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Research Article

Solution and Stability of Euler-Lagrange-Rassias Quartic Functional Equations in Various Quasinormed Spaces

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We obtain the general solution of Euler-Lagrange-Rassias quartic functional equation of the following $f(ax+by)+f(bx+ay)+(1/2)ab(a-b)^2 f(x-y)=(a^2-b^2)^2[f(x)+f(y)]+(1/2)ab(a+b)^2 f(x+y)$. We also prove the Hyers-Ulam-Rassias stability in various quasinormed spaces when $b=1$.

1. Introduction

One of the interesting questions concerning the stability problems of functional equations is as follows: when is it true that a mapping satisfying a functional equation approximately must be close to the solution of the given functional equation? Such an idea was suggested in 1940 by Ulam [1] as follows. *Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$ then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?* In other words, we are looking for situations when the homomorphisms are stable; that is, if a mapping is almost a homomorphism, then there exists a true homomorphism near it. In 1941, Hyers [2] considered the case of approximately additive mappings in Banach spaces and satisfying the well-known weak Hyers inequality controlled by a positive constant. The famous Hyers stability result that appeared in [2] was generalized in the stability involving a sum of powers of norms by Aoki [3]. In 1978, Rassias [4] provided a generalization of Hyers Theorem which allows the Cauchy difference to be unbounded. During the last decades, stability problems of various functional equations have been extensively studied and generalized by a number of authors [5–10]. In particular, Rassias [11] introduced the Euler-Lagrange type quadratic functional equation

$$f(rx+sy)+f(sx-ry)=\left(r^2+s^2\right)\left[f(x)+f(y)\right], \quad (1)$$

for fixed reals r, s with $r \neq 0$, $s \neq 0$. Also, Jun and Kim [12] proved the Hyers-Ulam-Rassias stability of a Euler-Lagrange type cubic mapping as follows:

$$\begin{aligned} f(ax+by)+f(bx+ay) &= (a+b)(a-b)^2\left[f(x)+f(y)\right] \\ &\quad + ab(a+b)f(x+y), \end{aligned} \quad (2)$$

where $a \neq 0$, $b \neq 0$, $a \pm b \neq 0$, for all $x, y \in X$. Several Euler-Lagrange type functional equations have been investigated by numerous mathematicians; c.f. for example, [13–15].

And Rassias [16] investigated stability properties of the following quartic functional equation:

$$\begin{aligned} &f(x+2y)+f(x-2y)+6f(x) \\ &\quad = 4f(x+y)+4f(x-y)+24f(y). \end{aligned} \quad (3)$$

It is easy to see that $f(x)=x^4$ is a solution of (3) by virtue of the identity

$$\begin{aligned} &(x+2y)^4+(x-2y)^4+x^4 \\ &\quad = 4(x+y)^4+4(x-y)^4+24y^4. \end{aligned} \quad (4)$$

For this reason, (3) is called a quartic functional equation. Also, Chung and Sahoo [17] determined the general solution of (3) without assuming any regularity conditions on the unknown function. In fact, they proved that the function

$f : \mathbb{R} \to \mathbb{R}$ is a solution of (3) if and only if $f(x) = A(x, x, x, x)$, where the function $A : \mathbb{R}^4 \to \mathbb{R}$ is symmetric and additive in each variable. Lee and Chung [18] introduced a quartic functional equation as follows:

$$f(ax+y) + f(ax-y) = a^2 f(x+y) + a^2 f(x-y) + 2a^2(a^2-1) f(x) - 2(a^2-1) f(y), \tag{5}$$

for fixed integer a with $a \neq 0, \pm 1$.

In this paper, we consider the following a generalized quartic functional equation:

$$f(ax+by) + f(bx+ay) + \frac{1}{2}ab(a-b)^2 f(x-y) = (a^2-b^2)^2 [f(x) + f(y)] + \frac{1}{2}ab(a+b)^2 f(x+y), \tag{6}$$

for fixed integers a and b such that $a \neq 0$, $b \neq 0$, $a \pm b \neq 0$, for all $x, y \in X$. In fact, the generalized quartic functional equation (6) is following from the spirit of the pioneering Euler-Lagrange quartic functional equation (3) as well as Euler-Lagrange quadratic functional equation (1) introduced by Rassias: see [16] and [11], respectively. For the same reason as (1), (2), and (3), we call (6) a Euler-Lagrange-Rassias quartic functional equation. First of all, we obtain the general solution of Euler-Lagrange-Rassias quartic functional equation. To prove the stability problem for the Euler-Lagrange-Rassias quartic functional equation on various quasi-normed spaces, we may consider the following:

$$f(ax+y) + f(x+ay) + \frac{1}{2}a(a-1)^2 f(x-y) = (a^2-1)^2 [f(x) + f(y)] + \frac{1}{2}a(a+1)^2 f(x+y), \tag{7}$$

for fixed integer a with $a \neq 0$, $a \neq \pm 1$, for all $x, y \in X$.

We will use the following definitions to prove Hyers-Ulam-Rassias stability for the Euler-Lagrange-Rassias quartic functional equation in the quasi-β-normed and quasi fuzzy β-normed spaces. Let β be a real number with $0 < \beta \leq 1$ and let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$.

Definition 1. Let X be a linear space over a field $\mathbb{K}$. A *quasi β-norm* $\|\cdot\|$ is a real-valued function on X satisfying the following statements:

(1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$,

(2) $\|\lambda x\| = |\lambda|^\beta \cdot \|x\|$ for all $\lambda \in \mathbb{K}$ and all $x \in X$,

(3) there is a constant $K \geq 1$ such that $\|x+y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called a *quasi-β-normed space* if $\|\cdot\|$ is a quasi-β-norm on X. The smallest possible K is called the *modulus of concavity* of $\|\cdot\|$. A *quasi-β-Banach space* is a complete quasi-β-normed space.

A quasi β-norm $\|\cdot\|$ is called a (β, p)-norm $(0 < p \leq 1)$ if (3) takes the form $\|x+y\|^p \leq \|x\|^p + \|y\|^p$ for all $x, y \in X$. In this case, a quasi β-Banach space is called a (β, p)-Banach space; see [19, 20].

In 1984, Katsaras [21] and Wu and Fang [22] independently introduced a notion of a fuzzy norm and they gave the generalization of the Kolmogoroff normalized theorem for a fuzzy topological linear space. Since then, some mathematicians have defined fuzzy metrics and norms on a linear space from various points of view; see [23–27]. In 2003, Bag and Samanta [23] modified the definition of Cheng and Mordeson [28]. Bag and Samanta [23] introduced the following definition of fuzzy normed spaces. The notion of fuzzy stability of functional equations was given in the paper [29].

Definition 2. Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0, 1]$ is called a *fuzzy norm* on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$

(N_1) $N(x, t) = 0$ for $t \leq 0$;

(N_2) $x = 0$ if and only if $N(x, t) = 1$ for all $t > 0$;

(N_3) $N(cx, t) = N(x, t/|c|)$ if $c \neq 0$;

(N_4) $N(x+y, s+t) \geq \min\{N(x, s), N(y, t)\}$;

(N_5) $N(x, \cdot)$ is a nondecreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x, t) = 1$;

(N_6) for $x \neq 0$, $N(x, \cdot)$ is continuous on $\mathbb{R}$.

The pair (X, N) is called a *fuzzy normed vector space.*

Mirmostafaee [30] introduced a notion for a quasi fuzzy p-normed space as follows.

Definition 3. By *a quasi fuzzy norm*, one means a real vector space X, with a fuzzy subset N of $X \times \mathbb{R}$ and some $K \geq 1$ such that all axioms of fuzzy normed space in Definition 2 except (N_4) and

(N_4')

$$N(x+y, K(s+t)) \geq \min\{N(x, s), N(y, t)\} \quad (x, y \in X, s, t > 0) \tag{8}$$

hold.

A quasi fuzzy normed space (X, N) which satisfies

(N_4'')

$$N\left(x+y, \sqrt[p]{s+t}\right) \geq \min\left\{N\left(x, \sqrt[p]{s}\right), N\left(y, \sqrt[p]{t}\right)\right\} \quad (x, y \in X, s, t > 0), \tag{9}$$

for some $0 < p \leq 1$, is called a *quasi fuzzy p-norm*.

Definition 4. Let X be a real vector space. A quasi fuzzy p-norm $N : X \times \mathbb{R} \to [0, 1]$ is called a *quasi fuzzy (β, p)-norm* on X if (N_3) in Definition 2 takes the form

(N_3')

$$N(cx,t) = N\left(x, \frac{t}{|c|^{\beta}}\right) \quad (c \neq 0, 0 < \beta \leq 1). \tag{10}$$

Example 5. Let $(X, \|\cdot\|)$ be a real normed space. Define

$$N(x,t) = \begin{cases} \dfrac{t}{t+\|x\|} & \text{when } t > 0,\ t \in \mathbb{R} \\ 0 & \text{when } t \leq 0, \end{cases} \tag{11}$$

where $x \in X$. Then (X, N) is a quasi fuzzy (β, p)-normed space.

Note that when $p = 1$, we call the quasi fuzzy (β, p)-norm a quasi fuzzy β-norm.

Definition 6. Let (X, N) be a quasi fuzzy β-normed vector space. A sequence $\{x_n\}$ in X is said to be *convergent* or *converge* if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and one denotes it by $N - \lim_{n\to\infty} x_n = x$.

Definition 7. Let (X, N) be a quasi fuzzy β-normed vector space. A sequence $\{x_n\}$ in X is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$ and all integer $d > 0$, one has $N(x_{n+d} - x_n, t) > 1 - \varepsilon$.

It is well known that every convergent sequence in a quasi fuzzy β-normed vector space is Cauchy. If each Cauchy sequence is convergent, then the quasi fuzzy β-normed space is said to be *quasi fuzzy complete* and the quasi fuzzy β-normed vector space is called a *quasi fuzzy Banach space*.

2. Euler-Lagrange-Rassias Quartic Functional Equations

Let X, Y be real vector spaces. In this section, we will investigate that the functional equation (3) is equivalent to the presented functional equation (6).

Lemma 8. *A mapping $f : X \to Y$ satisfies the functional equation* (3) *if and only if f satisfies*

$$\begin{aligned} f(2x+y) + f(x+2y) = {} & 9f(x) + 9f(y) \\ & + 9f(x+y) - f(x-y), \end{aligned} \tag{12}$$

for all $x, y \in X$.

Proof. It follows from [31, 32]. □

Theorem 9. *A mapping $f : X \to Y$ satisfies the functional equation* (3) *if and only if f satisfies the functional equation* (7).

Proof. It is easy to verify that $f(0) = 0$ by letting $x = y = 0$ in (3). We will show this induction on a. Lemma 8 implies that it is true when $a = 2$, and we may assume it holds for all a. Now, letting $x = (a-1)x + y$ and $y = x$ in (3), we have

$$\begin{aligned} & f((a+1)x+y) + f((a-3)x+y) \\ & \quad = 4f(ax+y) + 4f((a-2)x+y) \\ & \qquad + 24f(x) - 6f((a-1)x+y), \end{aligned} \tag{13}$$

for all $x, y \in X$. After the switching x and y in the previous equation (13),

$$\begin{aligned} & f(x+(a+1)y) + f(x+(a-3)y) \\ & \quad = 4f(x+ay) + 4f(x+(a-2)y) \\ & \qquad + 24f(y) - 6f(x+(a-1)y), \end{aligned} \tag{14}$$

for all $x, y \in X$. Adding two equations (13) and (14), we have

$$\begin{aligned} & f(x+(a+1)y) + f(x+(a+1)y) \\ & \quad = -[f((a-3)x+y) + f(x+(a-3)y)] \\ & \qquad + 4[f(ax+y) + f(x+ay)] \\ & \qquad + 4[f((a-2)x+y) + f(x+(a-2)y)] \\ & \qquad - 6[f((a-1)x+y) + f(x+(a-1)y)] \\ & \qquad + 24[f(x) + f(y)], \end{aligned} \tag{15}$$

for all $x, y \in X$. The induction steps imply that

$$\begin{aligned} & f(x+(a+1)y) + f(x+(a+1)y) \\ & \quad = -\frac{1}{2}a^3 f(x-y) - \frac{1}{2}a^2 f(x-y) \\ & \qquad + \frac{1}{2}a^3 f(x+y) + \frac{5}{2}a^2 f(x+y) \\ & \qquad + 4af(x+y) + 2f(x+y) \\ & \qquad + \left(a^4 + 4a^3 + 4a^2\right)[f(x) + f(y)] \\ & \quad = -\frac{1}{2}(a+1)((a+1)-1)^2 f(x-y) \\ & \qquad + \frac{1}{2}(a+1)((a+1)+1)^2 f(x+y) \\ & \qquad + ((a+1)^2 - 1)^2 [f(x) + f(y)], \end{aligned} \tag{16}$$

for all $x, y \in X$. Hence we have

$$\begin{aligned} & f((a+1)x+y) + f(x+(a+1)y) \\ & \qquad + \frac{1}{2}(a+1)((a+1)-1)^2 f(x-y) \\ & \quad = ((a+1)^2 - 1)^2 [f(x) + f(y)] \\ & \qquad + \frac{1}{2}(a+1)((a+1)+1)^2 f(x+y), \end{aligned} \tag{17}$$

for all $x, y \in X$, as desired. □

Note that $f(ax) = a^4 f(x)$ by letting $y = 0$ in (7).

Lemma 10. *A mapping $f : X \to Y$ satisfies the functional equation* (7) *if and only if f satisfies the functional equation* (6).

Proof. It is easy to show that $f(0) = 0$ and $f(ax) = a^4 f(x)$ by putting $x = y = 0$ and $y = 0$ in (7), respectively. By letting $y = by$ in (7), we have

$$f(ax+by) + f(x+aby) + \frac{1}{2}a(a-1)^2 f(x-by) = \left(a^2-1\right)^2 [f(x) + f(by)] + \frac{1}{2}a(a+1)^2 f(x+by). \tag{18}$$

Also, switching x and y in the above equation and then adding two equations, we get

$$\begin{aligned} f(ax+by) + f(bx+ay) &= -[f(abx+y) + f(x+aby)] + \left(a^2-1\right)^2 \\ &\quad \times \left[f(x) + f(y) + b^4 f(x) + b^4 f(y)\right] \\ &\quad + \frac{1}{2}a(a+1)^2 [f(bx+y) + f(x+by)] \\ &\quad - \frac{1}{2}a(a-1)^2 [f(bx-y) + f(x-by)]. \end{aligned} \tag{19}$$

Then (7) implies that

$$f(ax+by) + f(bx+ay) + \frac{1}{2}ab(a-b)^2 f(x-y) = \left(a^2-b^2\right)^2 [f(x) + f(y)] + \frac{1}{2}ab(a+b)^2 f(x+y). \tag{20}$$

□

Corollary 11. *A mapping $f : X \to Y$ satisfies the functional equation* (3) *if and only if f satisfies the functional equation* (6).

3. Stability in Quasi-β-Normed Spaces

Throughout this section, let X be a quasi-β-normed space and let Y be a quasi β-Banach space with a quasi β-norm $\|\cdot\|_Y$. Let K be the modulus of concavity of $\|\cdot\|_Y$. We will investigate the Hyers-Ulam-Rassias stability problem for the functional equation (7). For a given mapping $f : X \to Y$ and all fixed integers a with $a \neq 0$, $a \neq \pm 1$, let

$$\begin{aligned} D_a f(x,y) := {} & f(ax+y) + f(x+ay) \\ & + \frac{1}{2}a(a-1)^2 f(x-y) \\ & - \left(a^2-1\right)^2 [f(x) + f(y)] \\ & - \frac{1}{2}a(a+1)^2 f(x+y), \end{aligned} \tag{21}$$

for x and y in X.

Theorem 12. *Suppose that there exists a mapping $\phi : X^2 \to \mathbb{R}^+ := [0,\infty)$ for which a mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$\|D_a f(x,y)\|_Y \le \phi(x,y), \tag{22}$$

and the series $\sum_{j=0}^{\infty} (K/|a|^{4\beta})^j \phi(a^j x, a^j y)$ converges for all $x, y \in X$. Then there exists a unique Euler-Lagrange-Rassias quartic mapping $Q : X \to Y$ which satisfies (7) *and the inequality*

$$\|f(x) - Q(x)\|_Y \le \frac{K}{|a|^{4\beta}} \sum_{j=0}^{\infty} \left(\frac{K}{|a|^{4\beta}}\right)^j \phi\left(a^j x, 0\right), \tag{23}$$

for all $x \in X$.

Proof. By letting $y = 0$ in (22) and $f(0) = 0$, we have

$$\begin{aligned} &\left\| f(ax) + f(x) + \frac{1}{2}a(a-1)^2 f(x) - \frac{1}{2}a(a+1)^2 \right. \\ &\quad \left. - \left(a^2-1\right)^2 f(x) \right\|_Y \\ &= \left\| f(ax) - a^4 f(x) \right\|_Y \\ &= |a|^{4\beta} \left\| f(x) - \frac{1}{a^4} f(ax) \right\|_Y \le \phi(x,0), \end{aligned} \tag{24}$$

that is,

$$\left\| f(x) - \frac{1}{a^4} f(ax) \right\|_Y \le \frac{1}{|a|^{4\beta}} \phi(x,0), \tag{25}$$

for all $x \in X$. For any positive integer m, we have

$$\begin{aligned} &\left\| \left(\frac{1}{a^4}\right)^m f(a^m x) - \left(\frac{1}{a^4}\right)^{m+1} f\left(a^{m+1} x\right) \right\|_Y \\ &\quad \le \frac{1}{|a|^{4\beta}} \left(\frac{1}{|a|^{4\beta}}\right)^m \phi(a^m x, 0), \end{aligned} \tag{26}$$

for all $x \in X$. For any positive integers n and m with $m < n$,

$$\begin{aligned} &\left\| \left(\frac{1}{a^4}\right)^m f(a^m x) - \left(\frac{1}{a^4}\right)^n f(a^n x) \right\|_Y \\ &\quad \le \frac{1}{K^{m-1}} \frac{1}{|a|^{4\beta}} \sum_{j=m}^{n-1} \left(\frac{K}{|a|^{4\beta}}\right)^j \phi\left(a^j x, 0\right), \end{aligned} \tag{27}$$

for all $x \in X$. By letting $m = 0$, we have

$$\begin{aligned} &\left\| f(x) - \left(\frac{1}{a^4}\right)^n f(a^n x) \right\|_Y \\ &\quad \le \frac{K}{|a|^{4\beta}} \sum_{j=0}^{n-1} \left(\frac{K}{|a|^{4\beta}}\right)^j \phi\left(a^j x, 0\right), \end{aligned} \tag{28}$$

for all $x \in X$ and $n \in \mathbb{N}$. Since the right-hand side of the previous inequality tends to 0 as $n \to \infty$, $\{(1/a^4)^n f(a^n x)\}$ is a Cauchy sequence in the quasi β-Banach space Y. Thus we may define

$$Q(x) = \lim_{n\to\infty}\left(\frac{1}{a^4}\right)^n f(a^n x), \tag{29}$$

for all $x \in X$. Hence we have the inequality (23). Since $K \geq 1$, replacing x and y by $a^n x$ and $a^n y$, respectively, and dividing by $|a|^{4\beta n}$ in (22), we have

$$\left(\frac{1}{|a|^{4\beta}}\right)^n \|D_a f(a^n x, a^n y)\|_Y \leq \left(\frac{K}{|a|^{4\beta}}\right)^n \phi(a^n x, a^n y), \tag{30}$$

for all $x, y \in X$. By taking $n \to \infty$, the definition of Q implies that Q satisfies (7) for all $x, y \in X$; that is, Q is the Euler-Lagrange-Rassias quartic mapping. It is left to show that the quadratic mapping Q is unique. Assume that there exists $T : X \to Y$ satisfying (7) and (23). Then

$$\begin{aligned}\|T(x) - Q(x)\|_Y &= \left(\frac{1}{|a|^{4\beta}}\right)^n \|T(a^n x) - Q(a^n x)\|_Y \\ &\leq \left(\frac{1}{|a|^{4\beta}}\right)^n K\Big(\|T(a^n x) - f(a^n x)\|_Y \\ &\qquad + \|f(a^n x) - Q(a^n x)\|_Y\Big) \\ &\leq \frac{2}{|a|^{4\beta n}}\sum_{j=n}^{\infty}\left(\frac{K}{|a|^{4\beta}}\right)^j \phi(a^j x, 0),\end{aligned} \tag{31}$$

for all $x \in X$. By letting $n \to \infty$, we immediately have the uniqueness of Q. □

Theorem 13. *Suppose that there exists a mapping $\phi : X^2 \to \mathbb{R}^+ := [0,\infty)$ for which a mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$\|D_a f(x, y)\|_Y \leq \phi(x, y), \tag{32}$$

and the series $\sum_{j=1}^{\infty}(|a|^{4\beta}K)^j \phi(a^{-j}x, a^{-j}y)$ converges for all $x, y \in X$. Then there exists a unique Euler-Lagrange-Rassias quartic mapping $Q : X \to Y$ which satisfies (7) and the inequality

$$\|f(x) - Q(x)\|_Y \leq \sum_{j=1}^{\infty}\left(|a|^{4\beta}K\right)^j \phi\left(a^{-j}x, 0\right), \tag{33}$$

for all $x \in X$.

Proof. If x is replaced by $(1/a)x$ in inequality (25), we have

$$\left\|f(x) - a^4 f\left(\frac{1}{a}x\right)\right\|_Y \leq \phi(x, 0), \tag{34}$$

for all $x \in X$. The remains of the proof follow from the proof of Theorem 12. □

4. Stability in Quasi Fuzzy β-Normed Spaces

Let us fix some notations which will be used throughout this section. We assume X is a vector space and (Y, N) is a quasi fuzzy β-Banach space. We will prove the Hyers-Ulam-Rassias stability of the functional equation satisfying equation (7) in quasi fuzzy β-Banach space.

Theorem 14. *Let $\phi : X^2 \to [0,\infty)$ be a function such that for some $0 < |\alpha| < |a|^4$*

$$\begin{aligned}&N'(\phi(ax, 0), t) \geq N'(\alpha\phi(x, 0), t), \\ &\lim_{n\to\infty} N'\left(\phi(a^n x, a^n y), |a|^{4n\beta}t\right) = 1,\end{aligned} \tag{35}$$

for all $x, y \in X$ and all $t > 0$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$N(D_a f(x, y), t) \geq N'(\phi(x, y), t), \tag{36}$$

for all $x, y \in X$ and all $t > 0$.

Then $Q(x) := N - \lim_{n\to\infty}(1/a^{4n}) f(a^n x)$ exists for each $x \in X$ and defines a unique Euler-Lagrange-Rassias quartic mapping $Q : X \to Y$ such that

$$N(f(x) - Q(x), t) \geq N'\left(\phi(x, 0), \frac{|a|^{4\beta} - |\alpha|^{\beta}}{2}t\right), \tag{37}$$

for all $x \in X$ and all $t > 0$.

Proof. Let $y = 0$ in inequality (36). Since $f(0) = 0$, we have

$$\begin{aligned}&N(D_a f(x, 0), t) \\ &\quad = N\left(\frac{1}{a^4} f(ax) - f(x), \frac{t}{|a|^{4\beta}}\right) \geq N'(\phi(x, 0), t),\end{aligned} \tag{38}$$

for all $x \in X$ and all $t > 0$. Replacing x by $a^n x$ in inequality (38),

$$N\left(\frac{1}{a^4} f\left(a^{n+1}x\right) - f(a^n x), \frac{t}{|a|^{4\beta}}\right) \geq N'(\phi(a^n x, 0), t), \tag{39}$$

that is,

$$\begin{aligned}&N\left(\left(\frac{1}{a^4}\right)^{n+1} f\left(a^{n+1}x\right) - \left(\frac{1}{a^4}\right)^n f(a^n x), \frac{t}{|a|^{4\beta}}\frac{1}{|a|^{4n\beta}}\right) \\ &\quad \geq N'(\phi(a^n x, 0), t),\end{aligned} \tag{40}$$

for all $x \in X$, $t > 0$ and $n \geq 0$. Since $N'(\phi(a^n x, 0), t) \geq N'(\phi(x, 0), (t/|\alpha|^{n\beta}))$,

$$\begin{aligned}&N\left(\left(\frac{1}{a^4}\right)^{n+1} f\left(a^{n+1}x\right) - \left(\frac{1}{a^4}\right)^n f(a^n x), \frac{t}{|a|^{4\beta}}\frac{1}{|a|^{4n\beta}}\right) \\ &\quad \geq N'\left(\phi(x, 0), \frac{t}{|\alpha|^{n\beta}}\right).\end{aligned} \tag{41}$$

By letting $t = |\alpha|^{n\beta} t$ in the previous inequality,

$$N\left(\left(\frac{1}{a^4}\right)^{n+1} f\left(a^{n+1}x\right) - \left(\frac{1}{a^4}\right)^{n} f\left(a^{n}x\right), \frac{t}{|a|^{4\beta}} \cdot \left(\frac{|\alpha|}{|a|^4}\right)^{n\beta}\right) \geq N'\left(\phi\left(x,0\right),t\right), \tag{42}$$

for all $x \in X$ and all $t > 0$. Hence we get

$$\begin{aligned} N&\left(\left(\frac{1}{a^4}\right)^{n} f\left(a^{n}x\right) - f\left(x\right), \sum_{j=0}^{n-1} \frac{t}{|a|^{4\beta}} \cdot \left(\frac{|\alpha|}{|a|^4}\right)^{j\beta}\right) \\ &\geq \min \bigcup_{j=0}^{n-1} \left[N\left(\left(\frac{1}{a^4}\right)^{j+1} f\left(a^{j+1}x\right) - \left(\frac{1}{a^4}\right)^{j} f\left(a^{j}x\right), \frac{t}{|a|^{4\beta}} \cdot \left(\frac{|\alpha|}{|a|^4}\right)^{j\beta}\right)\right] \\ &\geq N'\left(\phi\left(x,0\right),t\right), \end{aligned} \tag{43}$$

for all $x \in X$ and all $t > 0$. Letting $x = a^m x$ in the previous inequality, we have

$$\begin{aligned} N&\left(\left(\frac{1}{a^4}\right)^{n} f\left(a^{n+m}x\right) - f\left(a^{m}x\right), \sum_{j=0}^{n-1} \frac{t}{|a|^{4\beta}} \cdot \left(\frac{|\alpha|}{|a|^4}\right)^{j\beta}\right) \\ &\geq N'\left(\phi\left(a^m x,0\right),t\right) \geq N'\left(\phi\left(x,0\right), \frac{t}{|\alpha|^{m\beta}}\right), \end{aligned} \tag{44}$$

that is,

$$\begin{aligned} N&\left(\left(\frac{1}{a^4}\right)^{n+m} f\left(a^{n+m}x\right) - \left(\frac{1}{a^4}\right)^{m} f\left(a^{m}x\right), \frac{t}{|a|^{4m\beta}} \sum_{j=0}^{n-1} \frac{1}{|a|^{4\beta}} \cdot \left(\frac{|\alpha|}{|a|^4}\right)^{j\beta}\right) \\ &\geq N'\left(\phi\left(x,0\right), \frac{t}{|\alpha|^{m\beta}}\right), \end{aligned} \tag{45}$$

for all $x \in X$ and all $t > 0$. Letting $t = |\alpha|^{m\beta} t$, we have

$$\begin{aligned} N&\left(\left(\frac{1}{a^4}\right)^{n+m} f\left(a^{n+m}x\right) - \left(\frac{1}{a^4}\right)^{m} f\left(a^{m}x\right), \frac{t}{|a|^{4\beta}} \sum_{j=m}^{n+m-1} \left(\frac{|\alpha|}{|a|^4}\right)^{j\beta}\right) \\ &\geq N'\left(\phi\left(x,0\right),t\right), \end{aligned} \tag{46}$$

for all $x \in X$, $t > 0$ and $n, m \geq 0$. Hence $\{(1/a^{4n}) f(a^n x)\}$ is a Cauchy sequence in the quasi fuzzy β-Banach space (Y, N). Thus, we may define

$$Q\left(x\right) = N - \lim_{n \to \infty} \frac{1}{a^{4n}} f\left(a^n x\right), \tag{47}$$

for all $x \in X$. Hence inequality (43) implies that

$$\begin{aligned} N&\left(Q\left(x\right) - f\left(x\right),t\right) \\ &\geq \min\left\{N\left(Q\left(x\right) - \left(\frac{1}{a^4}\right)^{n} f\left(a^n x\right), \frac{t}{2}\right), N\left(\left(\frac{1}{a^4}\right)^{n} f\left(a^n x\right) - f\left(x\right), \frac{t}{2}\right)\right\} \\ &\geq N'\left(\phi\left(x,0\right), \frac{t}{\left(2/|a|^{4\beta}\right)\sum_{j=0}^{n-1}\left(|\alpha|/|a|^4\right)^{j\beta}}\right), \end{aligned} \tag{48}$$

for n large enough and all $x \in X$. Taking the limit as $n \to \infty$ and using (N_6), we have

$$N\left(Q\left(x\right) - f\left(x\right),t\right) \geq N'\left(\phi\left(x,0\right), \frac{|a|^{4\beta} - |\alpha|^{\beta}}{2} t\right), \tag{49}$$

for all $x \in X$. Hence it satisfies inequality (37). Now letting $x = a^n x$ and $y = a^n y$ in (36),

$$N\left(D_a f\left(a^n x, a^n y\right),t\right) \geq N'\left(\phi\left(a^n n, a^n y\right),t\right), \tag{50}$$

for all $x \in X$ and all $t > 0$. This implies that

$$N\left(\left(\frac{1}{a^4}\right)^{n} D_a f\left(a^n x, a^n y\right),t\right) \geq N'\left(\phi\left(a^n n, a^n y\right), |a|^{4\beta} t\right), \tag{51}$$

for all $x \in X$ and all $t > 0$. Since $N'(\phi(a^n n, a^n y), |a|^{4\beta} t) = 1$, we may conclude that the mapping Q satisfies (7); that is, Q is the Euler-Lagrange-Rassias quartic mapping. It is left to show that the quartic mapping Q is unique. Assume there is another $T : X \to Y$ satisfying (7) and inequality (37). For each $x \in X$, clearly $Q(a^n x) = a^{4n} Q(x)$ and $T(a^n x) = a^{4n} T(x)$ for all $n \in \mathbb{N}$.

$$\begin{aligned} N\left(T\left(x\right) - Q\left(x\right),t\right) &= N\left(\frac{1}{a^{4n}} T\left(a^n x\right) - \frac{1}{a^{4n}} Q\left(a^n x\right),t\right) \\ &= N\left(T\left(a^n x\right) - Q\left(a^n x\right), |a|^{4n\beta} t\right) \end{aligned}$$

$$\geq \min\left\{N\left(T(a^n x) - f(a^n x), \frac{|a|^{4n\beta}t}{2}\right), N\left(f(a^n x) - Q(a^n x), \frac{|a|^{4n\beta}t}{2}\right)\right\}$$
$$\geq N'\left(\phi(a^n x, 0), \frac{|a|^{4\beta} - |\alpha|^{\beta}}{2} \cdot \frac{|a|^{4n\beta}t}{2}\right)$$
$$\geq N'\left(\phi(x,0), \left(\frac{|a|^4}{|\alpha|}\right)^{n\beta} \cdot \frac{|a|^{4\beta} - |\alpha|^{\beta}}{4}t\right), \quad (52)$$

for all $x \in X$ and $t > 0$. Since $0 < |\alpha| < |a|^4$, we have $\lim_{n\to\infty}(|a|^4/|\alpha|)^{n\beta} = \infty$. Hence $N(T(x) - Q(x), t) = 1$; that is, the mapping Q is unique, as desired. □

Theorem 15. *Let $\phi : X^2 \to [0,\infty)$ be a function such that for some $|\alpha| > |a|^4$*

$$N'\left(\phi\left(\frac{1}{a}x, 0\right), t\right) \geq N'(\alpha\phi(x,0), t),$$
$$\lim_{n\to\infty} N'\left(\phi(a^{-n}x, a^{-n}y), \frac{1}{|a|^{4n\beta}}t\right) = 1, \quad (53)$$

for all $x, y \in X$ and all $t > 0$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$N(D_a f(x,y), t) \geq N'(\phi(x,y), t), \quad (54)$$

for all $x, y \in X$ and all $t > 0$.

Then $Q(x) := N - \lim_{n\to\infty} a^{4n} f((1/a^n)x)$ exists for each $x \in X$ and defines a unique Euler-Lagrange-Rassias quartic mapping $Q : X \to Y$ such that

$$N(f(x) - Q(x), t) \geq N'\left(\phi(x,0), \frac{|\alpha|^{\beta} - |a|^{4\beta}}{2|\alpha|^{2\beta}}t\right), \quad (55)$$

for all $x \in X$ and all $t > 0$.

Proof. The techniques are completely similar to the proof of Theorem 14. Hence we present some key idea of this proof. Let $y = 0$ in inequality (54). Since $f(0) = 0$, we have

$$N(D_a f(x,0), t) = N\left(f(ax) - a^4 f(x), t\right) \geq N'(\phi(x,0), t), \quad (56)$$

for all $x \in X$ and all $t > 0$. Replacing x by $(1/a)x$ in inequality (56), we have

$$N\left(f(x) - a^4 f\left(\frac{1}{a}x\right), t\right) \geq N'\left(\phi\left(\frac{1}{a}x, 0\right), t\right) \geq N'\left(\phi(x,0), \frac{1}{|\alpha|^{\beta}}t\right) \quad (57)$$

or

$$N\left(f(x) - a^4 f\left(\frac{1}{a}x\right), |\alpha|^{\beta}t\right) \geq N'(\phi(x,0), t), \quad (58)$$

for all $x \in X$ and all $t > 0$. For positive integers n and m,

$$N\left(a^{4(n+m)} f\left(\frac{1}{a^{n+m}}x\right) - a^{4m} f\left(\frac{1}{a^m}x\right), |\alpha|^{\beta} \sum_{j=m}^{n+m-1}\left(\frac{|a|^4}{|\alpha|}\right)^{j\beta} t\right) \geq N'(\phi(x,0), t), \quad (59)$$

for all $x \in X$ and $t > 0$. Hence we may conclude that $\{a^{4n} f((1/a^n)x)\}$ is a Cauchy sequence in the quasi fuzzy β-Banach space (Y, N). Thus we may define

$$Q(x) = N - \lim_{n\to\infty} a^{4n} f\left(\frac{1}{a^n}x\right), \quad (60)$$

for all $x \in X$. Also, for any positive integer n, we get

$$N\left(a^{4n} f\left(\frac{1}{a^n}x\right) - f(x), t\right) \geq N'\left(\phi(x,0), \frac{t}{|\alpha|^{\beta}\sum_{j=0}^{n-1}\left(|a|^4/|\alpha|\right)^{j\beta}}\right), \quad (61)$$

for all $x \in X$ and all $t > 0$. This implies inequality (55). □

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References

[1] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience Publishers, New York, NY, USA, 1960.

[2] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, pp. 222–224, 1941.

[3] T. Aoki, "On the stability of the linear transformation in Banach spaces," *Journal of the Mathematical Society of Japan*, vol. 2, pp. 64–66, 1950.

[4] T. M. Rassias, "On the stability of the linear mapping in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 72, no. 2, pp. 297–300, 1978.

[5] Z. Gajda, "On stability of additive mappings," *International Journal of Mathematics and Mathematical Sciences*, vol. 14, no. 3, pp. 431–434, 1991.

[6] S. Czerwik, "On the stability of the quadratic mapping in normed spaces," *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, vol. 62, pp. 59–64, 1992.

[7] T. M. Rassias, "On the stability of functional equations in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 251, no. 1, pp. 264–284, 2000.

[8] T. M. Rassias and P. Šemrl, "On the Hyers-Ulam stability of linear mappings," *Journal of Mathematical Analysis and Applications*, vol. 173, no. 2, pp. 325–338, 1993.

[9] T. M. Rassias and K. Shibata, "Variational problem of some quadratic functionals in complex analysis," *Journal of Mathematical Analysis and Applications*, vol. 228, no. 1, pp. 234–253, 1998.

[10] J.-H. Bae and W.-G. Park, "On the generalized Hyers-Ulam-Rassias stability in Banach modules over a C^x-algebra," *Journal of Mathematical Analysis and Applications*, vol. 294, no. 1, pp. 196–205, 2004.

[11] J. M. Rassias, "On the stability of the Euler-Lagrange functional equation," *Chinese Journal of Mathematics*, vol. 20, no. 2, pp. 185–190, 1992.

[12] K.-W. Jun and H.-M. Kim, "On the stability of Euler-Lagrange type cubic mappings in quasi-Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 332, no. 2, pp. 1335–1350, 2007.

[13] H.-M. Kim and M.-Y. Kim, "Generalized stability of Euler-Lagrange quadratic functional equation," *Abstract and Applied Analysis*, vol. 2012, Article ID 219435, 16 pages, 2012.

[14] A. Zivari-Kazempour and M. E. Gordji, "Generalized Hyers-Ulam stabilities of an Euler-Lagrange-Rassias quadratic functional equation," *Asian-European Journal of Mathematics*, vol. 5, no. 1, Article ID 1250014, 7 pages, 2012.

[15] T. Z. Xu and J. M. Rassias, "Stability of general multi-Euler-Lagrange quadratic functional equations in non-archimedean fuzzy normed spaces," *Advances in Difference Equations*, vol. 2012, no. 1, 2012.

[16] J. M. Rassias, "Solution of the Ulam stability problem for quartic mappings," *Glasnik Matematički*, vol. 34, no. 2, pp. 243–252, 1999.

[17] J. K. Chung and P. K. Sahoo, "On the general solution of a quartic functional equation," *Bulletin of the Korean Mathematical Society*, vol. 40, no. 4, pp. 565–576, 2003.

[18] Y.-S. Lee and S.-Y. Chung, "Stability of quartic functional equations in the spaces of generalized functions," *Advances in Difference Equations*, vol. 2009, Article ID 838347, 16 pages, 2009.

[19] Y. Benyamini and J. Lindenstrauss, *Geometric Nonlinear Functional Analysis. Vol. 1*, vol. 48, American Mathematical Society, Providence, RI, USA, 2000.

[20] S. Rolewicz, *Metric Linear Spaces*, Polish Scientific Publishers, Warsaw, Poland, 2nd edition, 1984.

[21] A. K. Katsaras, "Fuzzy topological vector spaces. II," *Fuzzy Sets and Systems*, vol. 12, no. 2, pp. 143–154, 1984.

[22] C. Wu and J. Fang, "Fuzzy generalization of Klomogoros theorem," *Journal of Harbin Institute of Technology*, vol. 1, pp. 1–7, 1984.

[23] T. Bag and S. K. Samanta, "Finite dimensional fuzzy normed linear spaces," *Journal of Fuzzy Mathematics*, vol. 11, no. 3, pp. 687–705, 2003.

[24] C. Felbin, "Finite-dimensional fuzzy normed linear space," *Fuzzy Sets and Systems*, vol. 48, no. 2, pp. 239–248, 1992.

[25] S. V. Krishna and K. K. M. Sarma, "Separation of fuzzy normed linear spaces," *Fuzzy Sets and Systems*, vol. 63, no. 2, pp. 207–217, 1994.

[26] J.-Z. Xiao and X.-H. Zhu, "Fuzzy normed space of operators and its completeness," *Fuzzy Sets and Systems*, vol. 133, no. 3, pp. 389–399, 2003.

[27] A. K. Mirmostafaee, M. Mirzavaziri, and M. S. Moslehian, "Fuzzy stability of the Jensen functional equation," *Fuzzy Sets and Systems*, vol. 159, no. 6, pp. 730–738, 2008.

[28] S. C. Cheng and J. N. Mordeson, "Fuzzy linear operators and fuzzy normed linear spaces," *Bulletin of the Calcutta Mathematical Society*, vol. 86, no. 5, pp. 429–436, 1994.

[29] A. K. Mirmostafaee and M. S. Moslehian, "Fuzzy versions of Hyers-Ulam-Rassias theorem," *Fuzzy Sets and Systems*, vol. 159, no. 6, pp. 720–729, 2008.

[30] A. K. Mirmostafaee, "A fixed point approach to almost quartic mappings in quasi fuzzy normed spaces," *Fuzzy Sets and Systems*, vol. 160, no. 11, pp. 1653–1662, 2009.

[31] H.-M. Kim, "On the stability problem for a mixed type of quartic and quadratic functional equation," *Journal of Mathematical Analysis and Applications*, vol. 324, no. 1, pp. 358–372, 2006.

[32] K. Jun and H. Kim, "Solution of Ulam stability problem for approximately biquadratic mappings and functional inequalities," *Journal of Inequalities and Applications*. In press.