

Research Article

Lazer-Leach Type Conditions on Periodic Solutions of Liénard Equation with a Deviating Argument at Resonance

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We study the existence of periodic solutions of Liénard equation with a deviating argument $x'' + f(x)x' + n^2x + g(x(t - \tau)) = p(t)$, where $f, g, p : \mathbf{R} \rightarrow \mathbf{R}$ are continuous and p is 2π -periodic, $0 \leq \tau < 2\pi$ is a constant, and n is a positive integer. Assume that the limits $\lim_{x \rightarrow \pm\infty} g(x) = g(\pm\infty)$ and $\lim_{x \rightarrow \pm\infty} F(x) = F(\pm\infty)$ exist and are finite, where $F(x) = \int_0^x f(u)du$. We prove that the given equation has at least one 2π -periodic solution provided that one of the following conditions holds: $2\cos(n\tau)[g(+\infty) - g(-\infty)] \neq \int_0^{2\pi} p(t)\sin(\theta + nt)dt$, for all $\theta \in [0, 2\pi]$, $2n\cos(n\tau)[F(+\infty) - F(-\infty)] \neq \int_0^{2\pi} p(t)\sin(\theta + nt)dt$, for all $\theta \in [0, 2\pi]$, $2[g(+\infty) - g(-\infty)] - 2n\sin(n\tau)[F(+\infty) - F(-\infty)] \neq \int_0^{2\pi} p(t)\sin(\theta + nt)dt$, for all $\theta \in [0, 2\pi]$, $2n[F(+\infty) - F(-\infty)] - 2\sin(n\tau)[g(+\infty) - g(-\infty)] \neq \int_0^{2\pi} p(t)\sin(\theta + nt)dt$, for all $\theta \in [0, 2\pi]$.

1. Introduction

We are concerned with the existence of periodic solutions of Liénard equation with a deviating argument as follows:

$$x'' + f(x)x' + n^2x + g(x(t - \tau)) = p(t), \quad (1)$$

where $f, g, p : \mathbf{R} \rightarrow \mathbf{R}$ are continuous and p is 2π -periodic, $0 \leq \tau < 2\pi$ is a constant, and n is a positive integer.

In recent years, the periodic problem of Liénard equations with a deviating argument has been widely studied because of its background in applied sciences (see [1–8] and the references cited therein).

In the case when $f(x) \equiv 0$, for all $x \in \mathbf{R}$ and $\tau = 0$, (1) becomes

$$x'' + n^2x + g(x) = p(t). \quad (2)$$

Assume that limits

$$(h_1) \lim_{x \rightarrow \pm\infty} g(x) = g(\pm\infty)$$

exist and are finite. Lazer and Leach [9] proved that (2) has at least one 2π -periodic solution provided that the following condition holds:

$$2[g(+\infty) - g(-\infty)] \neq \int_0^{2\pi} p(t)\sin(\theta + nt)dt, \quad \forall \theta \in [0, 2\pi]. \quad (3)$$

Assume that, besides (h_1) , the limits

$$(h_2) \lim_{x \rightarrow \pm\infty} F(x) = F(\pm\infty)$$

exist and are finite, where $F(x) = \int_0^x f(u)du$. It was proved in [10] that the following equation:

$$x'' + f(x)x' + n^2x + g(x(t - \tau)) = p(t) \quad (4)$$

has at least 2π -periodic solution provided that one of the following conditions holds:

$$\begin{aligned}
 & 2[g(+\infty) - g(-\infty)] \\
 & \neq \int_0^{2\pi} p(t) \sin(\theta + nt) dt, \quad \forall \theta \in [0, 2\pi], \\
 & 2n[F(+\infty) - F(-\infty)] \\
 & \neq \int_0^{2\pi} p(t) \sin(\theta + nt) dt, \quad \forall \theta \in [0, 2\pi].
 \end{aligned} \tag{5}$$

In the case when $f(x) \equiv 0$, for all $x \in \mathbf{R}$ and $\tau \neq 0$, (1) becomes as follows:

$$x'' + n^2x + g(x(t - \tau)) = p(t). \tag{6}$$

When the condition (h_1) holds, it was proved in [5] that (6) has at least one 2π -periodic solution provided that the condition (3) holds.

In the present paper, we deal with the existence of periodic solutions of (1) by assuming (h_1) and (h_2) . By using the continuation theorem [11], we prove the following result.

Theorem 1. *Assume that the conditions (h_1) and (h_2) hold. Then (1) has at least one 2π -periodic solution provided that one of the following conditions is satisfied:*

$$\begin{aligned}
 & 2 \cos(n\tau) [g(+\infty) - g(-\infty)] \\
 & \neq \int_0^{2\pi} p(t) \sin(\theta + nt) dt, \quad \forall \theta \in [0, 2\pi], \\
 & 2n \cos(n\tau) [F(+\infty) - F(-\infty)] \\
 & \neq \int_0^{2\pi} p(t) \sin(\theta + nt) dt, \quad \forall \theta \in [0, 2\pi], \\
 & 2[g(+\infty) - g(-\infty)] - 2n \sin(n\tau) [F(+\infty) - F(-\infty)] \\
 & \neq \int_0^{2\pi} p(t) \sin(\theta + nt) dt, \quad \forall \theta \in [0, 2\pi], \\
 & 2n[F(+\infty) - F(-\infty)] - 2 \sin(n\tau) [g(+\infty) - g(-\infty)] \\
 & \neq \int_0^{2\pi} p(t) \sin(\theta + nt) dt, \quad \forall \theta \in [0, 2\pi].
 \end{aligned} \tag{7}$$

Remark 2. Let us denote by Φ the function on the right-hand side of four inequalities above, namely,

$$\Phi(\theta) = \int_0^{2\pi} p(t) \sin(\theta + nt) dt, \quad \theta \in [0, 2\pi]. \tag{8}$$

Then Φ can be expressed in the following form:

$$\Phi(\theta) = A \sin \theta + B \cos \theta, \quad \theta \in [0, 2\pi], \tag{9}$$

where

$$A = \int_0^{2\pi} p(t) \cos nt dt, \quad B = \int_0^{2\pi} p(t) \sin nt dt. \tag{10}$$

Obviously, the value of Φ forms a closed interval $[-\varrho, \varrho]$ with $\varrho = \sqrt{A^2 + B^2}$. Therefore, the four conditions in Theorem 1 are equivalent to the following conditions, respectively:

$$\begin{aligned}
 & 2 \cos(n\tau) [g(+\infty) - g(-\infty)] \notin [-\varrho, \varrho], \\
 & 2n \cos(n\tau) [F(+\infty) - F(-\infty)] \notin [-\varrho, \varrho], \\
 & 2[g(+\infty) - g(-\infty)] \\
 & \quad - 2n \sin(n\tau) [F(+\infty) - F(-\infty)] \notin [-\varrho, \varrho], \\
 & 2n[F(+\infty) - F(-\infty)] \\
 & \quad - 2 \sin(n\tau) [g(+\infty) - g(-\infty)] \notin [-\varrho, \varrho].
 \end{aligned} \tag{11}$$

Remark 3. In the case when $\tau = 0$, the four conditions in Theorem 1 reduce to the conditions (5). Therefore, Theorem 1 generalizes the result in [10].

Throughout this paper, we always use \mathbf{R} to denote the real number set. For a multivariate function ζ depending on r , the notation $\zeta = o(1)$ always means that, for $r \rightarrow \infty$, $\zeta \rightarrow 0$ holds uniformly with respect to other variables, whereas $\zeta = O(1)$ (or $\zeta = O(r^{-1})$) always means that ζ (or $r \cdot \zeta$) is bounded for r large enough. For any continuous 2π -periodic function $\psi(t)$, we always set $\|\psi\|_\infty = \max_{0 \leq t \leq 2\pi} |\psi(t)|$.

2. Basic Lemmas

It is well known that continuation theorems play an important role in studying the existence of periodic solutions of differential equations. We now introduce a continuation theorem which will be used to prove the existence of periodic solutions of (1).

Let X and Y be two real Banach spaces and let $L : D(L) \subset X \rightarrow Y$ be a Fredholm operator with index zero, where $D(L)$ denotes the domain of L . This means that $\text{Im } L$ is a closed subspace of Y and $\dim \ker L = \text{codim } \text{Im } L < +\infty$. Let $P : X \rightarrow \text{Ker } L$, $Q : Y \rightarrow Y$ be two linear continuous projectors satisfying the following:

$$\text{Im } P = \text{Ker } L, \quad \text{Ker } Q = \text{Im } L. \tag{12}$$

Then we have the following:

$$X = \text{Ker } L \oplus \text{Ker } P, \quad Y = \text{Im } L \oplus \text{Im } Q. \tag{13}$$

Clearly, $L_P = L|_{D(L) \cap \text{Ker } P} \rightarrow \text{Im } L$ is invertible. Denote by K_P the inverse of L_P . Let $\Omega \subset X$ be an open bounded set. A continuous map $N : \overline{\Omega} \rightarrow Y$ is said to be L -compact on $\overline{\Omega}$ if both $QN : \overline{\Omega} \rightarrow Y$ and $K_P(I - Q)N : \overline{\Omega} \rightarrow X$ are compact.

Lemma 4 (see [11]). *Let X and Y be two real Banach spaces. Suppose that $L : D(L) \subset X \rightarrow Y$ is a Fredholm operator with index zero and $N : \overline{\Omega} \rightarrow Y$ is L -compact on $\overline{\Omega}$, where Ω is an open bounded subset of X . Moreover, assume that all the following conditions are satisfied:*

- (1) $Lx \neq \lambda Nx$, for all $x \in \partial\Omega \cap D(L)$, $\lambda \in (0, 1)$;
- (2) $Nx \notin \text{Im } L$, for all $x \in \partial\Omega \cap \text{Ker } L$;

(3) *The Brouwer degree $\text{deg}\{JQN, \Omega \cap \text{Ker } L, 0\} \neq 0$, where $J : \text{Im } Q \rightarrow \text{Ker } L$ is an isomorphism.*

Then equation $Lx = Nx$ has at least one solution on $D(L) \cap \bar{\Omega}$.

3. Main Results

In this section, we will use the continuation theorem introduced in Section 2 to prove the existence of periodic solutions of (1). To this end, we first quote some notations and definitions.

Let X and Y be two Banach spaces defined by the following:

$$\begin{aligned} X &= \{x \in C^1(\mathbf{R}, \mathbf{R}) : x(t + 2\pi) = x(t), \forall t \in \mathbf{R}\}, \\ Y &= \{y \in C(\mathbf{R}, \mathbf{R}) : y(t + 2\pi) = y(t), \forall t \in \mathbf{R}\} \end{aligned} \quad (14)$$

with the following norms

$$\|x\|_X = \max\{\|x\|_\infty, \|x'\|_\infty\}, \quad \|y\|_Y = \|y\|_\infty. \quad (15)$$

Define a linear operator

$$L : D(L) \subset X \rightarrow Y, \quad Lx = x'' + n^2x, \quad (16)$$

where $D(L) = \{x \in X : x'' \in C(\mathbf{R}, \mathbf{R})\}$, and a nonlinear operator

$$\begin{aligned} N : X &\rightarrow Y, \\ (Nx)(t) &= -f(x(t))x'(t) - g(x(t - \tau)) + p(t). \end{aligned} \quad (17)$$

It is easy to see that

$$\begin{aligned} \text{Ker } L &= \text{Span}\{\sin nt, \cos nt\}, \\ \text{Im } L &= \left\{y \in Y : \int_0^{2\pi} y(t) \sin ntdt = 0, \right. \\ &\quad \left. \int_0^{2\pi} y(t) \cos ntdt = 0\right\}. \end{aligned} \quad (18)$$

On the other hand, for any $y \in Y$, we can write the following:

$$y(t) = \bar{y}(t) + \tilde{y}(t), \quad \forall t \in [0, 2\pi], \quad (19)$$

where $\bar{y}(t)$ is defined by the following:

$$\bar{y}(t) = a \sin nt + b \cos nt \quad (20)$$

with

$$a = \frac{1}{\pi} \int_0^{2\pi} y(t) \sin ntdt, \quad b = \frac{1}{\pi} \int_0^{2\pi} y(t) \cos ntdt, \quad (21)$$

whereas $\tilde{y}(t)$ satisfies the following:

$$\int_0^{2\pi} \tilde{y}(t) \sin ntdt = 0, \quad \int_0^{2\pi} \tilde{y}(t) \cos ntdt = 0. \quad (22)$$

Therefore,

$$Y = \text{Ker } L \oplus \text{Im } L. \quad (23)$$

It follows that L is a Fredholm map of index zero.

Let us define two continuous projectors $P : X \rightarrow \text{Ker } L$ and $Q : Y \rightarrow Y$ by setting the following:

$$\begin{aligned} (Px)(t) &= a_1 \sin nt + b_1 \cos nt, \\ (Qy)(t) &= \bar{y}(t) = a \sin nt + b \cos nt, \quad \forall t \in [0, 2\pi] \end{aligned} \quad (24)$$

for any $x \in X$ and $y \in Y$, where constants a_1 and b_1 are defined as constants a and b . Obviously, $\text{Im } Q = \text{Ker } L$.

Set $L_P = L|_{D(L) \cap \text{Ker } P} \rightarrow \text{Im } L$. Then L_P is an algebraic isomorphism and we define $K_P : \text{Im } L \rightarrow D(L) \cap \text{Ker } P$ by the following:

$$K_P = L_P^{-1}. \quad (25)$$

Clearly, we have that, for any $y \in \text{Im } L$,

$$\begin{aligned} (K_P y)(t) &= \frac{1}{n} \int_0^t y(s) \sin n(t-s) ds \\ &\quad + \frac{1}{2n\pi} \int_0^{2\pi} sy(s) \sin n(t-s) ds. \end{aligned} \quad (26)$$

For any open bounded set $\Omega \subset X$, we can prove by standard arguments that $K_P(I - Q)N$ and QN are compact on the closure $\bar{\Omega}$. Therefore, N is L -compact on $\bar{\Omega}$.

It is noted that (1) is equivalent to the operator equation

$$Lx = Nx. \quad (27)$$

To use Lemma 4, we embed this operator equation into an equation family with a parameter $\lambda \in (0, 1)$,

$$Lx = \lambda Nx, \quad (28)$$

which is equivalent to the equation as follows:

$$\begin{aligned} x'' + n^2x + \lambda f(x(t))x'(t) + \lambda g(x(t - \tau)) \\ = \lambda p(t), \quad \lambda \in (0, 1). \end{aligned} \quad (29)$$

In the following, we will prove some new results on the existence of periodic solutions of (1) by using the continuation theorem. Consider the equivalent system of (29):

$$\begin{aligned} x' &= y - \lambda F(x), \quad y' = -n^2x - \lambda g(x(t - \tau)) + \lambda p(t). \end{aligned} \quad (30)$$

Let $x(t)$ be any (possible) 2π -periodic solution of (29). Set $y(t) = x'(t) + \lambda F(x(t))$. Then $(x(t), y(t))$ is a 2π -periodic solution of (30).

Now, let us introduce a transformation $\Phi : (r, \theta) \in \mathbf{R}^+ \times S^1 \rightarrow (x, y) \in \mathbf{R}^2 \setminus \{0\}$ with $S^1 = \mathbf{R}/2\pi\mathbf{Z}$,

$$x = \frac{1}{n} r^{1/2} \sin \theta, \quad y = r^{1/2} \cos \theta. \quad (31)$$

Under the transformation Φ , if $|x(t)| + |y(t)| \neq 0$, for $t \in [0, 2\pi]$, then the 2π -periodic solution $(x(t), y(t))$ of (30) can be expressed in the form $(r(t), \theta(t))$ satisfying the equations as follows:

$$\begin{aligned} \frac{d\theta}{dt} &= n + \lambda r^{-1/2} g\left(\frac{1}{n} r^{1/2} (t - \tau) \sin \theta (t - \tau)\right) \sin \theta \\ &\quad - \lambda n r^{-1/2} F\left(\frac{1}{n} r^{1/2} \sin \theta\right) \cos \theta - \lambda r^{-1/2} p(t) \sin \theta, \\ \frac{dr}{dt} &= -2\lambda r^{1/2} g\left(\frac{1}{n} r^{1/2} (t - \tau) \sin \theta (t - \tau)\right) \cos \theta \\ &\quad - 2\lambda n r^{1/2} F\left(\frac{1}{n} r^{1/2} \sin \theta\right) \sin \theta + 2\lambda r^{1/2} p(t) \cos \theta. \end{aligned} \quad (32)$$

Let us set $(r_0, \theta_0) = (r(0), \theta(0))$ with $r_0 = n^2 x^2(0) + y^2(0)$. Without loss of generality, we always assume $\theta(0) \in [0, 2\pi]$. Dividing the second equation of (32) by $r^{1/2}$, we get the following:

$$\begin{aligned} \frac{dr^{1/2}}{dt} &= -\lambda g\left(\frac{1}{n} r^{1/2} (t - \tau) \sin \theta (t - \tau)\right) \cos \theta \\ &\quad - \lambda n F\left(\frac{1}{n} r^{1/2} \sin \theta\right) \sin \theta + \lambda p(t) \cos \theta. \end{aligned} \quad (33)$$

Integrating (33) and applying conditions (h_1) and (h_2) , we get the following:

$$r(t)^{1/2} = r_0^{1/2} + O(1), \quad \forall t \in [0, 2\pi]. \quad (34)$$

Furthermore,

$$r(t)^{-1/2} = r_0^{-1/2} + O(r_0^{-1}), \quad t \in [0, 2\pi]. \quad (35)$$

On the other hand, it follows from the first equation of (32) and (35) that

$$\frac{d\theta}{dt} = n + O(r_0^{-1/2}), \quad t \in [0, 2\pi]. \quad (36)$$

Therefore, we get the following:

$$\theta(t) = \theta_0 + nt + O(r_0^{-1/2}), \quad t \in [0, 2\pi]. \quad (37)$$

The estimations (34)–(37) will be used to obtain apriori bounds of 2π -periodic solutions of (29). Multiplying both

sides of (29) by $\sin(\theta_0 + nt)$ and $\cos(\theta_0 + nt)$, respectively, and integrating over the interval $[0, 2\pi]$, we obtain the following:

$$\begin{aligned} &\int_0^{2\pi} f(x(t)) x'(t) \sin(\theta_0 + nt) dt \\ &\quad + \int_0^{2\pi} g(x(t - \tau)) \sin(\theta_0 + nt) dt \\ &= \int_0^{2\pi} p(t) \sin(\theta_0 + nt) dt, \end{aligned} \quad (38)$$

$$\begin{aligned} &\int_0^{2\pi} f(x(t)) x'(t) \cos(\theta_0 + nt) dt \\ &\quad + \int_0^{2\pi} g(x(t - \tau)) \cos(\theta_0 + nt) dt \\ &= \int_0^{2\pi} p(t) \cos(\theta_0 + nt) dt. \end{aligned}$$

Hence,

$$\begin{aligned} &-n \int_0^{2\pi} F(x(t)) \cos(\theta_0 + nt) dt \\ &\quad + \int_0^{2\pi} g(x(t - \tau)) \sin(\theta_0 + nt) dt \\ &= \int_0^{2\pi} p(t) \sin(\theta_0 + nt) dt \end{aligned} \quad (39)$$

$$\begin{aligned} &n \int_0^{2\pi} F(x(t)) \sin(\theta_0 + nt) dt \\ &\quad + \int_0^{2\pi} g(x(t - \tau)) \cos(\theta_0 + nt) dt \\ &= \int_0^{2\pi} p(t) \cos(\theta_0 + nt) dt. \end{aligned} \quad (40)$$

Multiplying both sides of (29) by $\sin(\theta_0 + n(t - \tau))$ and $\cos(\theta_0 + n(t - \tau))$, respectively, and integrating over the interval $[0, 2\pi]$, we obtain the following:

$$\begin{aligned} &\int_0^{2\pi} f(x(t)) x'(t) \sin(\theta_0 + n(t - \tau)) dt \\ &\quad + \int_0^{2\pi} g(x(t - \tau)) \sin(\theta_0 + n(t - \tau)) dt \\ &= \int_0^{2\pi} p(t) \sin(\theta_0 + n(t - \tau)) dt, \end{aligned} \quad (41)$$

$$\begin{aligned} &\int_0^{2\pi} f(x(t)) x'(t) \cos(\theta_0 + n(t - \tau)) dt \\ &\quad + \int_0^{2\pi} g(x(t - \tau)) \cos(\theta_0 + n(t - \tau)) dt \\ &= \int_0^{2\pi} p(t) \cos(\theta_0 + n(t - \tau)) dt. \end{aligned}$$

Hence,

$$\begin{aligned}
 & -n \int_0^{2\pi} F(x(t)) \cos(\theta_0 + n(t - \tau)) dt \\
 & \quad + \int_0^{2\pi} g(x(t - \tau)) \sin(\theta_0 + n(t - \tau)) dt \\
 & = \int_0^{2\pi} p(t) \sin(\theta_0 + n(t - \tau)) dt, \\
 & n \int_0^{2\pi} F(x(t)) \sin(\theta_0 + n(t - \tau)) dt \\
 & \quad + \int_0^{2\pi} g(x(t - \tau)) \cos(\theta_0 + n(t - \tau)) dt \\
 & = \int_0^{2\pi} p(t) \cos(\theta_0 + n(t - \tau)) dt.
 \end{aligned} \tag{42}$$

Proof of Theorem 1. We shall prove the existence of periodic solutions of (1) provided that either

$$\begin{aligned}
 & 2 \cos n\tau [g(+\infty) - g(-\infty)] \\
 & \neq \int_0^{2\pi} p(t) \sin(\theta + nt) dt, \quad \forall \theta \in [0, 2\pi],
 \end{aligned} \tag{43}$$

or

$$\begin{aligned}
 & 2n [F(+\infty) - F(-\infty)] - 2 \sin n\tau [g(+\infty) - g(-\infty)] \\
 & \neq \int_0^{2\pi} p(t) \sin(\theta + nt) dt, \quad \forall \theta \in [0, 2\pi]
 \end{aligned} \tag{44}$$

holds by using (39) and (40). The other cases can be handled similarly by using (43) and (44). We proceed in three steps.

(1) We prove that there exist positive constants M_1 and M_2 such that, for any 2π -periodic solution $x(t)$ of (29),

$$\|x\|_\infty < M_1, \quad \|x'\|_\infty < M_2. \tag{45}$$

Assume by contradiction that (45) does not hold. Then there exists a sequence of 2π -periodic solutions $\{x_k(t)\}_{k=1}^\infty$ of (29) with $\lambda = \lambda_k \in (0, 1)$ such that

$$\|x_k\|_\infty + \|x'_k\|_\infty \longrightarrow +\infty, \quad \text{for } k \rightarrow \infty. \tag{46}$$

Write $y_k(t) = x'_k(t) + \lambda_k F(x_k(t))$. Since F is bounded on the interval $(-\infty, +\infty)$, we have the following:

$$\|x_k\|_\infty + \|y_k\|_\infty \longrightarrow +\infty, \quad \text{for } k \rightarrow \infty. \tag{47}$$

Let $(r_k(t), \theta_k(t))$ be the 2π -periodic solution of (32) related to $(x_k(t), y_k(t))$. Obviously, $r_k(t) = n^2 x_k^2(t) + y_k^2(t)$. Then we have the following:

$$\|r_k\|_\infty \longrightarrow +\infty, \quad \text{for } k \longrightarrow \infty. \tag{48}$$

Without loss of generality, we also assume $\theta_k(0) \in [0, 2\pi]$. It follows from (39) that

$$\begin{aligned}
 & -n \int_0^{2\pi} F(x_k(t)) \cos(\theta_k(0) + nt) dt \\
 & \quad + \int_0^{2\pi} g(x_k(t - \tau)) \sin(\theta_k(0) + nt) dt \\
 & = \int_0^{2\pi} p(t) \sin(\theta_k(0) + nt) dt.
 \end{aligned} \tag{49}$$

From (34) and (37) we get that, for $t \in [0, 2\pi]$,

$$\begin{aligned}
 & g(x_k(t - \tau)) \\
 & = g\left(\frac{1}{n} r_k(t - \tau)^{1/2} \sin \theta_k(t - \tau)\right) \\
 & = g\left(\frac{1}{n} r_k(0)^{1/2} \sin(\theta_k(0) + n(t - \tau)) + O(1)\right).
 \end{aligned} \tag{50}$$

Therefore,

$$\begin{aligned}
 & \int_0^{2\pi} g(x_k(t - \tau)) \sin(\theta_k(0) + nt) dt \\
 & = \int_0^{2\pi} g\left(\frac{1}{n} r_k(0)^{1/2} \sin(\theta_k(0) + n(t - \tau)) + O(1)\right) \\
 & \quad \times \sin(\theta_k(0) + nt) dt.
 \end{aligned} \tag{51}$$

Obviously, we have the following:

$$\begin{aligned}
 & \int_0^{2\pi} g\left(\frac{1}{n} r_k(0)^{1/2} \sin(\theta_k(0) + n(t - \tau)) + O(1)\right) \\
 & \quad \times \sin(\theta_k(0) + nt) dt \\
 & = \frac{1}{n} \int_{\theta_k(0) - n\tau}^{2n\pi + \theta_k(0) - n\tau} g\left(\frac{1}{n} r_k(0)^{1/2} \sin s + O(1)\right) \\
 & \quad \times \sin(s + n\tau) ds \\
 & = \frac{\cos n\tau}{n} \\
 & \quad \times \int_{\theta_k(0) - n\tau}^{2n\pi + \theta_k(0) - n\tau} g\left(\frac{1}{n} r_k(0)^{1/2} \sin s + O(1)\right) \sin s ds \\
 & \quad + \frac{\sin n\tau}{n} \\
 & \quad \times \int_{\theta_k(0) - n\tau}^{2n\pi + \theta_k(0) - n\tau} g\left(\frac{1}{n} r_k(0)^{1/2} \sin s + O(1)\right) \cos s ds.
 \end{aligned} \tag{52}$$

Since $\theta_k(0) \in [0, 2\pi]$, there exists a subsequence $\{\theta_{k_j}(0)\}$ such that $\theta_{k_j} \rightarrow \theta_*$, $j \rightarrow \infty$. By using Lebesgue

dominated convergent theorem and the condition (h_1) , we get the following:

$$\begin{aligned} & \lim_{j \rightarrow +\infty} \int_{\theta_{k_j}(0) - n\tau}^{2n\pi + \theta_{k_j}(0) - n\tau} g\left(\frac{1}{n}r_{k_j}(0)^{1/2} \sin s + O(1)\right) \sin s \, ds \\ &= 2n(g(+\infty) - g(-\infty)), \\ & \lim_{j \rightarrow +\infty} \int_{\theta_{k_j}(0) - n\tau}^{2n\pi + \theta_{k_j}(0) - n\tau} g\left(\frac{1}{n}r_{k_j}(0)^{1/2} \sin s + O(1)\right) \cos s \, ds = 0. \end{aligned} \tag{53}$$

Therefore,

$$\begin{aligned} & \lim_{j \rightarrow +\infty} \int_0^{2\pi} g(x_{k_j}(t - \tau)) \sin(\theta_{k_j}(0) + nt) \, dt \\ &= 2 \cos n\tau [g(+\infty) - g(-\infty)]. \end{aligned} \tag{54}$$

Similarly, we can get the following:

$$\lim_{j \rightarrow +\infty} \int_0^{2\pi} F(x_{k_j}(t)) \cos(\theta_{k_j}(0) + nt) \, dt = 0. \tag{55}$$

Hence, we obtain the following:

$$2 \cos n\tau [g(+\infty) - g(-\infty)] = \int_0^{2\pi} p(t) \sin(\theta_* + nt) \, dt, \tag{56}$$

which contradicts with (43).

On the other hand, it follows from (40) that

$$\begin{aligned} & n \int_0^{2\pi} F(x_k(t)) \sin(\theta_k(0) + nt) \, dt \\ &+ \int_0^{2\pi} g(x_k(t - \tau)) \cos(\theta_k(0) + nt) \, dt \\ &= \int_0^{2\pi} p(t) \cos(\theta_k(0) + nt) \, dt. \end{aligned} \tag{57}$$

According to (50), we have the following:

$$\begin{aligned} & \int_0^{2\pi} g(x_k(t - \tau)) \cos(\theta_k(0) + nt) \, dt \\ &= \int_0^{2\pi} g\left(\frac{1}{n}r_k(0)^{1/2} \cos(\theta_k(0) + n(t - \tau)) + O(1)\right) \\ &\quad \times \cos(\theta_k(0) + nt) \, dt \\ &= \frac{1}{n} \int_{\theta_k(0) - n\tau}^{2n\pi + \theta_k(0) - n\tau} g\left(\frac{1}{n}r_k(0)^{1/2} \sin s + O(1)\right) \\ &\quad \times \cos(s + n\tau) \, ds \\ &= \frac{\cos n\tau}{n} \\ &\quad \times \int_{\theta_k(0) - n\tau}^{2n\pi + \theta_k(0) - n\tau} g\left(\frac{1}{n}r_k(0)^{1/2} \sin s + O(1)\right) \cos s \, ds \\ &\quad - \frac{\sin n\tau}{n} \\ &\quad \times \int_{\theta_k(0) - n\tau}^{2n\pi + \theta_k(0) - n\tau} g\left(\frac{1}{n}r_k(0)^{1/2} \sin s + O(1)\right) \sin s \, ds. \end{aligned} \tag{58}$$

From (53) we obtain the following:

$$\begin{aligned} & \lim_{j \rightarrow +\infty} \int_0^{2\pi} g(x_{k_j}(t - \tau)) \cos(\theta_{k_j}(0) + nt) \, dt \\ &= -2 \sin n\tau [g(+\infty) - g(-\infty)]. \end{aligned} \tag{59}$$

Similarly, we have the following:

$$\begin{aligned} & \lim_{j \rightarrow +\infty} \int_0^{2\pi} F(x_{k_j}(t)) \sin(\theta_{k_j}(0) + nt) \, dt \\ &= 2 [F(+\infty) - F(-\infty)]. \end{aligned} \tag{60}$$

It follows from (57)–(60) that

$$\begin{aligned} & 2n [F(+\infty) - F(-\infty)] - 2 \sin(n\tau) [g(+\infty) - g(-\infty)] \\ &= \int_0^{2\pi} p(t) \cos(\theta_* + nt) \, dt, \end{aligned} \tag{61}$$

which contradicts with (44). Therefore, there exist positive constants M_1 and M_2 such that (45) holds.

(2) Let $x(t) = \rho \sin(nt + \alpha)$, where α is an arbitrary constant. We will prove that there exists $M_0 > 0$ such that, for $\rho \geq M_0$, $Nx \notin \text{Im } L$. Otherwise, there exists a sequence

$\{Q_k\}$ satisfying $\lim_{k \rightarrow \infty} Q_k = +\infty$ such that $Nx_k \in \text{Im } L$ with $x_k(t) = Q_k \sin(nt + \alpha)$. We will prove the following:

$$\begin{aligned} 2 \cos n\tau [g(+\infty) - g(-\infty)] &= \int_0^{2\pi} p(t) \sin(nt + \alpha) dt, \\ 2n [F(+\infty) - F(-\infty)] - 2 \sin(n\tau) [g(+\infty) - g(-\infty)] \\ &= \int_0^{2\pi} p(t) \sin(nt + \alpha) dt. \end{aligned} \tag{62}$$

In fact, since $Nx_k \in \text{Im } L$, we have the following:

$$\begin{aligned} \int_0^{2\pi} [f(x_k(t)) x'_k(t) + g(x_k(t - \tau))] \sin(nt + \alpha) dt \\ = \int_0^{2\pi} p(t) \sin(nt + \alpha) dt, \\ \int_0^{2\pi} [f(x_k(t)) x'_k(t) + g(x_k(t - \tau))] \cos(nt + \alpha) dt \\ = \int_0^{2\pi} p(t) \cos(nt + \alpha) dt. \end{aligned} \tag{63}$$

Using the same method as in step 1, we have the following:

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_0^{2\pi} [f(x_k(t)) x'_k(t) + g(x_k(t - \tau))] \sin(nt + \alpha) dt \\ = 2 \cos n\tau [g(+\infty) - g(-\infty)], \\ \lim_{k \rightarrow \infty} \int_0^{2\pi} [f(x_k(t)) x'_k(t) + g(x_k(t - \tau))] \cos(nt + \alpha) dt \\ = 2n [F(+\infty) - F(-\infty)] \\ - 2 \sin(n\tau) [g(+\infty) - g(-\infty)]. \end{aligned} \tag{64}$$

As a consequence, (62) holds. Thus, we get a contradiction.

(3) Let $M > \max\{nM_0, M_1, M_2\}$ be a sufficiently large constant (if it is necessary, M can be enlarged). Set

$$\Omega = \{x \in X : \|x\|_\infty < M, \|x'\|_\infty < M\}. \tag{65}$$

From the conclusion in step 1 we know that

$$Lx \neq \lambda Nx, \quad \forall x \in \partial\Omega \cap D(L), \lambda \in (0, 1). \tag{66}$$

From the conclusion in step 2 we know that

$$Nx \notin \text{Im } L, \quad \forall x \in \partial\Omega \cap \text{Ker } L, \tag{67}$$

which implies $QNx \neq 0$ for any $x \in \partial\Omega \cap \text{Ker } L$. Since $\text{Im } Q = \text{Ker } L$, we can take an isomorphism $J = \text{identity} : \text{Im } Q \rightarrow \text{Ker } L$. In what follows, we will prove the following:

$$\deg\{JQN, \Omega \cap \text{Ker } L, 0\} \neq 0. \tag{68}$$

To this end, let us define $\phi : \text{Ker } L \rightarrow \mathbf{R}^2, x = a \sin nt + b \cos nt \rightarrow (a, b)$, namely,

$$\phi x = (a, b). \tag{69}$$

Obviously, ϕ is a linear isomorphism. For any $x = a \sin nt + b \cos nt$, set

$$(JQNx)(t) = h_1(a, b) \sin nt + h_2(a, b) \cos nt, \tag{70}$$

where

$$\begin{aligned} h_1(a, b) &= \frac{1}{\pi} \int_0^{2\pi} [-f(x(t)) x'(t) - g(x(t - \tau)) + p(t)] \\ &\quad \times \sin nt dt \\ h_2(a, b) &= \frac{1}{\pi} \int_0^{2\pi} [-f(x(t)) x'(t) - g(x(t - \tau)) + p(t)] \\ &\quad \times \cos nt dt. \end{aligned} \tag{71}$$

Define $h : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ as follows:

$$h(a, b) = \phi \circ Q \circ N \circ \phi^{-1}(a, b) = (h_1(a, b), h_2(a, b)). \tag{72}$$

Then we have the following:

$$\deg\{JQN, \Omega \cap \text{Ker } L, 0\} = \deg\{h, \phi(\Omega \cap \text{Ker } L), 0\}. \tag{73}$$

To calculate $\deg\{h, \phi(\Omega \cap \text{Ker } L), 0\}$, we first estimate l_1 and l_2 as follows:

$$\begin{aligned} l_1(a, b) &= \frac{1}{\pi} \int_0^{2\pi} [-f(x(t)) x'(t) - g(x(t - \tau))] \sin nt dt, \\ l_2(a, b) &= \frac{1}{\pi} \int_0^{2\pi} [-f(x(t)) x'(t) - g(x(t - \tau))] \cos nt dt. \end{aligned} \tag{74}$$

Write $x = \rho \sin(nt + \vartheta)$ with $\rho = \sqrt{a^2 + b^2}, \vartheta = \arctan(b/a)$ or $\vartheta = \pi + \arctan(b/a)$. Then we have that, for $\rho \rightarrow \infty$,

$$\begin{aligned} \int_0^{2\pi} f(x(t)) x'(t) \sin nt dt \\ = -n \int_0^{2\pi} F(x(t)) \cos nt dt \\ = -n \int_0^{2\pi} F(\rho \sin(nt + \vartheta)) \cos nt dt \\ = - \int_9^{2n\pi + \vartheta} F(\rho \sin s) (\cos s \cos \vartheta + \sin s \sin \vartheta) ds \\ = -\cos \vartheta \int_9^{2n\pi + \vartheta} F(\rho \sin s) \cos s ds \\ - \sin \vartheta \int_9^{2n\pi + \vartheta} F(\rho \sin s) \sin s ds \\ = -2n \sin \vartheta [F(+\infty) - F(-\infty)] + o(1). \end{aligned} \tag{75}$$

On the other hand, we have that, for $\rho \rightarrow \infty$,

$$\begin{aligned} & \int_0^{2\pi} g(x(t-\tau)) \sin ntdt \\ &= \int_0^{2\pi} g(\rho \sin(n(t-\tau) + \vartheta)) \sin ntdt \\ &= \frac{1}{n} \int_{\vartheta-n\tau}^{2n\pi+\vartheta-n\tau} g(\rho \sin s) \sin(s+n\tau-\vartheta) ds \\ &= \frac{\cos(n\tau-\vartheta)}{n} \int_{\vartheta-n\tau}^{2n\pi+\vartheta-n\tau} g(\rho \sin s) \sin s ds \\ &\quad + \frac{\sin(n\tau-\vartheta)}{n} \int_{\vartheta-n\tau}^{2n\pi+\vartheta-n\tau} g(\rho \sin s) \cos s ds \\ &= 2 \cos(n\tau-\vartheta) [g(+\infty) - g(-\infty)] + o(1). \end{aligned} \tag{76}$$

Therefore, we get the following:

$$\begin{aligned} l_1(a, b) &= \frac{2}{\pi} \{n \sin \vartheta [F(+\infty) - F(-\infty)] \\ &\quad - \cos(n\tau - \vartheta) [g(+\infty) - g(-\infty)]\} \\ &\quad + o(1). \end{aligned} \tag{77}$$

To estimate l_2 , we have that, for $\rho \rightarrow \infty$,

$$\begin{aligned} & \int_0^{2\pi} f(x(t)) x'(t) \cos ntdt \\ &= n \int_0^{2\pi} F(x(t)) \sin ntdt \\ &= n \int_0^{2\pi} F(\rho \sin(nt + \vartheta)) \sin ntdt \\ &= \cos \vartheta \int_{\vartheta}^{2n\pi+\vartheta} F(\rho \sin s) \sin s ds \\ &\quad - \sin \vartheta \int_{\vartheta}^{2n\pi+\vartheta} F(\rho \sin s) \cos s ds \\ &= 2n \cos \vartheta [F(+\infty) - F(-\infty)] + o(1). \end{aligned} \tag{78}$$

Meanwhile, we get that, for $\rho \rightarrow \infty$,

$$\begin{aligned} & \int_0^{2\pi} g(x(t-\tau)) \cos ntdt \\ &= \int_0^{2\pi} g(\rho \sin(n(t-\tau) + \vartheta)) \cos ntdt \\ &= \frac{1}{n} \int_{\vartheta-n\tau}^{2n\pi+\vartheta-n\tau} g(\rho \sin s) \cos(s+n\tau-\vartheta) ds \\ &= \frac{\cos(n\tau-\vartheta)}{n} \int_{\vartheta-n\tau}^{2n\pi+\vartheta-n\tau} g(\rho \sin s) \cos s ds \\ &\quad - \frac{\sin(n\tau-\vartheta)}{n} \int_{\vartheta-n\tau}^{2n\pi+\vartheta-n\tau} g(\rho \sin s) \sin s ds \\ &= -2 \sin(n\tau-\vartheta) [g(+\infty) - g(-\infty)] + o(1). \end{aligned} \tag{79}$$

Hence, we obtain the following:

$$\begin{aligned} l_2(a, b) &= \frac{2}{\pi} \{ -n \cos \vartheta [F(+\infty) - F(-\infty)] \\ &\quad + \sin(n\tau - \vartheta) [g(+\infty) - g(-\infty)] \} \\ &\quad + o(1). \end{aligned} \tag{80}$$

Set

$$\begin{aligned} \hat{h}_1(a, b) &= \frac{2}{\pi} \{ n \sin \vartheta [F(+\infty) - F(-\infty)] \\ &\quad - \cos(n\tau - \vartheta) [g(+\infty) - g(-\infty)] \}, \\ \hat{h}_2(a, b) &= \frac{2}{\pi} \{ -n \cos \vartheta [F(+\infty) - F(-\infty)] \\ &\quad + \sin(n\tau - \vartheta) [g(+\infty) - g(-\infty)] \}, \\ \hat{h}(a, b) &= (\hat{h}_1(a, b), \hat{h}_2(a, b)). \end{aligned} \tag{81}$$

Replacing ϑ in $x = \rho \sin(nt + \vartheta)$ with $\pi + \vartheta$, we get the following:

$$\begin{aligned} \hat{h}_1(-a, -b) &= -\frac{2}{\pi} \{ n \sin \vartheta [F(+\infty) - F(-\infty)] \\ &\quad - \cos(n\tau - \vartheta) [g(+\infty) - g(-\infty)] \}, \\ \hat{h}_2(-a, -b) &= -\frac{2}{\pi} \{ -n \cos \vartheta [F(+\infty) - F(-\infty)] \\ &\quad + \sin(n\tau - \vartheta) [g(+\infty) - g(-\infty)] \}. \end{aligned} \tag{82}$$

As a consequence,

$$\begin{aligned} \hat{h}_1(-a, -b) &= -\hat{h}_1(a, b), \\ \hat{h}_2(-a, -b) &= -\hat{h}_2(a, b), \\ \hat{h}(-a, -b) &= -\hat{h}(a, b). \end{aligned} \tag{83}$$

We note that, for $a^2 + b^2 \rightarrow +\infty$,

$$\begin{aligned} h_1(a, b) &= \hat{h}_1(a, b) + c_1 + o(1), \\ h_2(a, b) &= \hat{h}_2(a, b) + c_2 + o(1), \end{aligned} \tag{84}$$

where

$$\begin{aligned} c_1 &= \frac{1}{\pi} \int_0^{2\pi} p(t) \sin ntdt, \\ c_2 &= \frac{1}{\pi} \int_0^{2\pi} p(t) \cos ntdt. \end{aligned} \tag{85}$$

Let us consider the map $H : \phi(\Omega \cap \text{Ker } L) \times [0, 1] \rightarrow \mathbf{R}^2 : (a, b, \mu) \rightarrow (h_1(a, b, \mu), h_2(a, b, \mu))$ with

$$\begin{aligned} h_1(a, b, \mu) &= \hat{h}_1(a, b) + \mu c_1, \\ h_2(a, b, \mu) &= \hat{h}_2(a, b) + \mu c_2. \end{aligned} \tag{86}$$

Obviously, H is continuous. Next, we shall prove that, for any $(a, b, \mu) \in \partial\phi(\Omega \cap \text{Ker } L) \times [0, 1]$,

$$(h_1(a, b, \mu), h_2(a, b, \mu)) \neq (0, 0). \tag{87}$$

Otherwise, there exists some $(a, b, \mu) \in \partial\phi(\Omega \cap \text{Ker } L) \times [0, 1]$ such that

$$h_1(a, b, \mu) = 0, \quad h_2(a, b, \mu) = 0. \tag{88}$$

Then we have the following:

$$\begin{aligned} & 2n \sin \vartheta [F(+\infty) - F(-\infty)] \\ & \quad - 2 \cos(n\tau - \vartheta) [g(+\infty) - g(-\infty)] \\ & = -\mu\pi c_1, \\ & - 2n \cos \vartheta [F(+\infty) - F(-\infty)] \\ & \quad + 2 \sin(n\tau - \vartheta) [g(+\infty) - g(-\infty)] \\ & = -\mu\pi c_2. \end{aligned} \tag{89}$$

Therefore, we get the following:

$$\begin{aligned} 2 \cos n\tau [g(+\infty) - g(-\infty)] &= \mu \int_0^{2\pi} p(t) \sin(nt + \vartheta) dt, \\ 2n [F(+\infty) - F(-\infty)] - 2 \sin(n\tau) [g(+\infty) - g(-\infty)] \\ &= \mu \int_0^{2\pi} p(t) \cos(nt + \vartheta) dt. \end{aligned} \tag{90}$$

Since $\mu \in [0, 1]$, we know from (90) that

$$2 \cos n\tau [g(+\infty) - g(-\infty)] \in [-\varrho, \varrho], \tag{91}$$

$$\begin{aligned} 2n [F(+\infty) - F(-\infty)] \\ - 2 \sin(n\tau) [g(+\infty) - g(-\infty)] \in [-\varrho, \varrho], \end{aligned} \tag{92}$$

where ϱ is given in Remark 2. From Remark 2 we know that (91) and (92) contradict with (43) and (44).

In particular, we have that, for $(a, b) \in \partial\phi(\Omega \cap \text{Ker } L)$,

$$(\widehat{h}_1(a, b), \widehat{h}_2(a, b)) \neq (0, 0), \tag{93}$$

$$(\widehat{h}_1(a, b) + c_1, \widehat{h}_2(a, b) + c_2) \neq (0, 0). \tag{94}$$

Since $\widehat{h} : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ is odd and (93) holds, we know from Borsuk Theorem [12] that

$$\deg(\widehat{h}, \phi(\Omega \cap \text{Ker } L), 0) = 2m + 1 \neq 0, \tag{95}$$

where m is an integer.

On the other hand, we know from (94) and the expressions of \widehat{h}_1 and \widehat{h}_2 that there exists a positive constant ν , which is independent of a and b , such that, for $(a, b) \in \mathbf{R}^2$

$$|\widehat{h}_1(a, b) + c_1| + |\widehat{h}_2(a, b) + c_2| \geq \nu. \tag{96}$$

Consequently, we infer from the homotopy invariance of degree that, if $M > \max\{M_0, M_1, M_2\}$ is large enough; then

$$\begin{aligned} & \deg(JQN, \Omega \cap \text{Ker } L, 0) \\ & = \deg(h, \phi(\Omega \cap \text{Ker } L), 0) \\ & = \deg(H(\cdot, 1), \phi(\Omega \cap \text{Ker } L), 0) \\ & = \deg(H(\cdot, 0), \phi(\Omega \cap \text{Ker } L), 0) \\ & = \deg(\widehat{h}, \phi(\Omega \cap \text{Ker } L), 0) \\ & = 2m + 1 \neq 0. \end{aligned} \tag{97}$$

Therefore, all conditions of Lemma 4 are satisfied. Thus, (1) has at least one 2π -periodic solution. \square

4. Remarks

We can use the method developed in Section 3 to deal with the existence of 2π -periodic solutions of the following equation:

$$x'' + f(x') + n^2x + g(x(t - \tau)) = p(t). \tag{98}$$

Assume that the limits

$$(h_3) \lim_{x \rightarrow \pm\infty} f(x) = f(\pm\infty)$$

exist and are finite. We can prove the following theorem.

Theorem 5. *Assume that the conditions (h_1) and (h_3) hold. Then (98) has at least one 2π -periodic solution provided that one of the following conditions holds:*

$$\begin{aligned} & 2 \cos(n\tau) [g(+\infty) - g(-\infty)] \\ & \neq \int_0^{2\pi} p(t) \sin(\theta + nt) dt, \quad \forall \theta \in [0, 2\pi], \\ & 2 \cos(n\tau) [f(+\infty) - f(-\infty)] \\ & \neq \int_0^{2\pi} p(t) \sin(\theta + nt) dt, \quad \forall \theta \in [0, 2\pi], \\ & 2 [g(+\infty) - g(-\infty)] - 2 \sin(n\tau) [f(+\infty) - f(-\infty)] \\ & \neq \int_0^{2\pi} p(t) \sin(\theta + nt) dt, \quad \forall \theta \in [0, 2\pi], \\ & 2 [f(+\infty) - f(-\infty)] - 2 \sin(n\tau) [g(+\infty) - g(-\infty)] \\ & \neq \int_0^{2\pi} p(t) \sin(\theta + nt) dt, \quad \forall \theta \in [0, 2\pi]. \end{aligned} \tag{99}$$

Remark 6. In the case when $n = 1$, the third and the fourth condition in Theorem 5 are identical to the related conditions in [6]. But the first and the second condition in Theorem 5 did not appear in [6].

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