Research Article

The Existence of Positive Nonconstant Steady States in a Reaction: Diffusion Epidemic Model

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We investigate the disease's dynamics of a reaction-diffusion epidemic model. We first give a priori estimates of upper and lower bounds for positive solutions to model and then give the conditions of the existence and nonexistence of the positive nonconstant steady states, which guarantees the existence of the stationary patterns.

1. Introduction

Infectious diseases are the second leading cause of death worldwide, after heart disease, and are responsible for more deaths annually than cancer [1]. Since the pioneer work of Kermark and McKendrick [2], mathematical models have been contributing to improve our understanding of infectious disease dynamics and help us develop preventive measures to control infection spread qualitatively and quantitatively.

Many studies indicate that spatial epidemiology with self-diffusion has become a principal scientific discipline aiming at understanding the causes and consequences of spatial heterogeneity in disease transmission [3]. In these studies, reaction-diffusion equations have been intensively used to describe spatiotemporal dynamics. In particular, the spatial spread of infections has been studied by analyzing traveling wave solutions and calculating spread rates [4–10].

Besides, there has been some research on pattern formation in the spatial epidemic model, starting with Turing's seminal paper [11]. Turing's revolutionary idea was that the passive diffusion could interact with chemical reaction in such a way that even if the reaction by itself has no symmetry-breaking capabilities, diffusion can destabilize the symmetric solutions with the result that the system with diffusion has them [12]. In these studies [3, 13–20], via standard linear analysis, the authors obtained the conditions of Turing instability, and, via numerical simulation, they showed the pattern

formation induced by self-diffusion or cross-diffusion and found that model dynamics exhibits a diffusion controlled formation growth to stripes, spots, and coexistence or chaos pattern replication.

Recently, the researchers are interested in research on the stationary patterns due to the existence and nonexistence nonconstant solutions of the reaction-diffusion model [21–29]. But the research on the existence and nonexistence nonconstant solutions of reaction-diffusion epidemic model, seems rare [3].

In this paper, we will focus on the disease's dynamics through studying the existence of the constant and nonconstant steady states of a simple reaction-diffusion epidemic model.

The rest of this paper is organized as follows. In Section 2, we derive a reaction-diffusion epidemic model. In Section 3, we give a priori estimates of upper and lower bounds for positive solutions to model. In Section 4, we give the main results on the existence and nonexistence of positive nonconstant steady states of the model. The paper ends with a brief discussion in Section 5.

2. Basic Model

In [30], Berezovsky and coworkers introduced a simple epidemic model through the incorporation of variable population, disease induced mortality, and emigration into the

classic model of Kermark and McKendrick [2]. The total population (N) is divided into two groups susceptible (S) and infectious (I); that is, $N = S + I$. The model describing the relations between the state variables is

$$\frac{dS}{dt} = rN\left(1 - \frac{N}{K}\right) - \beta\frac{SI}{N} - (\mu + \theta)S,$$
$$\frac{dI}{dt} = \beta\frac{SI}{N} - (\mu + d)I, \tag{1}$$

where the birth process incorporates density dependent effects via a logistic equation with the intrinsic growth rate r and the carrying capacity K; $S(t)$, $I(t)$ represent population densities of susceptible and infected population, respectively; β denotes the transmission rate (the infection rate constant); μ is the natural mortality; d denotes the disease-induced mortality; θ is the per-capita emigration rate of noninfective.

For model (1), the epidemic threshold of basic reproduction number R_0 is then computed as

$$R_0 = \frac{\beta}{\mu + d}. \tag{2}$$

The basic demographic reproductive number R_d is given by

$$R_d = \frac{r}{\mu + \theta}. \tag{3}$$

For simplicity, rescalling the model (1) by letting $S \to S/K$, $I \to I/K$, and $t \to t/(\mu + d)$ leads to the following model:

$$\frac{dS}{dt} = \nu R_d (S + I)(1 - (S + I)) - R_0\frac{SI}{S + I} - \nu S,$$
$$\frac{dI}{dt} = R_0\frac{SI}{S + I} - I, \tag{4}$$

where $\nu = (\mu + \theta)/(\mu + d)$ defined by the ratio of the average life-span of susceptibles to that of infections and $S + I \le 1$.

See [30] for more details.

Assume that the habitat $\Omega \subset \mathbb{R}^m$ ($m \ge 1$) is a bounded domain with smooth boundary $\partial\Omega$ (when $m > 1$), and $\mathbf{n}$ is the outward unit normal vector on $\partial\Omega$. We consider the following reaction-diffusion SI epidemic model:

$$\frac{\partial S}{\partial t} - d_S\Delta S = \nu R_d (S + I)(1 - (S + I))$$
$$- R_0\frac{SI}{S + I} - \nu S, \quad x \in \Omega, t > 0,$$
$$\frac{\partial I}{\partial t} - d_I\Delta I = R_0\frac{SI}{S + I} - I, \quad x \in \Omega, t > 0,$$
$$S(x, 0) = S_0(x) > 0, \quad I(x, 0) = I_0(x) \ge 0, \; x \in \Omega,$$
$$\frac{\partial S}{\partial \mathbf{n}} = \frac{\partial I}{\partial \mathbf{n}} = 0, \quad x \in \partial\Omega, \; t > 0, \tag{5}$$

where the nonnegative constants d_S and d_I are the diffusion coefficients of S and I, respectively. The symbol Δ is

the Laplacian operator. The homogeneous Neumann boundary condition implies that the above model is self-contained and there is no infection across the boundary.

The corresponding kinetic model (5) with $m = 2$ has been investigated by Wang et al. [20].

In this paper, we concentrated on the steady states of model (5) which satisfy

$$-d_S\Delta S = \nu R_d (S + I)(1 - (S + I)) - R_0\frac{SI}{S + I} - \nu S, \quad x \in \Omega,$$
$$-d_I\Delta I = R_0\frac{SI}{S + I} - I, \quad x \in \Omega,$$
$$\frac{\partial S}{\partial \mathbf{n}} = \frac{\partial I}{\partial \mathbf{n}} = 0, \quad x \in \partial\Omega. \tag{6}$$

Throughout this paper, the positive solution (S, I) satisfying model (6) refers to a classical one with $S > 0$, $I > 0$ on $\overline{\Omega}$. Clearly, model (6) has a unique positive constant solution (endemic equilibrium) $E^* = (S^*, I^*)$ if $R_d > (\nu + R_0 - 1)/R_0\nu$ and $R_0 > 1$, where

$$S^* = \frac{\nu R_0 R_d - R_0 + 1 - \nu}{\nu R_0^2 R_d}, \qquad I^* = (R_0 - 1)S^*. \tag{7}$$

3. A Priori Estimates for Positive Solutions to Model (6)

The main purpose of this section is to give a priori upper and lower positive bounds for positive solution of model (6). To this aim, we first cite two known results. The first is due to Lin et al. [31] and the second to Lou and Ni [32]. In the following, let us denote the constants ν, R_d, and R_0 collectively by Λ. The positive constants C, $\underline{C}$, $\overline{C}$, C^*, and so forth will depend only on the domains Ω and Λ.

Lemma 1 (Harnack inequality [31]). *Let $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a positive solution to $\Delta w(x) + c(x)w(x) = 0$, where $c \in C(\overline{\Omega})$, satisfying the homogeneous Neumann boundary conditions. Then there exists a positive constant $C^* = C^*(\|c\|_\infty, \Omega)$, such that*

$$\max_{\overline{\Omega}} w \le C^* \min_{\overline{\Omega}} w. \tag{8}$$

Lemma 2 (maximum principle [32]). *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^m$ and $g \in C(\overline{\Omega} \times \mathbb{R})$.*

(a) *Assume that $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and satisfies*

$$\Delta w(x) + g(x, w(x)) \ge 0, \quad x \in \Omega,$$
$$\frac{\partial w}{\partial \mathbf{n}} \le 0, \quad x \in \partial\Omega. \tag{9}$$

If $w(x_0) = \max_{\overline{\Omega}} w(x)$, then $g(x_0, w(x_0)) \ge 0$.

(b) *Assume that $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and satisfies*

$$\Delta w(x) + g(x, w(x)) \leq 0, \quad x \in \Omega,$$

$$\frac{\partial w}{\partial \mathbf{n}} \geq 0, \quad x \in \partial\Omega. \tag{10}$$

If $w(x_1) = \min_{\overline{\Omega}} w(x)$, then $g(x_1, w(x_1)) \leq 0$.

Theorem 3. *If $R_0 > 1$, then the positive solution $(S(x), I(x))$ of model (6) satisfies*

$$\max_{\overline{\Omega}} S(x) < \frac{1}{4} R_d, \qquad \max_{\overline{\Omega}} I(x) < \frac{1}{4} R_d (R_0 - 1). \tag{11}$$

Proof. Assume that $(S(x), I(x))$ is a positive solution of model (6). We set

$$S(x_1) = \max_{\overline{\Omega}} S(x), \qquad I(x_2) = \max_{\overline{\Omega}} I(x). \tag{12}$$

By applying Lemma 2, we have

$$\frac{1}{4} R_d - \nu S(x_1)$$

$$\geq \nu R_d (S(x_1) + I(x_1))(1 - (S(x_1) + I(x_1))) - \nu S(x_1)$$

$$\geq \frac{R_0 S(x_1) I(x_1)}{S(x_1) + I(x_1)} > 0, \tag{13}$$

and $R_0 S(x_2) I(x_2)/(S(x_2) + I(x_2)) \geq I(x_2)$. This clearly gives

$$S(x_1) < \frac{1}{4} R_d, \quad I(x_2) \leq (R_0 - 1) S(x_2) < \frac{1}{4} R_d (R_0 - 1). \tag{14}$$

$\square$

Theorem 4. *Assume that $R_d > 1$ and $R_0 > 1$. Let d and D be fixed positive constants. Then there exists a positive constant $\underline{C} = \underline{C}(\Lambda, d)$ such that, if $d_S, d_I > d$, every positive solution $(S(x), I(x))$ of model (6) satisfies*

$$\min_{\overline{\Omega}} S(x) > \underline{C}, \qquad \min_{\overline{\Omega}} I(x) > \underline{C}. \tag{15}$$

Proof. Let

$$c_1(x) = \frac{1}{d_S}\left(\nu R_d \left(1 + \frac{I}{S}\right)(1 - (S + I)) - \frac{R_0 I}{S + I} - \nu\right),$$

$$c_2(x) = \frac{1}{d_I}\left(\frac{R_0 S}{S + I} - 1\right). \tag{16}$$

In view of Theorem 3, there exists a positive constant $C = C(\Lambda)$ such that $\|c_1(x)\|_\infty \leq C$, $\|c_2(x)\|_\infty \leq C$ provided that $d_S, d_I > d$. As S and I satisfy

$$\Delta S(x) + c_1(x)S = 0, \quad x \in \Omega,$$

$$\Delta I(x) + c_2(x)I = 0, \quad x \in \Omega,$$

$$\frac{\partial S}{\partial \mathbf{n}} = \frac{\partial I}{\partial \mathbf{n}} = 0, \quad x \in \partial\Omega. \tag{17}$$

It follows from Lemma 1 that there exists a positive constant $C^* = C^*(\Lambda, d)$ such that

$$\max_{\overline{\Omega}} S \leq C^* \min_{\overline{\Omega}} S, \qquad \max_{\overline{\Omega}} I \leq C^* \min_{\overline{\Omega}} I \tag{18}$$

for $d_S, d_I \geq d$.

Now, on the contrary, suppose that (15) is not true, then there exist sequences $\{d_{S,i}\}_{i=1}^\infty$, $\{d_{I,i}\}_{i=1}^\infty$ with $(d_{S,i}, d_{I,i}) \in [d, \infty) \times [d, \infty)$ and the positive solution (S_i, I_i) of model (6) corresponding to $(d_S, d_I) = (d_{S,i}, d_{I,i})$, such that

$$\min_{\overline{\Omega}} S_i(x) \longrightarrow 0 \quad \text{or} \quad \min_{\overline{\Omega}} I_i(x) \longrightarrow 0 \quad \text{as } i \longrightarrow \infty. \tag{19}$$

It follows from Lemma 1 that

$$S_i(x) \longrightarrow 0 \quad \text{or} \quad I_i(x) \longrightarrow 0$$

$$\text{uniformly on } \overline{\Omega} \text{ as } i \longrightarrow \infty. \tag{20}$$

(S_i, I_i) satisfies

$$-d_{S,i}\Delta S_i = \nu R_d (S_i + I_i)(1 - (S_i + I_i))$$

$$- \frac{R_0 S_i I_i}{S_i + I_i} - \nu S_i, \quad x \in \Omega,$$

$$-d_{I,i}\Delta I_i = \frac{R_0 S_i I_i}{S_i + I_i} - I_i, \quad x \in \Omega,$$

$$\frac{\partial S_i}{\partial \mathbf{n}} = \frac{\partial I_i}{\partial \mathbf{n}} = 0, \quad x \in \partial\Omega. \tag{21}$$

Integrating by parts, we obtain that, for $i = 1, 2, \ldots$,

$$\int_\Omega \left(\nu R_d (S_i + I_i)(1 - (S_i + I_i)) - \frac{R_0 S_i I_i}{S_i + I_i} - \nu S_i\right) dx = 0,$$

$$\int_\Omega I_i \left(\frac{R_0 S_i}{S_i + I_i} - 1\right) dx = 0. \tag{22}$$

By the regularity theory for elliptic equations [33], we see that there exist a subsequence of $\{(S_i, I_i)\}_i^\infty$, which we will still denote by $\{(S_i, I_i)\}_i^\infty$, and two nonnegative functions $\widetilde{S}, \widetilde{I} \in C^2(\Omega)$, such that $(S_i, I_i) \rightarrow (\widetilde{S}, \widetilde{I})$ in $[C^2(\Omega)]^2$ as $i \rightarrow \infty$. By (20), we have that $\widetilde{S} \equiv 0$ or $\widetilde{I} \equiv 0$.

Letting $i \rightarrow \infty$ in (22) we obtain that

$$\int_\Omega \left(\nu R_d (\widetilde{S} + \widetilde{I})(1 - (\widetilde{S} + \widetilde{I})) - \frac{R_0 \widetilde{S}\widetilde{I}}{\widetilde{S} + \widetilde{I}} - \nu\widetilde{S}\right) dx = 0,$$

$$\int_\Omega \widetilde{I}\left(\frac{R_0 \widetilde{S}}{\widetilde{S} + \widetilde{I}} - 1\right) dx = 0. \tag{23}$$

Case 1 ($\widetilde{S} \equiv 0$, $\widetilde{I} \not\equiv 0$ *or* $\widetilde{S} \equiv 0$, $\widetilde{I} \equiv 0$). Since I_i satisfies the second inequality of (18), $I_i > 0$ on $\overline{\Omega}$. Therefore, $R_0 S_i/(S_i + I_i) - 1 \rightarrow -1 < 0$ on $\overline{\Omega}$ as $i \rightarrow \infty$. Hence, $\int_\Omega I_i(R_0 S_i/(S_i + I_i) - 1)dx < 0$ for sufficiently large i which contradicts the second integral identity of (22).

Case 2 ($\widetilde{I} \equiv 0$, $\widetilde{S} \not\equiv 0$). As above, $\widetilde{S} > 0$ on $\overline{\Omega}$. It follows from the first integral identity of (23) that

$$\int_\Omega \widetilde{S}\left(\nu R_d\left(1 - \widetilde{S}\right) - \nu\right) dx = 0. \qquad (24)$$

This fact combines with $0 < \widetilde{S} \leq (1/4)R_d$ yielding to $\widetilde{S} = 1 - 1/R_d$, which implies that $R_0 S_i/(S_i + I_i) \to R_0$ uniformly on $\overline{\Omega}$ as $i \to \infty$, since $I_i \to 0$ uniformly on $\overline{\Omega}$. As $R_0 > 1$, this contradicts the second integral identity of (23) and the fact that $I_i > 0$. This completes the proof. □

4. Existence and Nonexistence of Positive Nonconstant Steady States

In this section, we provide some sufficient conditions for the existence and nonexistence of nonconstant positive solution of model (6) by using the Leray-Schauder degree theory [34]. From now on, we denote by

$$0 = \mu_0 < \mu_1 < \mu_2 < \mu_3 < \cdots \qquad (25)$$

the eigenvalues of the operator $-\Delta$ on Ω with the zero-flux boundary conditions.

4.1. Nonexistence for Positive Nonconstant Steady States to Model (6). This section is devoted to the consideration of the nonexistence for the nonconstant positive solutions of model (6), and, in the following results, the diffusion coefficients do play a significant role.

Theorem 5. *Assume that $R_0 > 1$. Let D_2 be a fixed positive constant with $D_2 > (R_0 - 1)/\mu_1$. Then there exists a positive constant $D_1(\Lambda, D_2)$ such that model* (6) *has no positive nonconstant solution provided that $d_S \geq D_1$ and $d_I \geq D_2$.*

Proof. Let $(S(x), I(x))$ be any positive solution of model (6) and denote $\overline{g} = |\Omega|^{-1}\int_\Omega g\, dx$. Then, multiplying the first equation of model (6) by $(S - \overline{S})$, integrating over Ω, by virtue of Theorem 3, we have that

$$d_S \int_\Omega |\nabla S|^2 dx$$

$$= \int_\Omega \left(S - \overline{S}\right)$$

$$\times \left(\nu R_d\left(S + I\right)\left(1 - \left(S + I\right)\right) - \frac{R_0 SI}{S + I} - \nu S\right) dx$$

$$= \int_\Omega \left(S - \overline{S}\right)^2$$

$$\times \left(\nu\left(R_d - 1\right) - \nu R_d\left(S + \overline{S}\right) + 2\nu R_d I - \frac{R_0 \overline{I} I}{\left(\overline{S} + \overline{I}\right)\left(S + I\right)}\right) dx$$

$$+ \int_\Omega \left(\nu R_d\left(1 - \left(I + \overline{I}\right) + 2\overline{S}\right) - \frac{R_0 S\overline{S}}{\left(\overline{S} + \overline{I}\right)\left(S + I\right)}\right)$$

$$\times \left(S - \overline{S}\right)\left(I - \overline{I}\right) dx,$$

$$\leq C_1 \int_\Omega \left(S - \overline{S}\right)^2 + C_2 \int_\Omega \left|S - \overline{S}\right|\left|I - \overline{I}\right| dx,$$

$$(26)$$

where C_1, C_2 depend only on Λ. In a similar manner, we multiply the second equation in model (6) by $(I - \overline{I})$ to have

$$d_I \int_\Omega |\nabla I|^2 dx$$

$$= \int_\Omega \left(I - \overline{I}\right)\left(\frac{R_0 SI}{S + I} - I\right) dx$$

$$= \int_\Omega \left(I - \overline{I}\right)^2\left(-1 + \frac{R_0 \overline{S} S}{\left(\overline{S} + \overline{I}\right)\left(S + I\right)}\right) dx$$

$$+ \int_\Omega \frac{R_0 \overline{I} I}{\left(\overline{S} + \overline{I}\right)\left(S + I\right)}\left(S - \overline{S}\right)\left(I - \overline{I}\right) dx \qquad (27)$$

$$\leq \left(R_0 - 1\right)\int_\Omega \left(I - \overline{I}\right)^2 dx$$

$$+ R_0 \int_\Omega \left|S - \overline{S}\right|\left|I - \overline{I}\right| dx.$$

It follows from (26), (27) and the ε-Young inequality that

$$\int_\Omega \left(d_S|\nabla S|^2 + d_I |\nabla I|^2\right) dx$$

$$\leq \int_\Omega \left(\left(C_1 + \frac{C}{2\varepsilon}\right)\left(S - \overline{S}\right)^2 + \left(R_0 - 1 + \frac{\varepsilon C}{2}\right)\left(I - \overline{I}\right)^2\right) dx,$$

$$(28)$$

where $C = C_2 + R_0$. It follows from the well-known Poincaré inequality that

$$\int_\Omega \left(d_S|\nabla S|^2 + d_I |\nabla I|^2\right) dx$$

$$\leq \frac{1}{\mu_1}\left(C_1 + \frac{C}{2\varepsilon}\right)\int_\Omega |\nabla S|^2 dx \qquad (29)$$

$$+ \frac{1}{\mu_1}\left(R_0 - 1 + \frac{\varepsilon C}{2}\right)\int_\Omega |\nabla I|^2 dx.$$

Since $d_I \mu_1 > R_0 - 1$ from the assumption, we can find a sufficiently small ε_0 such that $d_I \mu_1 \geq R_0 - 1 + \varepsilon C/2$. Finally, by taking $D_1 := (1/\mu_1)(C_1 + C/2\varepsilon_0)$ one can conclude that $S = \overline{S}$ and $I = \overline{I}$, which asserts our results. □

4.2. Existence for Positive Nonconstant Steady States to Model (6). In this section, we discuss the global existence of nonconstant positive classical solutions to model (6), which guarantees the existence of the *stationary patterns* [21, 24, 26, 27].

Unless otherwise specified, in this section, we always require that $R_d > (\nu + R_0 - 1)/R_0\nu$ and $R_0 > 1$, which guarantees that model (6) has one positive constant steady state E^*. From now on, let us denote

$$\mathbf{u} = (S, I),$$

$$\mathbf{u}^* = (S^*, I^*) = \left(\frac{\nu R_0 R_d - R_0 + 1 - \nu}{\nu R_0^2 R_d}, (R_0 - 1) S^* \right). \quad (30)$$

Let $\mathbf{X} = \{\mathbf{u} \in [C^2(\Omega)]^2 \mid \partial\mathbf{u}/\partial\mathbf{n} = 0, x \in \partial\Omega\}$ and $\mathbf{X}^+ = \{\mathbf{u} \in \mathbf{X} \mid S, I > 0, x \in \overline{\Omega}\}$. Then we write model (6) in the form

$$-\Delta\mathbf{u} = \mathbf{G}(\mathbf{u}), \quad x \in \Omega,$$
$$\frac{\partial\mathbf{u}}{\partial\mathbf{n}} = 0, \quad x \in \partial\Omega, \quad (31)$$

where

$$\mathbf{G}(\mathbf{u}) = \begin{pmatrix} \dfrac{1}{d_S}\left(\nu R_d (S+I)(1-(S+I)) - \dfrac{R_0 SI}{S+I} - \nu S \right) \\ \dfrac{1}{d_I}\left(\dfrac{R_0 SI}{S+I} - I \right) \end{pmatrix}. \quad (32)$$

Define a compact operator $\mathscr{F} : \mathbf{X}^+ \to \mathbf{X}^+$ by

$$\mathscr{F}(\mathbf{u}) := (\mathbf{I} - \Delta)^{-1}\{\mathbf{G}(\mathbf{u}) + \mathbf{u}\}, \quad (33)$$

where $(\mathbf{I} - \Delta)^{-1}$ is the inverse operator of $\mathbf{I} - \Delta$ subject to the zero-flux boundary condition. Then $\mathbf{u}$ is a positive solution of model (31) if and only if $\mathbf{u}$ satisfies

$$(\mathbf{I} - \mathscr{F})\mathbf{u} = 0, \quad x \in \Omega. \quad (34)$$

To apply the index theory, we investigate the eigenvalue of the problem

$$-(\mathbf{I} - \mathscr{F}_{\mathbf{u}}(\mathbf{u}^*))\Psi = \lambda\Psi, \quad \Psi \neq \mathbf{0}, \quad (35)$$

where $\Psi = (\Psi_1, \Psi_2)^T$ and $\mathscr{F}_{\mathbf{u}}(\mathbf{u}^*) = (\mathbf{I} - \Delta)^{-1}(\mathbf{I} + \mathbf{A})$ with

$$\mathbf{A} = \begin{pmatrix} \dfrac{4R_0 + 2\nu - R_0^2 - \nu R_0 R_d - \nu R_0 - 3}{d_S R_0} & -\dfrac{\nu R_0 R_d + 3 - 2R_0 - 2\nu}{d_S R_0} \\ \dfrac{(R_0 - 1)^2}{d_I R_0} & -\dfrac{R_0 - 1}{d_I R_0} \end{pmatrix}$$
$$:= \begin{pmatrix} d_S^{-1} a_1 & -d_S^{-1} a_2 \\ d_I^{-1} a_3 & -d_I^{-1} a_4 \end{pmatrix}. \quad (36)$$

λ is an eigenvalue of (35) if and only if λ is an eigenvalue of the matrix $(\mu_i + 1)^{-1}(\mu_i\mathbf{I} - \mathbf{A})$ for any $i \geq 0$. Therefore, $\mathbf{I} - \mathscr{F}_{\mathbf{u}}(\mathbf{u}^*)$ is invertible if and only if, for any $i \geq 0$, the matrix

$$M_i := \mu_i\mathbf{I} - \mathbf{A} = \begin{pmatrix} \mu_i - d_S^{-1} a_1 & d_S^{-1} a_2 \\ -d_I^{-1} a_3 & \mu_i + d_I^{-1} a_4 \end{pmatrix} \quad (37)$$

is invertible. A straightforward computation yields

$$\det(M_i) = d_S^{-1} d_I^{-1}$$
$$\times \left(d_S d_I \mu_i^2 + (d_S a_4 - d_I a_1)\mu_i + \rho \right), \quad (38)$$

where $\rho = (1/R_0)(\nu R_0 R_d - R_0 + 1 - \nu)(R_0 - 1) > 0$. For the sake of convenience, we denote

$$H(d_S, d_I, \mu) = d_S d_I \mu_i^2 + (d_S a_4 - d_I a_1)\mu_i + \rho. \quad (39)$$

Then $H(d_S, d_I, \mu) = d_S d_I \det(M_i)$.

If $(d_S a_4 - d_I a_1)^2 > 4d_S d_2 \rho$, then $H(d_S, d_I, \mu) = 0$ has two real roots $\mu^{\pm}$ given by

$$\mu^+ (d_S, d_I)$$
$$= \frac{1}{2d_S d_2}\left(d_I a_1 - d_S a_4 + \sqrt{(d_S a_4 - d_I a_1)^2 - 4d_S d_2 \rho} \right),$$

$$\mu^- (d_S, d_I)$$
$$= \frac{1}{2d_S d_2}\left(d_I a_1 - d_S a_4 - \sqrt{(d_S a_4 - d_I a_1)^2 - 4d_S d_2 \rho} \right). \quad (40)$$

Set $B := B(d_S, d_I) = \{\mu : \mu \geq 0, \mu^-(d_S, d_I) < \mu < \mu^+(d_S, d_I)\}$, $S_p = \{\mu_0, \mu_1, \mu_2, \ldots\}$, and $m(\mu_i)$ the multiplicity of μ_i.

To compute index $(\mathbf{I} - \mathscr{F}, \mathbf{u}^*)$, we can assert the following conclusion by Pang and Wang [22].

Lemma 6 (see [22]). *Suppose $H(d_S, d_I, \mu_i) \neq 0$ for all $\mu_i \in S_p$. Then*

$$index\ (\mathbf{I} - \mathscr{F}, \mathbf{u}^*) = (-1)^\sigma, \quad (41)$$

where

$$\sigma = \begin{cases} \sum_{\mu \in B \cap S_j} m(u_i), & if\ B \cap S_p \neq \emptyset, \\ 0, & if\ B \cap S_p = \emptyset. \end{cases} \quad (42)$$

In particular, if $H(d_S, d_I, \mu) > 0$ for all $\mu \geq 0$, then $\sigma = 0$.

From Lemma 6, we see that to calculate the index of index$(\mathbf{I} - \mathscr{F}, \mathbf{u}^*)$, the key step is to determine the range of μ for which $H(d_S, d_I, \mu) < 0$.

Theorem 7. *Assume that $R_d > \max\{1, (\nu + R_0 - 1)/R_0\nu\}$. If $4R_0 + 2\nu - R_0^2 - \nu R_0 R_d - \nu R_0 - 3 > 0$, $(4R_0 + 2\nu - R_0^2 - \nu R_0 R_d - \nu R_0 - 3)/d_S R_0 \in (\mu_j, \mu_{j+1})$ for some $j \geq 1$, and $\sum_{i=1}^{j} m(\mu_i)$ is odd, then there exists a positive constant d^* such that model (6) has at least one nonconstant solution if $d_I > d^*$.*

Proof. Since $4R_0 + 2\nu - R_0^2 - \nu R_0 R_d - \nu R_0 - 3 > 0$, equivalently, $a_1 > 0$, it follows that if d_I is large enough, then $(d_S a_4 - d_I a_1)^2 > 4 d_S d_I \rho$ and $0 < \mu^-(d_S, d_I) < \mu^+(d_S, d_I)$. Furthermore,

$$\mu^-(d_S, d_I) \longrightarrow 0, \quad \mu^+(d_S, d_I) \longrightarrow \frac{a_1}{d_S}, \quad as\ d_I \longrightarrow \infty. \quad (43)$$

Since $a_1/d_S \in (\mu_j, \mu_{j+1})$ for some $j \geq 1$, there exists $d_0 \gg 1$ such that

$$\mu^+(d_S, d_I) \in (\mu_j, \mu_{j+1}), \qquad 0 < \mu^-(d_S, d_I) < \mu_1, \quad \forall d_I \geq d_0. \quad (44)$$

By Theorem 5, we know that there exists $d > d_0$ such that model (6) with diffusion coefficients $d_S = d$ and $d_I \geq d$ has no nonconstant solutions. Moreover, we can choose d so large that $a_1/d < \mu_1$. It follows that there exists $d^* > d$ such that

$$0 < \mu^-(d_S, d_I) < \mu^+(d_S, d_I) < \mu_1, \quad \forall d_I \geq d^*. \quad (45)$$

We shall prove that, for any $d_I \geq d^*$, model (6) has at least one nonconstant positive solution. On the contrary, suppose that this assertion is not true for some $d_I^* > d^*$. In the following, we will derive a contradiction by using a homotopy argument.

By virtue of Theorems 3 and 4, there exists a positive constant $C = C(\Lambda, d, d_S, d^*, d_I^*)$ such that the positive solution $(S(x), I(x))$ of model (6) satisfies $C^{-1} < S, I < C$.

Set

$$\mathscr{M} = \left\{(S, I) \in C(\overline{\Omega}) \times C(\overline{\Omega}) : C^{-1} < S, I < C, x \in \overline{\Omega}\right\}, \quad (46)$$

and define

$$\Phi : \mathscr{M} \times [0, 1] \longrightarrow C(\overline{\Omega}) \times C(\overline{\Omega}) \quad (47)$$

by

$$\Phi(\mathbf{u}, \theta) = (\mathbf{I} - \Delta)^{-1} \{\mathbf{G}(\mathbf{u}, \theta) + \mathbf{u}\}, \quad (48)$$

where

$$\mathbf{G}(\mathbf{u}, \theta)$$
$$= \begin{pmatrix} (\theta d_S + (1 - \theta) d)^{-1} \left(\nu R_d (S+I)(1 - (S+I)) - \dfrac{R_0 SI}{S + I} - \nu S\right) \\ (\theta d_I + (1 - \theta) d^*)^{-1} \left(\dfrac{R_0 SI}{S + I} - I\right) \end{pmatrix}. \quad (49)$$

It is clear that finding the positive solution of model (31) is equivalent to finding the fixed point of $\Phi(\mathbf{u}, 1)$ in $\mathscr{M}$. Further, by virtue of the definition of $\mathscr{M}$, we have that $\Phi(\mathbf{u}, \theta) = 0$ has no fixed point in $\partial\mathscr{M}$ for all $0 \leq \theta \leq 1$.

Since $\Phi(\mathbf{u}, t)$ is compact, the Leray-Schauder topological degree $\deg(\mathbf{I} - \Phi(\mathbf{u}, \theta), \mathscr{M}, 0)$ is well defined. From the invariance of Leray-Schauder degree at the homotopy, we deduce

$$\deg(\mathbf{I} - \Phi(\mathbf{u}, 1), \mathscr{M}, 0) = \deg(\mathbf{I} - \Phi(\mathbf{u}, 0), \mathscr{M}, 0). \quad (50)$$

In view of $\mu^- \in (\mu_i, \mu_{i+1})$ and $\mu^+ \in (\mu_j, \mu_{j+1})$, we have $B(d_S, d_I) \cap S_j = \{\mu_{i+1}, \mu_{i+2}, \ldots, \mu_p\}$. Clearly, $\mathbf{I} - \Phi(\mathbf{u}, 1) = \mathbf{I} - \mathscr{F}$. Thus, if model (6) has no other solutions except the constant one $\mathbf{u}^*$, then Lemma 6 shows that

$$\deg(\mathbf{I} - \Phi(\mathbf{u}, 1), \mathscr{M}, 0)$$
$$= \text{index}(\mathbf{I} - \mathscr{F}, \mathbf{u}^*) = (-1)^{\sum_{i=1}^{j} m(u_i)} = -1. \quad (51)$$

On the contrary, by the choice of d and d^*, we have that $B(\overline{d}_1, \overline{d}_2) \cap S_p = \emptyset$ and $\mathbf{u}^*$ is the only fixed point of $\Phi(\mathbf{u}, 0)$. It therefore follows from Lemma 6 that

$$\deg(\mathbf{I} - \Phi(\mathbf{u}, 0), \mathscr{M}, 0)$$
$$= \text{index}(\mathbf{I} - \mathscr{F}, \mathbf{u}^*) = (-1)^0 = 1. \quad (52)$$

From (50)–(52), we get a contradiction. Therefore, there exists a nonconstant solution of model (6). The proof is completed. □

5. Discussion

In this paper, we investigate the disease's dynamics through studying the existence and nonexistence positive constant steady states of a reaction-diffusion epidemic model. We give a priori estimates for positive solutions to model and show that the nonconstant positive steady states exist due to the emergence of diffusion, which demonstrates that *stationary patterns* can be found as a result of diffusion. The numerical results about the *stationary patterns* for model (5) can be found in [20].

On the other hand, there are plenty of papers which focus on the pattern formation of reaction-diffusion population models via standard linear analysis method and numerical simulations. But there is little literature analytically concerning the existence of a stationary patterns via theory and methods of partial differential equations infrequently. The methods and results in the present paper may enrich the research of pattern formation in the spatial epidemic model.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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