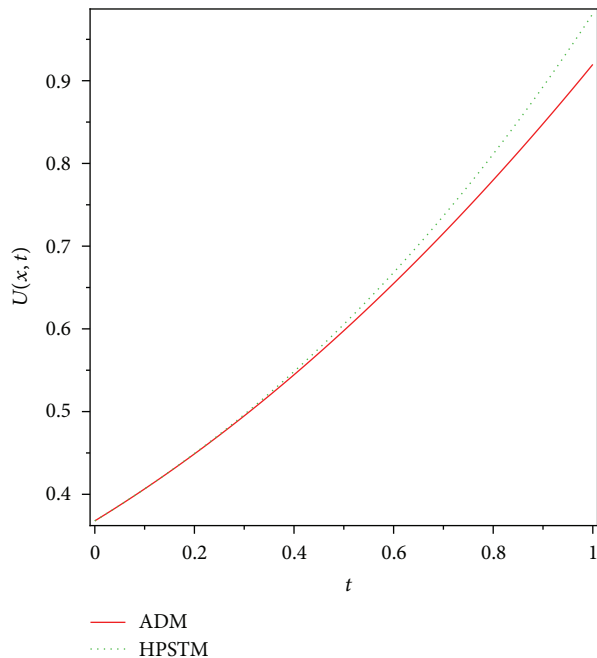


FIGURE 3: Comparison of the HPSTM and the ADM when $x = 1$ and $\alpha = 1$.

6. Conclusions

In this paper, the homotopy perturbation Sumudu transform method (HPSTM) and the Adomian decomposition method (ADM) are successfully applied for solving nonlinear time-fractional gas dynamics equation. The numerical solutions show that there is a good agreement between the two methods. Therefore, these two methods are very powerful and efficient techniques for solving different kinds of linear and nonlinear fractional differential equations arising in different fields of science and engineering. However, the HPSTM has an advantage over the ADM which is that it solves the nonlinear problems without using Adomian polynomials. In conclusion, the HPSTM and the ADM may be considered as a nice refinement in existing numerical techniques and might find the wide applications.

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