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Research Article

The Stability Criteria with Compound Matrices

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The bifurcation problem is one of the most important subjects in dynamical systems. Motivated by M. Li et al. who used compound matrices to judge the stability of matrices and the existence of Hopf bifurcations in continuous dynamical systems, we obtained some effective methods to judge the Schur stability of matrices on the base of the spectral property of compound matrices, which can be used to judge the asymptotical stability and the existence of Hopf bifurcations of discrete dynamical systems.

1. Introduction

The stability of matrices is intimately related to the stability of stationary solutions of various kinds in the theory and applications of dynamic systems [1]. Let A be an $n\times n$ matrix, and let $\sigma(A)$ be its spectrum. The stability modulus of A is $s(A) = \max\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\}$, and A is said to be stable if $s(A) < 0$. The asymptotic stability of the equilibrium point of a continuous dynamical system is closely related to the stability of the Jacobian matrix of the system at the equilibrium point; it is necessary that stability modulus is less than or equal to zero and it is sufficient that the Jacobian matrix is stable for the equilibrium point to be asymptotically stable. The stability of a matrix is related to the Routh-Hurwitz problem [2] on the number of zeros of a polynomial that have negative real parts. Li and Wang [3] have given a necessary and sufficient condition for the stability of an $n \times n$ matrix with real entries using the additive compound matrix. Accordingly, some criteria on the stability of the continuous dynamical systems are obtained.

The spectral radius of A is $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$, and A is said to be Schur stable if $\rho(A) < 1$. On the other hand, the asymptotic stability of a fixed point of a discrete dynamical system is closely related to the Schur stability of the linearized matrix of the system [4]; it is necessary that spectral radius of the linearized matrix is less than or equal to one and it is sufficient that the linearized matrix is Schur stable for the fixed point to be asymptotically stable. Corresponding to the Routh-Hurwitz criterion, the Jury criterion was derived by Jury in 1974. And Zheng et al. [5] obtained the extended Jury criterion on the coefficients of the characteristic polynomial of the matrix.

By the spectral property of the compound matrix, the compound matrix is an excellent tool to research the stability of matrices [6, 7]. In the present paper, using the additive compound matrix and the Lozinskii measure, some conclusions about the Schur stability of matrices and the stability of the discrete dynamical system are derived. Using the multiplicative compound matrix, we demonstrate how to judge the Schur stability of a matrix; at the same time we give the conditions that there exist no Hopf bifurcations in the discrete dynamical systems and the conditions for the fixed point to be asymptotically stable in the discrete dynamical systems.

We outline in the next section the preliminaries about the compound matrix and the Lozinskii measure and so on. In the Sections 3 and 4, the main results are demonstrated.

2. Preliminaries

Let $M_n(\mathbf{T})$ be the linear space of $n\times n$ matrices with entries in $\mathbf{T}$, where $\mathbf{T}$ is the field of real numbers $\mathbf{R}$ or complex numbers $\mathbf{C}$. An $A \in M_n(\mathbf{T})$ will be identified with the linear operator on $\mathbf{T}^n$ that it represents. Let $\wedge$ denote the exterior product in $\mathbf{T}^n$, and let k, $1 \le k \le n$, be an integer.

Definition 1 (see [6]). With respect to the canonical basis in the kth exterior product space $\wedge^k \mathbf{T}^n$, the kth additive compound matrix $A^{[k]}$ of A and the kth multiplicative compound matrix $A^{(k)}$ of A are, respectively, linear operators on $\wedge^k \mathbf{T}^n$ whose definitions on a decomposable element $u_1 \wedge \cdots \wedge u_k$ are

$$\begin{aligned} A^{[k]}(u_1 \wedge \cdots \wedge u_k) &= \sum_{j=1}^{k} u_1 \wedge \cdots \wedge A u_j \wedge \cdots \wedge u_k, \quad \forall u_1, \ldots, u_k \in \mathbf{T}^n, \\ A^{(k)}(u_1 \wedge \cdots \wedge u_k) &= A u_1 \wedge \cdots \wedge A u_k, \quad \forall u_1, \ldots, u_k \in \mathbf{T}^n. \end{aligned} \tag{1}$$

Definition over the whole $\wedge^k \mathbf{T}^n$ is done by linear extension. The entries of $A^{[k]}$ and $A^{(k)}$ are linear relations of those of A.

Let $A = (a_{ij})$, and let $N = C_n^k$. For any integer $i = 1, 2, \ldots, N$, let $i = (i_1, \ldots, i_k)$ be the ith member in the lexicographic ordering of integer k-tuples such that $1 \le i_1 < \cdots < i_k \le n$. Then the following conclusions are concluded.

Proposition 2 (see [6]). *We get the following:*

$$A^{(k)} = \left(a_{(i)}^{(j)}\right)_{N \times N}. \tag{2}$$

And let $Z = A^{[k]} = (z_{ij})$; then

$$z_{ij} = \begin{cases} a_{i_1 i_1} + \cdots + a_{i_k i_k}, & \text{if } (i) = (j), \\ (-1)^{r+s} a_{j_r i_s}, & \text{if exactly one entry } i_s \text{ of } (i) \\ & \text{does not occur in } (j) \text{ and } j_r \\ & \text{does not occur in } (i), \\ 0, & \text{if } (i) \text{ differs from } (j) \text{ in two} \\ & \text{or more entries.} \end{cases} \tag{3}$$

As special cases, we have $A^{[1]} = A$ and $A^{[n]} = \mathrm{tr}(A)$. The spectral property is as follows.

Proposition 3 (see [6]). *Let $\sigma(A) = \{\lambda_i : i = 1, \ldots, n\}$: then*

(1) $\sigma(A^{[k]}) = \{\lambda_{i_1} + \cdots + \lambda_{i_k} : 1 \le i_1 < \cdots < i_k \le n\}$,

(2) $\sigma(A^{(k)}) = \{\lambda_{i_1} \cdots \lambda_{i_k} : 1 \le i_1 < \cdots < i_k \le n\}$.

Pertinent to the purpose of the present paper is the norm property of the compound matrix.

The singular values $\sigma_1, \ldots, \sigma_n$ of an $n \times n$ matrix B are the eigenvalues of the symmetric matrix $\sqrt{B^* B}$, and let

$$\sigma_1 \ge \cdots \ge \sigma_n \ge 0. \tag{4}$$

Proposition 4 (see [7] (minimax-rule)). *The following is established:*

$$\begin{aligned} \sigma_j &= \max_{\substack{V \subset R^m \\ \dim V = j}} \min_{\substack{u \in V \\ \|u\| = 1}} \|Bu\| \\ &= \min_{\substack{W \subset R^m \\ \dim W = m - j + 1}} \max_{\substack{u \in W \\ \|u\| = 1}} \|Bu\|, \end{aligned} \tag{5}$$

$j = 1, \ldots, n$, where $\|\cdot\|$ are the matrix norm induced from the Euclidean norm of $\mathbf{R}^n$.

Corollary 5 (see [7]). *The following is established: $\sigma_1 = \|B\|$.*

Definition 6 (see [3]). Let $|\cdot|$ denote a vector norm in $\mathbf{T}^n$ and the operator norm it induces in $M_n(\mathbf{T})$. For $A \in M_n(\mathbf{T})$, the Lozinskii measure μ on $M_n(\mathbf{T})$ with respect to $|\cdot|$ is defined by

$$\mu(A) = \lim_{h \to 0^+} \frac{|I + hA| - 1}{h}. \tag{6}$$

The following are the stability criteria on matrices given by Li and Wang using the compound matrix and Lozinskii measure.

Proposition 7 (see [3]). *We get the following: $s(A) < 0 \Leftrightarrow s(A^{[2]}) < 0$ and $(-1)^n \det(A) > 0$.*

Proposition 8 (see [3]). *If $(-1)^n \det(A) > 0$, then A is stable $\Leftrightarrow$ there exists a Lozinskii measure μ on $M_N(\mathbf{R})$ such that $\mu(A^{[2]}) < 0$, $N = C_n^2$.*

Definition 9 (see [3]). Let $\alpha \mapsto A(\alpha) \in M_n(\mathbf{R})$ be a continuous function on (a, b). $\alpha_0 \in (a, b)$ is called a Hopf bifurcation point of $A(\alpha)$, if, when $\alpha < \alpha_0$, $A(\alpha)$ is stable. When $\alpha > \alpha_0$, there exist a pair of plural eigenvalues $\mathrm{Re}\,\lambda(\alpha) \pm i\,\mathrm{Im}\,\lambda(\alpha)$ such that $\mathrm{Re}\,\lambda(\alpha) > 0$, and the other eigenvalues have nonzero real parts; here $\mathrm{Im}\,\lambda(\alpha) \ne 0$.

Proposition 10 (see [3]). *Consider the system*

$$\begin{aligned} x' = f(x, \alpha), \quad \alpha \in (a, b), \ x \in D, \\ f \in C^1(\mathbf{R}^n \times (a, b), \mathbf{R}^n), \end{aligned} \tag{7}$$

where f, $\partial f / \partial x \in C(\mathbf{R}^n \times (a, b), \mathbf{R}^n)$ and D is an open set in $\mathbf{R}^n$. If, for all $\alpha \in (a, b)$, $f(x, \alpha) = 0$ have solutions $(\overline{x}(\alpha), \alpha)$, then $\overline{x}(\alpha)$ is called the equilibrium of the system (7). Let $J(\alpha) = (\partial f / \partial x)(\overline{x}(\alpha), \alpha)$, and if $\alpha = \alpha_0$ is the Hopf bifurcation point of $J(\alpha)$, then the Hopf bifurcation of system (7) occurs when $\alpha = \alpha_0$.

By the theorem, Li and Wang have given the methods to judge the existence of the Hopf bifurcation.

Proposition 11 (see [3]). *If there exists some Lozinskii measure μ such that*

$$\mu\left(J^{[2]}(\alpha)\right) \le 0, \quad \alpha \in (a, b), \tag{8}$$

then the system (7) *has no Hopf bifurcations from* $\overline{x}(\alpha)$ *in* (a, b). *Moreover, if*

$$(-1)^n \det (J(\alpha)) > 0, \quad \alpha \in (a, b), \tag{9}$$

then $\overline{x}(\alpha)$ *is an asymptotically stable equilibrium of the system* (7) *for* $\alpha \in (a, b)$.

3. The Schur Stability Criteria of Matrices Using the Additive Compound Matrix

Consider the $C^r (r \geq 1)$ map:

$$x \longmapsto g(x), \quad x \in \mathbf{R}^n. \tag{10}$$

If (10) has a fixed point $x = \overline{x}$, that is, $\overline{x} = g(\overline{x})$, then the linear map corresponding to (10) is

$$y \longmapsto Ay, \quad y \in \mathbf{R}^n, \tag{11}$$

where $A \equiv Dg(\overline{x})$, the Jacobian matrix of g at $\overline{x}$.

Similar to the continuous dynamical system, the stability of a discrete dynamical system can be judged by the Schur stability of the matrix.

Lemma 12 (see [4]). *If the matrix A of the system* (11) *is Schur stable, then the fixed point $\overline{x}$ of the system* (11) *is asymptotically stable.*

It is generally well known that it is a sufficient condition for the Hopf bifurcation to occur in the system (10) that the matrix A has a pair of conjugated eigenvalues whose spectral radius is equal to 1 (not equal to ± 1), and the spectral radius of the rest of the eigenvalues is less than 1. In addition, it is a sufficient condition for the k-codimension Hopf bifurcation to occur in the system (10) that the matrix A has k pairs of conjugated of the eigenvalues whose spectral radius is equal to 1 (not equal to ± 1), and the spectral radius of the rest eigenvalues is less than 1.

Thus, it is very necessary to investigate the following three questions:

(1) the conditions for the spectral radius of all the eigenvalues of the matrix A to be less than 1;

(2) the conditions for the spectral radius of a pair of the eigenvalues of the matrix A to be equal to 1 (not equal to ± 1) and the spectral radius of the rest of the eigenvalues of the matrix A to be less than 1;

(3) the conditions for the spectral radius of k pairs of the eigenvalues of the matrix A to be equal to 1 (not equal to ± 1) and the spectral radius of the rest of the eigenvalues of the matrix A to be less than 1.

For the characteristic polynomial of the matrix A:

$$f_n(z) = \det (zI_n - A) = z^n + a_1 z^{n-1} + \cdots + a_n, \tag{12}$$

we consider the operation

$$z = \frac{\omega + 1}{\omega - 1}, \tag{13}$$

and (12) is changed into

$$f_n = \omega^n + b_1 \omega^{n-1} + \cdots + b_n. \tag{14}$$

By operation (13), the inner of the unit circle and the unit circle are mapped to left plane and the imaginary axis of the ω complex plane. And the questions about the eigenvalues of the matrices of the discrete dynamical systems are transformed into the questions about the eigenvalues of the matrices of the continuous dynamical systems. Therefore the following conclusions are obtained.

Theorem 13. *If* $(-1)^n \det(I + (2/\det(A-I))(A-I)^*) > 0$, *then A is Schur stable $\Leftrightarrow$ there exists some Lozinskii measure μ such that*

$$\mu\left(I + \frac{2}{\det (A - I)} (A - I)^*\right)^{[2]} < 0. \tag{15}$$

Proof. The eigenvalues of $I + (2/\det(A-I))(A-I)^*$ are

$$1 + \frac{2}{\det (A - I)} \cdot \frac{\det (A - I)}{\lambda - 1} = 1 + \frac{2}{\lambda - 1} = \frac{\lambda + 1}{\lambda - 1}, \tag{16}$$

where λ is the eigenvalue of the matrix A.

Therefore by (13),

$$\rho(A) < 1 \Longleftrightarrow s\left(I + \frac{2}{\det (A - I)} (A - I)^*\right) < 0. \tag{17}$$

By Proposition 8, the conclusions are obtained. □

Theorem 14. *The spectral radius of a pair of conjugated eigenvalues of the matrix A is equal to 1 (not equal to ± 1) and the spectral radius of the rest eigenvalues is less than 1 $\Leftrightarrow$ there exists a Lozinskii measure μ such that* $\mu(I + (2/\det(A-I))(A-I)^*)^{[3]} < 0$, $(-1)^n \det(I + (2/\det(A-I))(A-I)^*) > 0$, $s(I + (2/\det(A-I))(A-I)^*)^{[2]} = 0$.

Proof. $\Rightarrow$ It is obvious.

$\Leftarrow$ Let

$$I + \frac{2}{\det (A - I)} (A - I)^* = \widetilde{A}. \tag{18}$$

By $s(\widetilde{A})^{[2]} = 0$, $\widetilde{A}$ has at most one eigenvalue that its real part is more than zero, and the rest of the eigenvalues have real parts less than zero. Without loss of generality, we assume that $\lambda_n > 0$ and λ_n is a real number. But

$$\operatorname{sign}\left(\det \widetilde{A}\right) = (-1)^{n-1}. \tag{19}$$

It contradicts

$$(-1)^n \det \widetilde{A} > 0. \tag{20}$$

And by

$$\mu\left(\widetilde{A}\right)^{[3]} < 0 \Longleftrightarrow s\left(\widetilde{A}\right)^{[3]} < 0, \tag{21}$$

it can be known that $\widetilde{A}$ has only one pair of eigenvalues whose real parts are zero. Thus $\widetilde{A}$ has only one pair of eigenvalues whose real parts are zero, and the real parts of the rest of the eigenvalues are less than zero. And the conclusions are established. □

Consider the $C^r (r \geq 1)$ map:

$$x \longmapsto g(x,\alpha), \quad x \in \mathbf{R}^n, \ \alpha \in (a,b). \tag{22}$$

If (22) has the fixed point $x = \overline{x}(\alpha)$, that is, $\overline{x}(\alpha) = g(\overline{x},\alpha)$, then the linear map corresponding to (22) is

$$y \longmapsto A(\alpha)\, y, \quad y \in \mathbf{R}^n, \ \alpha \in (a,b), \tag{23}$$

where $A \equiv Dg(\overline{x}(\alpha),\alpha)$, the Jacobian matrix of g at $\overline{x}(\alpha)$.

Definition 15. $\alpha_0 \in (a,b)$ is called the Hopf bifurcation point if, when $\alpha < \alpha_0$, $A(\alpha)$ is Schur stable; when $\alpha > \alpha_0$, there exist a pair of complex eigenvalues whose spectral radius is more than 1, and the rest of the eigenvalues are not 1 or -1.

Therefore the following theorem is obtained.

Theorem 16. *If there exists some Lozinskii measure μ such that $\mu(I + (2/det(J(\alpha) - I))(J(\alpha) - I)^*)^{[2]} \leq 0$, $\alpha \in (a,b)$, then the system* (22) *has no Hopf bifurcation points about its fixed point $\overline{x}(\alpha)$. And if $(-1)^n \det(I + (2/\det(J(\alpha) - I))(J(\alpha) - I)^*) > 0$, $\alpha \in (a,b)$, then $\overline{x}(\alpha)$ is an asymptotically stable fixed point of the system* (22)*, where $g, \partial g/\partial x \in C(\mathbf{R}^n \times (a,b), \mathbf{R}^n)$, $J(\alpha) = (\partial g/\partial x)(\overline{x}(\alpha),\alpha)$, and $\overline{x}(\alpha)$ is the fixed point of the system* (22).

4. The Schur Stability Criteria of Matrices Using the Multiplicative Compound Matrix

To research the Schur stability of the matrix and the stability of the discrete dynamical system, we find that the measure whose function is similar to that of the Lozinskii measure in the research of continuous dynamical systems is the norm of the matrix.

Let $M_n(\mathbf{C})$ denote all the $n \times n$ matrices in the complex field.

Lemma 17. *Let $A \in M_n(\mathbf{C})$; then $\rho(A) = \inf\{|A| : |\cdot| \text{ is the matrix norm in } M_n(\mathbf{C})\}$.*

Thus we have the following corollary.

Corollary 18. *Let $A \in M_n(\mathbf{C})$; then $\rho(A) < 1 \Leftrightarrow$ there exists some matrix norm in $M_n(\mathbf{C})$ such that $|A| < 1$.*

By the spectral property of the multiplicative compound matrix, the Schur stability of the matrix can be judged by the multiplicative compound matrix.

Theorem 19. *Let $A \in M_n(\mathbf{R})$; then $\rho(A) < 1 \Leftrightarrow \rho(A^{(2)}) < 1$ and $\det(I - A^2) > 0$.*

Proof. $\Rightarrow$ Because $\rho(A) < 1$, by Proposition 3, $\rho(A^{(2)}) < 1$. It might be as well that the n roots of $A^{(2)}$ are

$$\nu_1, \ldots, \nu_s, \nu_{s+1} = \overline{\nu}_1, \ldots, \nu_{2s} = \overline{\nu}_s, \nu_{2s+1}, \ldots, \nu_n, \tag{24}$$

where $\nu_{2s+1}, \ldots, \nu_n$ are real numbers. Then by $\rho(A) < 1$, we have

$$\begin{gathered} (1 - \nu_i)(1 - \overline{\nu}_i) > 0, \quad i = 1, \ldots, s, \\ 1 - \nu_j > 0, \quad j = 2s+1, \ldots, n. \end{gathered} \tag{25}$$

Thus

$$\det\left(I - A^2\right) = \prod_{i=1}^{s} (1 - \nu_i)(1 - \overline{\nu}_i) \prod_{j=2s+1}^{n} \left(1 - \nu_j\right) > 0. \tag{26}$$

$\Leftarrow$ By $\rho(A^{(2)}) < 1$ and Proposition 3, A has at most one eigenvalue whose spectral radius is more than 1. If $\rho(A) \geq 1$, by $A \in M_n(\mathbf{R})$, it can be known that A has just one real eigenvalue λ_0 whose absolute value is more than or equal to 1. Let

$$f(x) = |xI - A| = x^n - \operatorname{tr}(A)\, x^{n-1} + \cdots + (-1)^n |A| \tag{27}$$

be the characteristic polynomial of A.

By

$$\lim_{x \to +\infty} f(x) = +\infty, \qquad \lim_{x \to -\infty} (-1)^n f(x) = +\infty, \tag{28}$$

it can be known that if $\lambda_0 \geq 1$, then

$$\det(I - A) \leq 0, \qquad (-1)^n \det(-I - A) > 0. \tag{29}$$

Thus

$$\det\left(I - A^2\right) < 0, \tag{30}$$

a contradiction.

If $\lambda_0 \leq -1$, then

$$(-1)^n \det(-I - A) \leq 0, \qquad \det(I - A) > 0. \tag{31}$$

Thus

$$\det\left(I - A^2\right) < 0, \tag{32}$$

a contradiction.

Therefore the conclusions are established. □

By the previous conclusions and corollaries, we have the following.

Corollary 20. *Let $A \in M_n(\mathbf{R})$; then $\rho(A) < 1 \Leftrightarrow$ there exists some matrix norm in $M_N(\mathbf{R})$ such that $|A^{(2)}| < 1$ and $\det(I - A^2) > 0$, where $N = C_n^2$.*

By the property of the multiplicative compound matrix given in Section 2, the multiplicative compound matrix can be computed by the singular value of a matrix, and thus we have the following.

Corollary 21. *Let $A \in M_n(\mathbf{R})$; then $\rho(A) < 1 \Leftrightarrow \sigma_1\sigma_2 < 1$ and $\det(I - A^2) > 0$, $\sigma_1\sigma_2$ are the singular values of A.*

Corresponding to the stability of the fixed point and the Hopf bifurcation of the discrete dynamical systems, we have the following theorem.

Theorem 22. *If there exists some matrix norm in $M_N(\mathbf{R})$ such that $|J^{(2)}(\alpha)| \leq 1$, $\alpha \in (a,b)$, then the system* (22) *has no Hopf bifurcations about its fixed point $\overline{x}(\alpha)$. And if $\det(I - A^2) > 0$, $\alpha \in (a,b)$, then $\overline{x}(\alpha)$ is an asymptotically stable fixed point of the system* (22)*, where $g, \partial g/\partial x \in C(\mathbf{R}^n \times (a,b), \mathbf{R}^n)$, $J(\alpha) = (\partial g/\partial x)(\overline{x}(\alpha),\alpha)$, and $\overline{x}(\alpha)$ is the fixed point of the system* (22).

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