Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 925192, 6 pages
http://dx.doi.org/10.1155/2013/925192

Research Article

Solving a Class of Singular Fifth-Order Boundary Value Problems Using Reproducing Kernel Hilbert Space Method

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Received 4 May 2013; Accepted 13 June 2013

Academic Editor: Rodrigo Lopez Pouso

We use the reproducing kernel Hilbert space method to solve the fifth-order boundary value problems. The exact solution to the fifth-order boundary value problems is obtained in reproducing kernel space. The approximate solution is given by using an iterative method and the finite section method. The present method reveals to be more effective and convenient compared with the other methods.

1. Introduction

The reproducing kernel Hilbert space method has been shown [1–7] to solve effectively, easily, and accurately a large class of linear and nonlinear, ordinary, partial differential equations. However, in [1–7], it cannot be used directly boundary value problems with mixed boundary conditions, since it is very difficult to obtain a reproducing kernel function satisfying mixed nonlinear boundary conditions. The aim of this work is to fill this gap. In [8], we give a new reproducing kernel Hilbert space for solving singular linear fourth-order boundary value problems with mixed boundary conditions. In this paper, we use the new reproducing kernel Hilbert function space method to solve the nonlinear fifth-order boundary value problems.

Singular fifth-order boundary value problems arise in the fields of gas dynamics, Newtonian fluid mechanics, fluid mechanics, fluid dynamics, elasticity, reaction-diffusion processes, chemical kinetics, and other branches of applied mathematics.

Let us consider the following class of singular fifth-order mixed boundary value problems:

$$\begin{gathered} u^{(5)}(x) + \frac{p_1(x)}{x^{\alpha_1}(1-x)^{\beta_1}} u^{(4)}(x) + \cdots + \frac{p_4(x)}{x^{\alpha_4}(1-x)^{\beta_4}} u'(x) \\ + \frac{p_5(x)}{x^{\alpha_5}(1-x)^{\beta_5}} u(x) = f(x), \quad x \in (0,1), \\ B_i u = r_i, \quad (i = 1, 2, \ldots, 5), \end{gathered} \tag{1}$$

where $p_j(x), f(x) \in L^2[0,1]$ $(j = 1, \ldots, 5)$ are known functions. $B_i u$ $(i = 1, 2, \ldots, 5)$ are linear conditions. We assume that (1) has a unique solution which belongs to $W_2^6[0,1]$, where $W_2^6[0,1]$ is a reproducing kernel space.

Remark 1. If $B_i u = u^{(i)}(0)$ $(i = 1, \ldots, m)$, then (1) is an initial value problem. If $B_i u = u(x_i)$ $(i = 1, 2, \ldots, m)$, then (1) is a multipoint problem and so on. That is, problem (1) has a rather general form.

Let $\alpha = \max_{1\le i\le 5}\{\alpha_i\}$ and $\beta = \max_{1\le i\le 5}\{\beta_i\}$, $F(x) = x^{\alpha}(1-x)^{\beta} f(x)$.

Consider

$$\begin{aligned} (Lu)(x) = {} & x^{\alpha}(1-x)^{\beta} u^{(5)}(x) \\ & + x^{\alpha-\alpha_1}(1-x)^{\beta-\beta_1} p_1(x) u^{(4)}(x) \\ & + \cdots + x^{\alpha-\alpha_4}(1-x)^{\beta-\beta_4} p_4(x) u'(x) \\ & + x^{\alpha-\alpha_5}(1-x)^{\beta-\beta_5} p_5(x) u(x). \end{aligned} \tag{2}$$

It is easy to prove that $L : W_2^6[0,1] \to L^2[0,1]$ is a bounded linear operator. On the other hand, we suppose that the linear conditions can also always be homogenized; after homogenization of these conditions, we put these conditions into the reproducing kernel space $W_2^6[0,1]$ constructed in the following section. Equation (1) can be transformed into the following form in $W_2^6[0,1]$:

$$(Lu)(x) = F(x). \tag{3}$$

2. Reproducing Kernel Hilbert Space

Definition 2. Let H be a real Hilbert space of functions $f : \Omega \to R$. A function $K : \Omega \times \Omega \to R$ is called reproducing kernel for H if

(i) $K(x,\cdot) \in H$ for all $x \in \Omega$,

(ii) $f(x) = \langle f, K(\cdot,x)\rangle_H$ for all $f \in H$ and all $x \in \Omega$.

Definition 3. A real Hilbert space H of functions on a set Ω is called a reproducing kernel Hilbert space if there exists a reproducing kernel K of H.

One defines that the inner product space $\overline{W}_2^{m+1}[0,1] = \{u \mid u, u', \ldots, u^{(m)}$ are absolutely continuous function, $u^{(m+1)} \in L^2[0,1]\}$.

The inner product in $\overline{W}_2^{m+1}[0,1]$ is given by

$$\langle u(x), v(x)\rangle = \sum_{i=0}^{m} u^{(i)}(0)\, v^{(i)}(0) + \int_0^1 u^{(m+1)}(t)\, v^{(m+1)}(t)\, dt. \tag{4}$$

Theorem 4 (see [8]). *The space $\overline{W}_2^{m+1}[0,1]$ is a reproducing kernel space, and its reproducing kernel is*

$$R^{\{m+1\}}(x,y) = \begin{cases} \displaystyle\sum_{i=0}^{m} \frac{1}{(i!)^2} x^i y^i + \frac{1}{(m!)^2}\int_0^x (x-t)^m (y-t)^m dt, & x<y, \\ \displaystyle\sum_{i=0}^{m} \frac{1}{(i!)^2} x^i y^i + \frac{1}{(m!)^2}\int_0^y (x-t)^m (y-t)^m dt, & y<x. \end{cases} \tag{5}$$

For studying the solution of (1) in the homogenized form, we give space (6) as follows:

$$W_2^6[0,1] = \left\{u \mid u \in \overline{W}_2^6[0,1],\ B_i u = 0,\ i = 1,2,\ldots,5\right\}. \tag{6}$$

$W_2^6[0,1]$ is equipped with the same inner product $\overline{W}_2^6[0,1]$. In the following, we construct a reproducing kernel for the space $W_2^6[0,1]$, and we give Lemmas 5 and 6.

Lemma 5. *Let $A : H[a,b] \to L^2[a,b]$ be a bounded linear operator; function $R_x(y)$ is the reproducing kernel of space $H[a,b]$. Let $g_z(x) = (A_s R_x(s))(z)$; then $\|g_z(x)\|^2 = (A_s(A_t R_s(t))(z))(z)$, where $H[a,b]$ denotes any reproducing kernel space of functions over $[a,b]$, the symbol A_s indicates that the operator A applies to functions of the variable s, and the symbol $(A_s R_x(s))(z)$ indicates that the operator A applies to function $R_x(s)$ of the variable s and $s = z$.*

Proof. Consider

$$\begin{aligned} \|g_z(x)\|^2 &= \langle (A_s R_x(s))(z), (A_t R_x(t))(z)\rangle \\ &= (A_s(A_t\langle R_x(s), R_x(t)\rangle)(z))(z) \\ &= (A_s(A_t R_s(t))(z))(z). \end{aligned} \tag{7}$$

□

Lemma 6. *If A, $g_z(x)$, and $R_x(y)$ are defined as in Lemma 5, let $K_x(y) = R_x(y) - g_z(x)g_z(y)/\|g_z(x)\|^2$; consider the space $H_1 = \{u(y) \mid u(y) \in H[a,b]$, and $(A_y u(y))(z) = 0\}$, then, $K_x(y)$ is the reproducing kernel of space H_1.*

Proof. For any $u(y) \in H_1$, next, we will prove $\langle u(y), K_x(y)\rangle = u(x)$.

Consider

$$\begin{aligned} \langle u(y), K_x(y)\rangle &= \left\langle u(y), R_x(y) - \frac{g_z(x)g_z(y)}{\|g_z(x)\|^2}\right\rangle \\ &= \langle u(y), R_x(y)\rangle - \left\langle u(y), \frac{g_z(x)g_z(y)}{\|g_z(x)\|^2}\right\rangle \\ &= u(x) - g_z(x)\frac{\langle u(y), (A_s R_y(s))(z)\rangle}{\|g_z(x)\|^2} \\ &= u(x) - g_z(x)\frac{(A_s\langle u(y), R_y(s)\rangle)(z)}{\|g_z(x)\|^2} \\ &= u(x) - \frac{g_z(x)(A_s u(s))(z)}{\|g_z(x)\|^2} = u(x). \end{aligned} \tag{8}$$

□

Let $h_1(x) = B_{1y}R^{\{6\}}(x,y)$, $h_2(x) = B_{2y}(R^{\{6\}}(x,y) - h_1(x)h_1(y)/\|h_1(x)\|^2)$, $h_3(x) = B_{3y}(R^{\{6\}}(x,y) - h_1(x)h_1(y)/\|h_1(x)\|^2 - h_2(x)h_2(y)/\|h_2(x)\|^2)$, $h_4(x) = B_{4y}(R^{\{6\}}(x,y) - h_1(x)h_1(y)/\|h_1(x)\|^2 - h_2(x)h_2(y)/\|h_2(x)\|^2 - h_3(x)h_3(y)/\|h_3(x)\|^2)$, and $h_5(x) = B_{5y}(R^{\{6\}}(x,y) - h_1(x)h_1(y)/\|h_1(x)\|^2 - h_2(x)h_2(y)/\|h_2(x)\|^2 - h_3(x)h_3(y)/\|h_3(x)\|^2 - h_4(x)h_4(y)/\|h_4(x)\|^2)$, where the symbol B_{iy} $(i = 1,2,3,4,5)$ indicates that the operator B_i $(i = 1,2,3,4,5)$ applies to functions of the variable y. Using Lemma 6, we get Theorem 7.

Table 1: The numerical results of Example 12.

x	$u_T(x)$	$u_{100}(x)$	$\lvert u_{20}(x)-u_T(x)\rvert$	$\lvert u_{40}(x)-u_T(x)\rvert$	$\lvert u_{100}(x)-u_T(x)\rvert$	$\lvert u'_{100}(x)-u'_T(x)\rvert$
0	0	0	0	0	0	0
0.08	−0.00150556	−0.00149854	3.11533×10^{-4}	2.12134×10^{-4}	7.01613×10^{-6}	4.26411×10^{-5}
0.16	−0.00300419	−0.00299716	5.86206×10^{-4}	4.03105×10^{-4}	7.02601×10^{-6}	3.96519×10^{-5}
0.24	−0.000739712	−0.000738437	1.36893×10^{-4}	1.08655×10^{-4}	1.27471×10^{-6}	1.06360×10^{-4}
0.32	0.00926282	0.00925363	1.38316×10^{-3}	9.02756×10^{-4}	9.18729×10^{-6}	1.59425×10^{-4}
0.4	0.0316161	0.0315926	4.00907×10^{-3}	2.65016×10^{-3}	2.34247×10^{-5}	1.93257×10^{-4}
0.48	0.0717954	0.0717548	7.49348×10^{-3}	4.96714×10^{-3}	4.06485×10^{-5}	2.28313×10^{-4}
0.56	0.136337	0.136276	1.13395×10^{-3}	7.52215×10^{-3}	6.03524×10^{-5}	2.58664×10^{-4}
0.64	0.23304	0.232959	1.48395×10^{-3}	9.84568×10^{-3}	8.01808×10^{-5}	2.77257×10^{-4}
0.72	0.371189	0.371084	1.71134×10^{-3}	1.13541×10^{-3}	1.04886×10^{-4}	2.93384×10^{-4}
0.8	0.561801	0.561673	1.71481×10^{-3}	1.13757×10^{-3}	1.27883×10^{-4}	3.40704×10^{-4}
0.88	0.817902	0.817739	1.38345×10^{-3}	9.1769×10^{-3}	1.63033×10^{-4}	6.89140×10^{-4}
0.96	1.15484	1.15458	5.99608×10^{-3}	3.97735×10^{-3}	2.60877×10^{-4}	1.51714×10^{-3}

Theorem 7. *The space $W_2^6[0,1]$ is a reproducing kernel space, and its reproducing kernel is*

$$K(x,y)=R^{\{6\}}(x,y)-\frac{h_1(x)h_1(y)}{\|h_1(x)\|^2}-\frac{h_2(x)h_2(y)}{\|h_2(x)\|^2}-\frac{h_3(x)h_3(y)}{\|h_3(x)\|^2}-\frac{h_4(x)h_4(y)}{\|h_4(x)\|^2}-\frac{h_5(x)h_5(y)}{\|h_5(x)\|^2}. \tag{9}$$

3. Analytical Solution

Let $\psi_i(x)=(L_yK(x,y))(x_i)$, $i=1,2,\ldots$. Via Gram-Schmidt orthonormalization for $\{\psi_i(x)\}_{i=1}^{\infty}$, we get

$$\overline{\psi}_i(x)=\sum_{k=1}^{i}\beta_{ik}\psi_k(x), \tag{10}$$

where the β_{ik} are the coefficients resulting from Gram-Schmid orthonormalization.

Lemma 8. *If $\{x_i\}_{i=1}^{\infty}$ are distinct points dense in $[0,1]$ and L^{-1} is existent, then $\{\psi_i(x)\}_{i=1}^{\infty}$ is a complete function system in $W_2^{m+1}[0,1]$.*

Proof. For each fixed $u(x)\in W_2^{m+1}[0,1]$, if $\langle u(x),\psi_i(x)\rangle=0$, then

$$\langle u(x),\psi_i(x)\rangle = \left(L_y\langle u(x),K(x,y)\rangle\right)(x_i)=\left(L_yu(y)\right)(x_i)=0. \tag{11}$$

Taking into account the density of $\{x_i\}_{i=1}^{\infty}$, it results in $(L_yu(y))(x)=0$. It follows that $u(x)\equiv 0$ from the existence of L^{-1}. □

Theorem 9. *If $\{x_i\}_{i=1}^{\infty}$ are distinct points dense in $[0,1]$ and L^{-1} is existent, then*

$$u(x)=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}F(x_k)\overline{\psi}_i(x) \tag{12}$$

is an analytical solution of (3).

Proof. $u(x)$ can be expanded to Fourier series in terms of normal orthogonal basis $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ in $W_2^{m+1}[0,1]$ as follows:

$$\begin{aligned}u(x)&=\sum_{i=1}^{\infty}\langle u(x),\overline{\psi}_i(x)\rangle\overline{\psi}_i(x)\\&=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle u(x),\psi_k(x)\rangle\overline{\psi}_i(x)\\&=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle u(x),(L_sK_x(s))(x_k)\rangle\overline{\psi}_i(x)\\&=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left(L_s\langle u(x),K_x(s)\rangle\right)(x_k)\overline{\psi}_i(x)\\&=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left(L_su(s)\right)(x_k)\overline{\psi}_i(x)\\&=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}F(x_k)\overline{\psi}_i(x).\end{aligned} \tag{13}$$

□

4. Numerical Solution

We define an approximate solution $u_n(x)$ by

$$u_n(x)=\sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik}F(x_k)\overline{\psi}_i(x). \tag{14}$$

Table 2: Comparison of the absolute error of Example 13.

x	Solution			Absolute error		
	$u_T(t,x)$	Reference [9]	$u_{100}(x)$	$\lvert u_{50}(x)-u_T(x)\rvert$	Reference [9]	$\lvert u_{100}(x)-u_T(x)\rvert$
0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.1249	0.0000752	0.0000754	0.0000754	8.11621×10^{-7}	2×10^{-7}	2.74983×10^{-7}
0.2431	0.0013039	0.0013043	0.0013037	3.70493×10^{-6}	4×10^{-7}	1.13843×10^{-7}
0.3806	0.0080242	0.0080249	0.0080244	7.8819×10^{-6}	7×10^{-7}	2.32339×10^{-7}
0.4195	0.0116531	0.0116538	0.0116533	8.85202×10^{-6}	7×10^{-7}	2.59035×10^{-7}
0.5	0.0220970	0.0220978	0.0220972	1.01895×10^{-5}	8×10^{-7}	2.94629×10^{-7}
0.6923	0.0588207	0.0588201	0.0588209	8.5994×10^{-6}	6×10^{-7}	2.44068×10^{-7}
0.7854	0.0723723	0.0723726	0.0723724	5.68292×10^{-6}	4×10^{-7}	1.60302×10^{-7}
0.8917	0.0646361	0.0646363	0.0646366	1.96562×10^{-6}	2×10^{-7}	5.51306×10^{-7}
1.0	0.0	0.0	0.0	0.0	0.0	0.0

Table 3: The numerical results of Example 13.

x	$u'_T(x)$	$u'_{100}(x)$	$\lvert u'_{100}(x)-u'_T(x)\rvert$	$u''_T(x)$	$u''_{100}(x)$	$\lvert u''_{100}(x)-u''_T(x)\rvert$
0.0	0	0	0	0	0	0
0.1	0.0012491	0.00125306	3.95623×10^{-6}	0.0419792	0.0420291	4.98366×10^{-5}
0.2	0.0121642	0.0121721	7.93764×10^{-6}	0.193196	0.193223	2.63540×10^{-5}
0.3	0.0421473	0.0421562	8.95432×10^{-6}	0.410381	0.410375	6.05312×10^{-6}
0.4	0.0930975	0.0931043	6.87917×10^{-6}	0.591978	0.591944	3.40825×10^{-5}
0.5	0.15468	0.154682	2.49761×10^{-6}	0.596621	0.59657	5.13938×10^{-5}
0.6	0.200775	0.200773	2.91343×10^{-6}	0.250969	0.250915	5.41587×10^{-5}
0.7	0.186533	0.186526	7.76184×10^{-6}	−0.645692	−0.645732	3.97775×10^{-5}
0.8	0.0457947	0.0457844	1.023370×10^{-5}	−2.31836	−2.31836	6.35516×10^{-6}
0.9	−0.311216	−0.311224	8.34947×10^{-6}	−5.01403	−5.01398	4.75597×10^{-5}
1.0	−1	−1	6.12843×10^{-14}	−9	−8.99988	1.23122×10^{-5}

Theorem 10. *Let $\varepsilon_n^2 = \|u(x)-u_n(x)\|^2$, where $u(x)$ and $u_n(x)$ are given by* (12) *and* (14)*; then the sequence of real numbers $\varepsilon_n(x)$ is monotonously decreasing and $\varepsilon_n(x) \to 0$.*

Proof. We have

$$\begin{aligned}\varepsilon_n^2 = \|u(x)-u_n(x)\|^2 &= \left\|\sum_{i=n+1}^{\infty}\langle u(x),\overline{\psi}_i(x)\rangle\overline{\psi}_i(x)\right\|^2 \\ &= \sum_{i=n+1}^{\infty}\left(\langle u(x),\overline{\psi}_i(x)\rangle\right)^2.\end{aligned} \tag{15}$$

Clearly, $\varepsilon_{n-1} \geq \varepsilon_n$ and consequently $\{\varepsilon_n\}$ is monotonously decreasing in the sense of $\|\cdot\|$. By Theorem 9, we know that $\sum_{i=1}^{\infty}\langle u(x),\overline{\psi}_i(x)\rangle\overline{\psi}_i(x)$ is convergent in the norm of $\|\cdot\|$; then we have

$$\varepsilon_n^2 = \|u(x)-u_n(x)\|^2 \longrightarrow 0. \tag{16}$$

Hence, $\varepsilon_n \to 0$. □

Theorem 11 (convergence analysis). *$u_n(x)$ and $u_n^{(k)}(x)$ are uniformly convergent to $u(x)$ and $u^{(k)}(x)$, $k = 0,1,2,\ldots,m$, where $u(x)$ and $u_n(x)$ are given by* (12) *and* (14).

Proof. For any $x \in [0,1]$, $k = 0,1,2,\ldots,5$,

$$\begin{aligned}\left|u_n^{(k)}(x)-u^{(k)}(x)\right| &= \left|\left\langle u_n(t)-u(t),\frac{\partial^k K(x,t)}{\partial x^k}\right\rangle\right| \\ &\leq \|u_n(t)-u(t)\|\cdot\left\|\frac{\partial^k K(x,t)}{\partial x^k}\right\|.\end{aligned} \tag{17}$$

Then there exists $C_k > 0$ such that

$$\begin{aligned}\left|u_n^{(k)}(x)-u^{(k)}(x)\right| &\leq C_k\|u_n(t)-u(t)\| \\ &= C_k\varepsilon_n \longrightarrow 0.\end{aligned} \tag{18}$$

□

The numerical solution to (3) can be obtained using the following method:

$$u_n(x) = \sum_{i=1}^{n} d_i\psi_i(x), \tag{19}$$

where the coefficients d_i, $i = 1,\ldots,m$, are determined by the equation

$$\sum_{i=1}^{n} d_i L\psi_i(x)\big|_{x=x_j} = F\left(x_j\right), \quad j = 1,2,\ldots,n. \tag{20}$$

Using (19) and (20), we have $(Lu_m)(x_j) = F(x_j)$, $j = 1,2,\ldots,n$. So, $u_n(x)$ is the approximation solution of (3).

Table 4: The numerical results of Example 13.

x	$u_T^{(3)}(x)$	$u_{100}^{(3)}(x)$	$\lvert u_{100}^{(3)}(x) - u_T^{(3)}(x)\rvert$	$u_T^{(4)}(x)$	$u_{100}^{(4)}(x)$	$\lvert u_{100}^{(4)}(x) - u_T^{(4)}(x)\rvert$
0.1	0.971215	0.971117	9.72528×10^{-5}	11.8289	11.8244	4.50235×10^{-3}
0.2	1.97221	1.9719	3.12435×10^{-4}	7.04361	7.0429	7.15499×10^{-4}
0.3	2.19979	2.19947	3.16007×10^{-4}	−3.23499	−3.2345	4.83768×10^{-4}
0.4	1.19534	1.19511	2.34524×10^{-4}	−17.4321	−17.431	1.09059×10^{-3}
0.5	−1.39212	−1.39222	1.05489×10^{-4}	−34.8029	−34.8014	1.46428×10^{-3}
0.6	−5.85595	−5.8559	5.44543×10^{-5}	−54.8995	−54.8978	1.72026×10^{-3}
0.7	−12.4526	−12.4524	2.36294×10^{-4}	−77.4172	−77.4153	1.90772×10^{-3}
0.8	−21.4126	−21.4122	4.34553×10^{-4}	−102.132	−102.13	2.05168×10^{-3}
0.9	−32.9466	−32.9459	6.45658×10^{-4}	−128.873	−128.871	2.16643×10^{-3}
1.0	−47.25	−47.2491	9.67158×10^{-4}	−157.5	−157.498	2.26074×10^{-3}

5. Numerical Experiment

In this section, two numerical examples are studied to demonstrate the accuracy of the present method.

Example 12. Consider the following fifth-order boundary value problem with nonclassical side condition (the right-hand side of this problem has a singularity at $x = 0$, $x = 1$):

$$\begin{aligned} & u^{(5)}(x) - e^x \frac{x}{1-x} u''(x) + e^x \frac{x}{1-x} u(x) = f(x), \\ & \qquad\qquad\qquad\qquad 0 < x < 1, \\ & u(0) = u'(0) = u\left(\frac{1}{4}\right) = 0, \\ & 5u''(0) + 42 \int_0^1 e^{-x} u(x)\, dx = 0, \\ & 4u'''(1) + u'(1) = 10u''(1), \end{aligned} \tag{21}$$

where $f(x) = e^x(-45 + 195x - 750x^2 + 320x^{5/2} - 600x^3 - 32(5 + e^x)x^{7/2} + 40(20 + 3e^x)x^4 - 16(9 + 4e^x)x^{9/2} + 16(23 + 10e^x)x^5 - 16x^{11/2} + 32x^6)/32(-1+x)x^{5/2}$. The exact solution is $u_T(x) = x^2(\sqrt{x} - 1/2)e^x$. The numerical results are presented in Table 1.

Example 13 (see [9]). Consider the following fifth-order boundary value problem (the right-hand side of this problem has a singularity at $x = 0$):

$$\begin{aligned} & u^{(5)}(x) - e^{-x} u(x) \\ & \quad = -e^{-x} x^{\frac{9}{2}} (1 - x) + \frac{945(1 - 11x)}{32\sqrt{x}}, \quad 0 < x < 1, \\ & u(0) = u'(0) = u''(0) = u(1) = 0, \qquad u'(1) = -1, \end{aligned} \tag{22}$$

where the exact solution is $u_T(x) = x^{9/2}(1 - x)$. By the homogeneous process of the boundary condition, letting $v(x) = u(x) - x^3(1 - x)$, the problem can be transformed into the equivalent form

$$\begin{aligned} & v^{(5)}(x) - e^{-x} v(x) \\ & = -e^{-x} \left(x^{\frac{9}{2}} - x^3 \right)(1 - x) + \frac{945(1 - 11x)}{32\sqrt{x}}, \quad 0 \le x \le 1, \\ & \qquad v(0) = v'(0) = v''(0) = v(1) = v'(1) = 0. \end{aligned} \tag{23}$$

The numerical results are presented in Tables 2, 3, and 4.

6. Conclusions and Remarks

In this paper, a new reproducing kernel space satisfying mixed boundary value conditions is constructed skillfully. This makes it easy to solve such kind of problems. Furthermore, the exact solution of the problem can be expressed in series form. The numerical results demonstrate that the new method is quite accurate and efficient for singular problems of fifth-order ordinary differential equations. All computations have been performed using the Mathematica 7.0 software package.

Acknowledgments

The authors would like to express their thanks to the unknown referees for their careful reading and helpful comments. This paper is supported by the Natural Science Foundation of Inner Mongolia (2013MS0109) and Project Application Technology Research and Development Foundation of Inner Mongolia.

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