Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 891241, 7 pages
http://dx.doi.org/10.1155/2013/891241

Research Article

The Natural Filtration of Finite Dimensional Modular Lie Superalgebras of Special Type

Keli Zheng and Yongzheng Zhang

School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, China

Correspondence should be addressed to Yongzheng Zhang; zhyz@nenu.edu.cn

Received 21 May 2013; Accepted 8 July 2013

Academic Editor: Shi Weichen

This paper is concerned with the natural filtration of Lie superalgebra $S(n,m)$ of special type over a field of prime characteristic. We first construct the modular Lie superalgebra $S(n,m)$. Then we prove that the natural filtration of $S(n,m)$ is invariant under its automorphisms.

1. Introduction

Although many structural features of nonmodular Lie superalgebras (see [1–3]) are well understood, there seem to be very few general results on modular Lie superalgebras. The treatment of modular Lie superalgebras necessitates different techniques which are set forth in [4, 5]. In [6], four series of modular graded Lie superalgebras of Cartan type were constructed, which are analogous to the finite dimensional modular Lie algebras of Cartan type [7] or the four series of infinite dimensional Lie superalgebras of Cartan type defined by even differential forms over a field of characteristic zero [8]. Recent works on the modular Lie superalgebras of Cartan type can also be found in [9–13] and references therein.

It is well known that filtration techniques are of great importance in the structure and the classification theories of Lie (super)algebras (see [1, 2, 14, 15]). For some classes of modular Lie (super)algebras, the filtrations have been well investigated, for example, the natural filtrations of finite dimensional modular Lie algebras of Cartan type [16, 17] and of finite dimensional simple modular Lie superalgebras W, S, and H of Cartan type [18, 19].

The original motivation for this paper comes from the researches of structures for the finite dimensional modular Lie superalgebras $W(n,m)$ and $H(n,m)$, which were first introduced in [20, 21], respectively. The starting point of our studies is to construct a class of finite dimensional modular Lie superalgebras of special type, which is denoted by $S(n,m)$. A brief summary of the relevant concepts and notations in the finite dimensional modular Lie superalgebras $S(n,m)$ is presented in Section 2. In Section 3, by using the ad-nilpotent elements of $S(n,m)$, we show that the natural filtration of $S(n,m)$ is invariant under its automorphisms.

2. Preliminaries

Throughout this paper, $\mathbb{F}$ denotes an algebraic closed field of characteristic $p > 2$, and n is an integer greater than 3. In addition to the standard notation $\mathbb{Z}$, we write $\mathbb{N}$ and $\mathbb{N}_0$ to denote the sets of positive integers and nonnegative integers, respectively.

Let $\Lambda(n)$ be the Grassmann algebra over $\mathbb{F}$ in n variables $x_1, x_2, \ldots, x_n$. Set $\mathbb{B}_k = \{\langle i_1, i_2, \ldots, i_k\rangle \mid 1 \le i_1 < i_2 < \cdots < i_k \le n\}$ and $\mathbb{B}(n) = \bigcup_{k=0}^{n} \mathbb{B}_k$, where $\mathbb{B}_0 = \emptyset$. For $u = \langle i_1, i_2, \ldots, i_k\rangle \in \mathbb{B}_k$, set $|u| = k$, $\{u\} = \{i_1, i_2, \ldots, i_k\}$ and $x^u = x_{i_1} x_{i_2} \cdots x_{i_k}$ ($|\emptyset| = 0, x^{\emptyset} = 1$). Then $\{x^u \mid u \in \mathbb{B}(n)\}$ is an $\mathbb{F}$-basis of $\Lambda(n)$.

Let Π denote the prime field of $\mathbb{F}$; that is, $\Pi = \{0, 1, \ldots, p-1\}$. Suppose that the set $\{z_1, z_2, \ldots, z_m\}$ is a Π-linearly independent finite subset of $\mathbb{F}$. Let $G = \{\sum_{i=1}^{m} \lambda_i z_i \mid \lambda_i \in \Pi\}$. Then G is an additive subgroup of $\mathbb{F}$. Let $\mathbb{F}[y_1, y_2, \ldots, y_m]$ be the truncated polynomial algebra satisfying $y_i^p = 1$ for all $i = 1, 2, \ldots, m$. For every element $\lambda = \sum_{i=1}^{m} \lambda_i z_i \in G$, define $y^{\lambda} = y_1^{\lambda_1} y_2^{\lambda_2} \cdots y_m^{\lambda_m}$. Then $y^{\lambda} y^{\eta} = y^{\lambda+\eta}$ for all $\lambda, \eta \in G$. Let $\mathbb{T}(m)$ denote $\mathbb{F}[y_1, y_2, \ldots, y_m]$. Then $\mathbb{T}(m) = \{\sum_{\lambda\in G} a_{\lambda} y^{\lambda} \mid$

$a_\lambda \in \mathbb{F}\}$. Let $\mathscr{U} = \Lambda(n) \otimes \mathbb{T}(m)$. Then $\mathscr{U}$ is an associative superalgebra with $\mathbb{Z}_2$-gradation induced by the trivial $\mathbb{Z}_2$-gradation of $\mathbb{T}(m)$ and the natural $\mathbb{Z}_2$-gradation of $\Lambda(n)$; that is, $\mathscr{U} = \mathscr{U}_{\bar{0}} \oplus \mathscr{U}_{\bar{1}}$, where $\mathscr{U}_{\bar{0}} = \Lambda(n)_{\bar{0}} \otimes \mathbb{T}(m)$ and $\mathscr{U}_{\bar{1}} = \Lambda(n)_{\bar{1}} \otimes \mathbb{T}(m)$.

For $f \in \Lambda(n)$ and $\alpha \in \mathbb{T}(m)$, we abbreviate $f \otimes \alpha$ as $f\alpha$. Then the elements $x^u y^\lambda$ with $u \in \mathbb{B}(n)$ and $\lambda \in G$ form an $\mathbb{F}$-basis of $\mathscr{U}$. It is easy to see that $\mathscr{U} = \oplus_{i=0}^{n} \mathscr{U}_i$ is a $\mathbb{Z}$-graded superalgebra, where $\mathscr{U}_i = \text{span}_{\mathbb{F}}\{x^u y^\lambda \mid u \in \mathbb{B}(n), |u| = i, \lambda \in G\}$. In particular, $\mathscr{U}_0 = \mathbb{T}(m)$ and $\mathscr{U}_n = \text{span}_{\mathbb{F}}\{x^\pi y^\lambda \mid \lambda \in G\}$, where $\pi := \langle 1, 2, \ldots, n\rangle \in \mathbb{B}(n)$.

In this paper, if $A = A_{\bar{0}} \oplus A_{\bar{1}}$ is a superalgebra (or $\mathbb{Z}_2$-graded linear space), let $\text{Der}A$ be the derivation superalgebra of A (see [1] or [2] for the definition) and $hg(A) = A_{\bar{0}} \cup A_{\bar{1}}$; that is, $hg(A)$ is the set of all $\mathbb{Z}_2$-homogeneous elements of A. If $\deg x$ occurs in some expression, we regard x as a $\mathbb{Z}_2$-homogeneous element and $\deg x$ as the $\mathbb{Z}_2$-degree of x. Let $A = \oplus_{i=-r}^{n} A_i$ be a $\mathbb{Z}$-graded superalgebra. If $x \in A_i$, then we call x a $\mathbb{Z}$-homogeneous element and i the $\mathbb{Z}$-degree of x and set $zd(x) = i$.

Set $Y = \{1, 2, \ldots, n\}$. Given that $i \in Y$, let $\partial/\partial x_i$ be the partial derivative on $\Lambda(n)$ with respect to x_i. For $i \in Y$, let D_i be the linear transformation on $\mathscr{U}$ such that $D_i(x^u y^\lambda) = (\partial x^u/\partial x_i) y^\lambda$ for all $u \in \mathbb{B}(n)$ and $\lambda \in G$. Then $D_i \in \text{Der}_{\bar{1}}\mathscr{U}$ for all $i \in Y$ since $\partial/\partial x_i \in \text{Der}_{\bar{1}}(\Lambda(n))$.

Suppose that $u \in \mathbb{B}_k \subseteq \mathbb{B}(n)$ and $i \in Y$. When $i \in \{u\}$, we denote the uniquely determined element of $\mathbb{B}_{k-1}$ satisfying $\{u - \langle i\rangle\} = \{u\} \setminus \{i\}$ by $u - \langle i\rangle$ and denote the number of integers less than i in $\{u\}$ by $\tau(u, i)$. When $i \notin \{u\}$, we set $\tau(u, i) = 0$ and $x^{u-\langle i\rangle} = 0$. Therefore, $D_i(x^u) = (-1)^{\tau(u,i)} x^{u-\langle i\rangle}$ for any $i \in Y$ and $u \in \mathbb{B}(n)$.

We define $(fD)(g) = fD(g)$ for $f, g \in hg(\mathscr{U})$ and $D \in hg(\text{Der}\mathscr{U})$. Since the multiplication of $\mathscr{U}$ is supercommutative, it follows that fD is a derivation of $\mathscr{U}$. Let

$$W(n, m) = \text{span}_{\mathbb{F}} \left\{ x^u y^\lambda D_i \mid u \in \mathbb{B}(n), \lambda \in G, i \in Y \right\}. \quad (1)$$

Then $W(n, m)$ is a finite dimensional Lie superalgebra contained in $\text{Der}\mathscr{U}$. A direct computation shows that

$$\left[fD_i, gD_j\right] = fD_i(g) D_j - (-1)^{\deg fD_i \deg gD_j} gD_j(f) D_i, \quad (2)$$

where $f, g \in hg(\mathscr{U})$ and $i, j \in Y$.

Let $D_{r_1 r_2} : \mathscr{U} \to W(n, m)$ be the linear map such that for every $f \in hg(\mathscr{U})$ and $r_1, r_2 \in Y$,

$$D_{r_1 r_2}(f) = \sum_{i=1}^{2} f_{r_i} D_{r_i}, \quad (3)$$

where $f_{r_1} = -D_{r_2}(f)$ and $f_{r_2} = -D_{r_1}(f)$. It is easy to see that $D_{r_1 r_2}$ is an even linear map. Let $S(n, m) = \{D_{ij}(f) \mid f \in \mathscr{U}, i, j \in Y\}$. Then $S(n, m)$ is a finite dimensional Lie superalgebra with a $\mathbb{Z}$-gradation $S(n, m) = \oplus_{r=-1}^{n-2} S_r(n, m)$, where $S_r(n, m) = \{D_{ij}(x^u y^\lambda) \mid u \in \mathbb{B}(n), |u| = r + 2, \lambda \in G, i, j \in Y\}$. In this paper, $S(n, m)$ is called the Lie superalgebra of special type.

By the definition of linear map $D_{r_1 r_2}$, the following equalities are easy to verify:

$$\begin{gathered} D_{ii}(f) = -2D_i(f) D_i, \\ D_{ij}(f) = D_{ji}(f), \\ \left[D_k, D_{ij}(f)\right] = -D_{ij}(D_k(f)), \end{gathered} \quad (4)$$

$$\left[D_{s_1 s_2}(f), D_{r_1 r_2}(g)\right] = \sum_{i,j=1}^{2} (-1)^{\deg f} D_{s_i r_j}\left(f_{s_i} g_{r_j}\right), \quad (5)$$

where $f, g \in hg(\mathscr{U})$; $i, j, k \in Y$; and f_{s_i}, g_{r_j} and as in (3). The equality (5) shows that $S(n, m)$ is a subalgebra of $W(n, m)$. Hereafter, $S(n, m)$ and $S_i(n, m)$ will be simply denoted by S and S_i, respectively.

Put $A = \{D_{ij}(x^\pi y^\lambda) \mid i, j \in Y, \lambda \in G\}$ and $B = \{D_{ij}(x_k y^\eta) \mid i, j, k \in Y, \eta \in G\}$.

Proposition 1. *The Lie superalgebra S is generated by $A \cup B$.*

Proof. Suppose that $A \cup B$ generate the subalgebra Q of S. Since A and B are subsets of S, it follows that $Q \subseteq S$.

Next we will consider the reverse inclusion.

It is easy to see that $D_{ki}(x_k y^\lambda) = -y^\lambda D_i$ for all distinct elements i, k of Y and $\lambda \in G$. Therefore, $zd(D_{ki}(x_k y^\lambda)) = -1$ and $S_{-1} \subseteq Q$.

A direct calculation shows that

$$\begin{aligned} &\left[D_{ij}\left(x^\pi y^\lambda\right), D_{kl}\left(x_k y^\eta\right)\right] \\ &\quad = \left[-D_i\left(x^\pi y^\lambda\right) D_j - D_i\left(x^\pi y^\lambda\right) D_j, -y^\eta D_l\right] \\ &\quad = (-1)^n \left(D_i D_l\left(x^\pi y^{\lambda+\eta}\right) D_j + D_j D_l\left(x^\pi y^{\lambda+\eta}\right) D_i\right) \\ &\quad = -(-1)^n D_{ij}\left(D_l\left(x^\pi y^{\lambda+\eta}\right)\right) \in S, \end{aligned} \quad (6)$$

for all distinct elements i, j, k, l of Y and $\lambda, \eta \in G$. It follows from $zd(D_{ij}(D_l(x^\pi y^{\lambda+\eta}))) = n - 3$ that $S_{n-3} \subseteq Q$.

For distinct elements i, j, k, l, g of Y and $\lambda, \eta, \zeta \in G$, we have

$$\begin{aligned} &\left[D_{ij}\left(D_l\left(x^\pi y^{\lambda+\eta}\right)\right), D_{kg}\left(x_k y^\zeta\right)\right] \\ &\quad = (-1)^{n+1} D_{ij}\left(D_g D_l\left(x^\pi y^{\lambda+\eta+\zeta}\right)\right) \end{aligned} \quad (7)$$

and $zd(D_{ij}(D_g D_l(x^\pi y^{\lambda+\eta+\zeta}))) = n - 4$. Thus $S_{n-4} \subseteq Q$.

By the same methods above, we may obtain $D_{ij}(x^u y^\lambda) \in S$ for $u \in \mathbb{B}(n)$; that is, $S_i \subseteq Q$ for $1 \leq i \leq n - 5$.

According to $D_{ii}(x_i x_j x_k y^\lambda) = -2x_j x_k y^\lambda D_i \in S_1$ and $x_k y^{\lambda+\eta} D_i \in S_0$, we have

$$x_k y^{\lambda+\eta} D_i = \left[x_j x_k y^\lambda D_i, y^\eta D_j\right] \in Q. \quad (8)$$

Hence $S_0 \subseteq Q$.

In conclusion, $S \subseteq Q$. Therefore, the desired result follows immediately. □

3. The Natural Filtration of $S(n,m)$

Adopting the notion of [22], the element x of Lie superalgebra S is called ad-nilpotent if ad$x$ is a nilpotent linear transformation. The set of all ad-nilpotent elements of S is denoted by nil(S). Let $S_{(j)} = \oplus_{i\geq j} S_i$. Then

$$S = S_{(-1)} \supseteq S_{(0)} \supseteq S_{(1)} \supseteq \cdots \supseteq S_{(n-2)} \supseteq S_{(n-1)} = 0 \quad (9)$$

is a descending filtration of S, which is called the natural filtration of S. We also call $\{S_{(k)} \mid k \in \mathbb{Z}\}$ a filtration of S for short, where $S_{(k)} = S$ if $k \leq -1$ and $S_{(k)} = 0$ if $k \geq n-2$. Since S is $\mathbb{Z}$-graded and finite dimensional, we may easily obtain $S_{-1} \subseteq \text{nil}(S)$ and $S_{(1)} \subseteq \text{nil}(S)$.

Let $M_n(\mathbb{F})$ denote the set of all $n \times n$ matrices over $\mathbb{F}$. Notice that $\dim \mathbb{T}(m) = p^m$. Without loss of generality, we may suppose that $\{y_1, \ldots, y_{p^m}\}$ is a standard $\mathbb{F}$-basis of $\mathbb{T}(m)$. If $z = \sum_{i,j=1}^{n} \sum_{q=1}^{p^m} a_{ijq} x_i y_q D_j \in S_0$, where $a_{ijq} \in \mathbb{F}$, then let $\rho(z) = \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_{p^m} \end{pmatrix}_{np^m \times np^m}$, where $A_q = (a_{ijq})_{n\times n} \in M_n(\mathbb{F})$.

Lemma 2. *Suppose that $z = \sum_{i,j=1}^{n} \sum_{q=1}^{p^m} a_{ijq} x_i y_q D_j \in S_0$. If z is ad-nilpotent, then $\rho(z)$ is a nilpotent matrix.*

Proof. Let Γ be the representation of S_0 with values in S_{-1}. Then $\Gamma(z) = \text{ad}z$ and the matrix of $\Gamma(z)$ over the basis $\{y_1 D_1, \ldots, y_1 D_n, \ldots, y_{p^m} D_1, \ldots, y_{p^m} D_n\}$ of S_{-1} is $A = \begin{pmatrix} -(A_1)^t & & \\ & \ddots & \\ & & -(A_{p^m})^t \end{pmatrix}_{np^m \times np^m}$, where $A_q = (a_{ijq})_{n\times n} \in M_n(\mathbb{F})$. Since z is ad-nilpotent, the representation $\Gamma(z)$ is a nilpotent linear transformation. It implies that A is nilpotent. Therefore, $\rho(z) = -A^t$ is a nilpotent matrix. □

Lemma 3. *Let $z = \sum_{i=k}^{n-1} z_i$, where $z_i \in S_i$ and $k \leq n-1$. If $z \in$ nil (S) and $k \geq 0$, then $z_k \in$ nil (S).*

Proof. Suppose that $z = z_k + z'$, where $z_k \in S_k$ and $z' \in \oplus_{i=k+1}^{n-1} S_i \subseteq S_{(k+1)}$. Since $z \in \text{nil}(S)$, we may assume that $(\text{ad}z)^t = 0$. Let x be a $\mathbb{Z}$-homogeneous element of S with $\mathbb{Z}$-degree i. Then $(\text{ad}z)^t(x) = 0$. On the other hand,

$$(\text{ad}z)^t (x) = \left(\text{ad}\left(z_k + z'\right)\right)^t (x) = \left(\text{ad}z_k\right)^t (x) + h, \quad (10)$$

which implies $(\text{ad}z_k)^t(x) + h = 0$. It is easy to see that $(\text{ad}z_k)^t(x) \in S_{(kt+i)}$ and $h \in S_{(kt+i+1)} = \oplus_{j\geq kt+i+1} S_j$. Thus $(\text{ad}z_k)^t(x) = 0$. Since x is an arbitrary $\mathbb{Z}$-homogeneous element of S, we have $(\text{ad}z_k)^t(S) = 0$. Then $(\text{ad}z_k)^t = 0$; that is, $z_k \in \text{nil}(S)$. □

Suppose that E_{ij} denotes the $n \times n$ matrix whose (i,j) element is 1 and otherwise is zero. Obviously,

$$E_{ij} E_{kl} = \delta_{jk} E_{il}, \quad (11)$$

where δ_{jk} is the Kronecker delta.

If $z = \sum_{i,j=1}^{n} \sum_{q=1}^{p^m} a_{ijq} x_i y_q D_j \in S_0$, where $a_{ijq} \in \mathbb{F}$, then

$$\rho(z) = \sum_{i,j=1}^{n} a_{ij1} E_{ij} + \sum_{i,j=n+1}^{2n} a_{ij2} E_{ij} + \cdots + \sum_{i,j=n(p^m-1)+1}^{np^m} a_{ijp^m} E_{ij}. \quad (12)$$

Let $\Delta = \{z \in \text{nil}(S) \mid \text{ad}z(S) \subseteq \text{nil}(S)\}$.

Lemma 4. *Suppose that $z = \sum_{i=-1}^{n-2} z_i$, where $z_i \in S_i$. If $z \in \Delta$, then $z_{-1} = 0$.*

Proof. Suppose that $0 \neq z_{-1} = \sum_{i=1}^{n} \sum_{q=1}^{p^m} a_{iq} y_q D_i$, where $a_{iq} \in \mathbb{F}$. Let $a_{jq} \neq 0$ and $j, k, l \in Y$ such that i, j, k are distinct. We may assume that $d = [z_{-1}, D_{kl}(x_k x_l x_j)]$. A direct calculation shows that

$$\begin{aligned} d &= \left[\sum_{i=1}^{n}\sum_{q=1}^{p^m} a_{iq} y_q D_i, -x_l x_j D_l + x_k x_j D_k\right] \\ &= -\sum_{q=1}^{p^m} \left(a_{lq} x_j y_q D_l - a_{jq} x_l y_q D_l \right. \\ &\quad \left. - a_{kq} x_j y_q D_k + a_{jq} x_k y_q D_k\right). \end{aligned} \quad (13)$$

By equalities (11) and (12), we have

$$\begin{aligned} &(\rho(d))^t \\ &= (-1)^t \left((-1)^t \left(a_{j1}\right)^t E_{ll} + \left(a_{j1}\right)^t E_{kk}\right. \\ &\quad + (-1)^{t-1} a_{l1} \left(a_{j1}\right)^{t-1} E_{jl} - a_{k1} \left(a_{j1}\right)^{t-1} E_{jk} \\ &\quad + (-1)^t \left(a_{(j+n)2}\right)^t E_{(l+n)(l+n)} \\ &\quad + \left(a_{(j+n)2}\right)^t E_{(k+n)(k+n)} \\ &\quad + (-1)^{t-1} a_{(l+n)2} \left(a_{(j+n)1}\right)^{t-1} E_{(j+n)(l+n)} \\ &\quad - a_{(k+n)2} \left(a_{(j+n)2}\right)^{t-1} E_{(j+n)(k+n)} + \cdots \\ &\quad + (-1)^t \left(a_{(j+p^m-n)p^m}\right)^t E_{(l+p^m-n)(l+p^m-n)} \\ &\quad + \left(a_{(j+p^m-n)p^m}\right)^t E_{(k+p^m-n)(k+p^m-n)} \\ &\quad + (-1)^{t-1} a_{(l+p^m-n)p^m} \left(a_{(j+p^m-n)p^m}\right)^{t-1} \\ &\quad \times E_{(j+p^m-n)(l+p^m-n)} \\ &\quad \left. - a_{(k+p^m-n)p^m} \left(a_{(j+p^m-n)p^m}\right)^{t-1} E_{(j+p^m-n)(k+p^m-n)}\right). \end{aligned} \quad (14)$$

Since $(a_{j1})^t \neq 0$, we have $(\rho(d))^t \neq 0$. So $\rho(d)$ is not a nilpotent matrix. By Lemma 2, it follows that $d \notin \text{nil}(S)$. By Lemma 3, we have $[z, D_{kl}(x_k x_l x_j)] \notin \mathrm{nil}(S)$. Then $z \notin \Delta$. It contradicts $z \in \Delta$. This proves our assertion. □

Lemma 5. *Let* $z = \sum_{i=-1}^{n-2} z_i$, *where* $z_i \in S_i$. *If* $z \in \Delta$, *then* $z_0 = 0$.

Proof. Assume that $z_0 \neq 0$. Let $z_0 = \sum_{i,j=1}^{n} \sum_{q=1}^{p^m} a_{ijq} x_i y_q D_j$, $a_{ijq} \in \mathbb{F}$, and

$$l = \min\left\{i \mid a_{ij\lambda} \neq 0, i, j \in Y\right\},\qquad t = \min\left\{j \mid a_{ij\lambda} \neq 0, i, j \in Y\right\}. \tag{15}$$

(i) Suppose that $l \leq t$. Let

$$k = \max\left\{j \mid a_{lj\lambda} \neq 0, j \in Y\right\}. \tag{16}$$

Then $a_{lkq} \neq 0$. It is easy to see that $t \leq k$. Since $l \leq t$, we have $l \leq k$. Therefore,

$$z_0 = \sum_{j=t}^{k} \sum_{q=1}^{p^m} a_{ljq} x_l y_q D_j + \sum_{i=l+1}^{n} \sum_{j=t}^{n} \sum_{q=1}^{p^m} a_{ijq} x_i y_q D_j. \tag{17}$$

Assume that $l = k$. It follows from $t \leq k$ that $t \leq l$. Then we have $t = l$ which implies that

$$z_0 = \sum_{q=1}^{p^m} a_{llq} x_l y_q D_l + \sum_{i=l+1}^{n} \sum_{j=t}^{n} \sum_{q=1}^{p^m} a_{ijq} x_i y_q D_j. \tag{18}$$

Therefore,

$$\begin{aligned}\rho(z_0) = {} & a_{ll1} E_{ll} + \sum_{i=l+1}^{n} \sum_{j=t}^{n} a_{ij1} E_{ij} \\ & + a_{(l+n)(l+n)2} E_{(l+n)(l+n)} + \sum_{i=l+1+n}^{2n} \sum_{j=t+n}^{2n} a_{ij2} E_{ij} \\ & + \cdots + a_{(l+n(p^m-1))(l+n(p^m-1))p^m} E_{(l+n)(l+n)} \\ & + \sum_{i=l+1+n(p^m-1)}^{np^m} \sum_{j=t+n(p^m-1)}^{np^m} a_{ijp^m} E_{ij} \\ = {} & \begin{pmatrix} A_1 & & & & \\ B_1 & C_1 & & & \\ & & \ddots & & \\ & & & A_{p^m} & \\ & & & B_{p^m} & C_{p^m} \end{pmatrix}_{np^m \times np^m},\end{aligned} \tag{19}$$

where $A_k = a_{(l+(k-1)n)(l+(k-1)n)q} E_{(l+(k-1)n)(l+(k-1)n)}$ is an $(l+(k-1)n) \times (l+(k-1)n)$ matrix and $q \in \{1, \ldots, p^m\}$. Since $a_{ll1} \neq 0$, we have A_1 not being a nilpotent matrix. Then $\rho(z_0)$ is not a nilpotent matrix and $z_0 \notin \mathrm{nil}(S)$. Lemma 3 shows that $z \notin \mathrm{nil}(S)$. It is a contradiction of to $z \in \Delta$; that is, $l < k$.

Suppose that $h \in Y$ and $h \neq l, k$. Let $d = [z_0, x_k D_l]$. By equality (2), we obtain

$$d = \sum_{q=1}^{p^m} \left(a_{lkq} x_l y_q D_l + \sum_{i=l+1}^{n} a_{ikq} x_i y_q D_l - \sum_{j=t}^{k} a_{ljq} x_k y_q D_j \right). \tag{20}$$

Since $l < k$, $\rho(d)$ also has the matrix form as $\rho(z_0)$, it follows from $a_{lk1} \neq 0$ that A_1 is not a nilpotent matrix. Then $\rho(d)$ is not nilpotent. So $z \notin \mathrm{nil}(S)$ and $[z, x_k D_l] \notin \mathrm{nil}(S)$. It is a contradiction of $z \in \Delta$.

(ii) Suppose that $t < l$. Let $k = \max\{i \mid a_{it\lambda} \neq 0\}$ and $d' = [z, x_t D_k]$. Imitating (i), we may prove that $\rho(d')$ is also not nilpotent. Then the desired result follows. □

Lemma 6. (i) *If* $z \in S_0 \cap \mathrm{nil}(S)$ *and* $h \in S_{(1)}$, *then* $z + h \in \mathrm{nil}(S)$.

(ii) *Suppose that* i, j *are distinct elements of* Y; *then* $x_i y^\lambda D_j \in \mathrm{nil}(S)$ *for all* $\lambda \in G$.

(iii) *Suppose that* i, j, k *are distinct elements of* Y; *then* $a x_j y^\lambda D_k + b x_i y^\eta D_k \in \mathrm{nil}(S)$, *where* $a, b \in \mathbb{F}$ *and* λ, η *are arbitrary elements of* G.

Proof. (i) A direct verification shows that $\{\mathrm{ad}z\} \cup \{\mathrm{ad}S_{(1)}\}$ is a weakly closed subset of nilpotent elements of $pl(S)$, where $pl(S)$ is the general linear Lie superalgebra of S. It was shown in [23, Theorem 1 of Chapter II] that each element of $\mathrm{span}_{\mathbb{F}}(\{\mathrm{ad}z\} \cup \{\mathrm{ad}S_{(1)}\})$ is a nilpotent linear transformation of S. Then $\mathrm{ad}z + \mathrm{ad}h$ is nilpotent. So $z + h$ is ad-nilpotent.

(ii) To prove $(\mathrm{ad}x_i y^\lambda D_j)^p = 0$, we may assume without loss of generality that $i < j$. Set η to be an arbitrary element of G. If $k \neq i$, then

$$\begin{aligned}&\left(\mathrm{ad}x_i y^\lambda D_j\right)^2 (x^u y^\eta D_k) \\ &\quad = \left[x_i y^\lambda D_j, \left[x_i y^\lambda D_j, x^u y^\eta D_k\right]\right] \\ &\quad = (-1)^{\tau(u,j)} \left[x_i y^\lambda D_j, x_i x^{u-\langle j\rangle} y^{\lambda+\eta} D_k\right] \\ &\quad = 0.\end{aligned} \tag{21}$$

In the case of $k = i$, we have

$$\begin{aligned}&\left(\mathrm{ad}x_i y^\lambda D_j\right)^3 (x^u y^\eta D_k) \\ &\quad = \left[x_i y^\lambda D_j, \left[x_i y^\lambda D_j, \left[x_i y^\lambda D_j, x^u y^\eta D_i\right]\right]\right] \\ &\quad = \left[x_i y^\lambda D_j, \left[x_i y^\lambda D_j, (-1)^{\tau(u,j)} x_i x^{u-\langle j\rangle} y^\lambda D_i - x^u y^{\lambda+\eta} D_j\right]\right] \\ &\quad = (-1)^{\tau(u,j)} \left[x_i y^\lambda D_j, -x_i x^{u-\langle j\rangle} y^\lambda D_j - x_i x^{u-\langle j\rangle} y^{2\lambda+\eta} D_j\right] \\ &\quad = 0.\end{aligned} \tag{22}$$

For $p > 2$ we obtain $(\mathrm{ad}x_i y^\lambda D_j)^p (x^u y^\eta D_k) = 0$. Therefore $(\mathrm{ad}x_i y^\lambda D_j)^p (S) = 0$. This yields $(\mathrm{ad}x_i y^\lambda D_j)^p = 0$. Thus $x_i y^\lambda D_j \in \mathrm{nil}(S)$.

(iii) According to (ii) and $[x_j y^\lambda D_k, x_i y^\eta D_k] = 0$, $\{\mathrm{ad}x_j y^\lambda D_k, \mathrm{ad}x_i y^\eta D_k\}$ is a weakly closed subset of nilpotent elements of $pl(S)$. So $a x_j y^\lambda D_k + b x_i y^\eta D_k \in \mathrm{nil}(S)$, where a, $b \in \mathbb{F}$. □

Lemma 7. *If i, j, k are distinct elements of Y, then $x_ix_jy^{\lambda}D_k \in \Delta$ for all $\lambda \in G$.*

Proof. Suppose that $l \in Y \setminus \{i, j, k\}$. Then $x_ix_jy^{\lambda}D_k \in S_{(1)} \subseteq \mathrm{nil}(S)$. Let $z = \sum_{i=-1}^{n-2} z_i$, where $z_i \in S_i$. Assume that $[x_ix_jy^{\lambda}D_k, z] = f_0 + f_1$, where $f_0 = [x_ix_jy^{\lambda}D_k, z_{-1}] \in S_0$ and $f_1 \in S_{(1)}$. Let $z_{-1} = \sum_{l=1}^{n}\sum_{\eta\in G} a_{l\eta}y^{\eta}D_l$. Then

$$\begin{aligned} f_0 &= \left[x_ix_jy^{\lambda}D_k, \sum_{l=1}^{n}\sum_{\eta\in G} a_{l\eta}y^{\eta}D_l\right] \\ &= \sum_{\eta\in G}\left(a_{i\eta}x_jy^{\lambda+\eta}D_k - a_{j\eta}x_iy^{\lambda+\eta}D_k\right). \end{aligned} \tag{23}$$

By (iii) of Lemma 6, we have $f_0 \in S_0 \cap \mathrm{nil}(S)$. By (i) of Lemma 6, it follows that $f_0 + f_1 \in \mathrm{nil}(S)$. We finally obtain $x_ix_jy^{\lambda}D_k \in \Delta$ for all $\lambda \in G$. □

Let $Q = \{z \in \mathrm{nil}(S) \mid \mathrm{ad}z(\Delta) \subseteq \Delta\}$.

Lemma 8. *$Q = S_{(1)}$.*

Proof. By the definition of Δ, we have $S_{(2)} \subseteq \Delta$. Lemmas 4 and 5 show that $\Delta \subseteq S_{(1)}$. Then $[S_{(1)}, \Delta] \subseteq [S_{(1)}, S_{(1)}] \subseteq S_{(2)} \subseteq \Delta$. Thus $S_{(1)} \subseteq Q$.

Next we will prove $Q \subseteq S_{(1)}$. Let $z \in Q$ and $z = \sum_{i=-1}^{n-2} z_i$, where $z_i \in S_i$. Assume that $z_{-1} = \sum_{l=1}^{n}\sum_{\lambda\in G} a_{l\lambda}y^{\lambda}D_l \neq 0$, $a_{l\lambda} \in \mathbb{F}$. Without loss of generality, we may suppose that $a_i \neq 0$. Let $d = x_ix_jy^{\eta}D_k$, where i, j, k are distinct elements of Y and η is an arbitrary element of G. By Lemma 7, we have $d \in \Delta$. Let $[z, d] = h_0 + h_1$, where $h_0 = [z_{-1}, d] \in S_0$ and $h_1 \in S_{(1)}$. Since $a_i \neq 0$, we have $h_0 = \sum_{\lambda\in G}(a_{i\lambda}x_jy^{\lambda+\eta}D_k - a_{j\lambda}x_iy^{\lambda+\eta}D_k) \neq 0$. Lemma 5 implies that $h_0 + h_1 \notin \Delta$. It is a contradiction of $z \in Q$. Hence $z_{-1} = 0$.

Assume that $0 \neq z_0 = \sum_{i,j=1}^{n}\sum_{q=1}^{p^m} a_{ij\lambda}x_iy_qD_j$, $a_{ijq} \in \mathbb{F}$, and suppose that l and t are as the definitions in (15). We may suppose that $l \leq t$ (the proof is similar to the case $t < l$) and let k be as the definition in (16). In a similar way to the first part of the proof in Lemma 5, we have $l < k$. Suppose that $h \in Y \setminus \{l, k, t\}$ and $d_1 = x_kx_hD_l$. Lemma 7 shows that $d_1 \in \Delta$. Let $[z, d_1] = g_1 + g_2$, where $g_1 = [z_0, d_1] \in S_1$ and $g_2 \in S_{(2)}$. Using equality (2), we have

$$\begin{aligned} g_1 = \sum_{q=1}^{p^m}\Bigg(& a_{lkq}x_lx_hy_qD_l - \sum_{i=l+1}^{n} a_{ihq}x_ix_ky_qD_l \\ &- \sum_{j=t}^{k} a_{ljq}x_kx_hy_qD_j\Bigg). \end{aligned} \tag{24}$$

If $h < t$, then $a_{ihq} = 0$ in the above equality, where $i \in Y \setminus \{1, \ldots, l-1\}$. Thus

$$\begin{aligned} [D_h, g_1] = -\sum_{q=1}^{p^m}\Bigg(& a_{lkq}x_ly_qD_l + \sum_{i=l+1}^{n} a_{ihq}x_iy_qD_l \\ &+ a_{hhq}x_ky_qD_l - a_{ljq}x_ky_qD_j\Bigg). \end{aligned} \tag{25}$$

By equality (12), the matrix $\rho([D_h, g_1])$ has the matrix form as in Lemma 5. Since $a_{lkq} \neq 0$, A_1 is not a nilpotent matrix. It implies that $\rho([D_h, g_1])$ is not nilpotent. Hence $[D_h, g_1] \notin \mathrm{nil}(S)$. Lemma 3 shows that $[D_h, g_1 + g_2] \notin \mathrm{nil}(S)$; that is, $[D_h, g_1 + g_2] \notin \Delta$. It contradicts $z \in Q$. Thus $z_0 = 0$. Therefore, $z \in S_{(1)}$ and $Q \subseteq S_{(1)}$. □

According to the fact that Δ and Q are invariant subspaces under the automorphisms of S and Lemma 8, $S_{(1)}$ is also invariant under the automorphisms of S. Since

$$\begin{aligned} S_{(0)} &= \{x \in S \mid [x, S_{(1)}] \subseteq S_{(1)}\}, \\ S_{(i)} &= \{x \in S_{i-1} \mid [x, S] \subseteq S_{(i-1)}\}, \quad i \geq 1, \end{aligned} \tag{26}$$

we may easily obtain the following theorem.

Theorem 9. *The natural filtration of S is invariant under the automorphisms of S.*

Let $\mathfrak{S}_i = S_{(i)}/S_{(i+1)}$ for $-1 \leq i \leq n-2$. Then $\mathfrak{S}_i$ is a $\mathbb{Z}$-graded space. Suppose that $\mathfrak{S} := \oplus_{i=-1}^{n-2}\mathfrak{S}_i$; then $\mathfrak{S}$ is also a $\mathbb{Z}$-graded space. Let $x + S_{(i+1)} \in \mathfrak{S}_i$ and $y + S_{(j+1)} \in \mathfrak{S}_j$. Define

$$\left[x + S_{(i+1)}, y + S_{(j+1)}\right] := [x, y] + S_{(i+j+1)}. \tag{27}$$

It is easy to see that the definition above is reasonable. There exists a linear expansion such that $\mathfrak{S}$ has an operator $[,]$. A direct verification shows that $\mathfrak{S}$ is a Lie superalgebra with respect to the operator $[,]$. The Lie superalgebras $\mathfrak{S}$ is called a Lie superalgebra induced by the natural filtration of S.

Lemma 10. *$\mathfrak{S} \cong S$.*

Proof. Let $\phi : S \to \mathfrak{S}$ be a linear map such that $\phi(x) = x + S_{(i+1)}$, where $x \in S_{(i)} \setminus S_{(i+1)}$. A direct verification shows that ϕ is a homomorphism of Lie superalgebras. Suppose that $y \in \ker\phi$. If $y \neq 0$, then there exists $i \geq -1$ such that $y \in S_{(i)} \setminus S_{(i+1)}$. Since $\phi(y) = 0$, we have $y + S_{(i+1)} = 0$. Hence $y \in S_{(i+1)}$. That shows that $y = 0$. Thus, $\ker\phi = 0$. Therefore, ϕ is a monomorphism. It follows from the fact S is finite dimensional that ϕ is an isomorphism. □

The definition of ϕ shows that

$$\begin{aligned} \phi(S_i) &= \{x + S_{(i+1)} \mid x \in S_i\} = \{x + S_{(i+1)} \mid x \in S_{(i)}\} \\ &= S_{(i)}/S_{(i+1)} = \mathfrak{S}_i, \quad i \geq -1. \end{aligned} \tag{28}$$

Suppose that m, n, m', n' are elements of $\mathbb{N}_0$ and n, n' are greater than 3. In a similar way to S, the Lie superalgebra $S(n', m')$ will be simply denoted by S'. According to the definitions of Δ, Q, and $\mathfrak{S}$ in S, the Δ', Q', and $\mathfrak{S}'$ in S' are also defined by the same method, respectively.

Proposition 11. *Suppose that $S \cong S'$ and σ is an isomorphism from S to S'; then $\sigma(S_{(i)}) = S'_{(i)}$ for all $i \geq -1$.*

Proof. It is clear that $\sigma(S_{(-1)}) = S'_{(-1)}$ and $\sigma(\mathrm{nil}(S)) = \mathrm{nil}(S')$. A direct verification shows that $\sigma(\Delta) = \Delta'$. Hence $\sigma(Q) = Q'$.

By virtue of Lemma 8, we have $Q = S_{(1)}$ and $Q' = S'_{(1)}$. Thus $\sigma(S_{(1)}) = S'_{(1)}$. By equalities (26), the desired result $\sigma(S_{(i)}) = S'_{(i)}$ for all $i \geq -1$ is obtained. □

Lemma 12. *Suppose that $S \cong S'$ and σ is an isomorphism from S to S'; then σ induces an isomorphism $\widetilde{\sigma}$ from $\mathfrak{S}$ to $\mathfrak{S}'$ such that $\widetilde{\sigma}(\mathfrak{S}_i) = \mathfrak{S}'_i$ for all $i \geq -1$.*

Proof. Define a linear map $\widetilde{\sigma} : \mathfrak{S} \to \mathfrak{S}'$ such that

$$\widetilde{\sigma}\left(x + S_{(i+1)}\right) = \sigma(x) + S'_{(i+1)}, \tag{29}$$

where $x + S_{(i+1)} \in \mathfrak{S}_i$. Using Proposition 11, the definition of $\widetilde{\sigma}$ is reasonable and

$$\begin{aligned} &\widetilde{\sigma}\left(\left[x + S_{(i+1)}, y + S_{(j+1)}\right]\right) \\ &\quad = \sigma([x, y]) + S'_{(i+j+1)} \\ &\quad = \left[\sigma(x) + S'_{(i+1)}, \sigma(y) + S'_{(j+1)}\right] \\ &\quad = \left[\widetilde{\sigma}\left(x + S'_{(i+1)}\right), \widetilde{\sigma}\left(y + S'_{(j+1)}\right)\right]. \end{aligned} \tag{30}$$

Thus $\widetilde{\sigma}$ is a homomorphism from $\mathfrak{S}$ to $\mathfrak{S}'$. Clearly, $\widetilde{\sigma}(\mathfrak{S}_i) = \mathfrak{S}'_i$ for all $i \geq -1$. It follows that $\widetilde{\sigma}$ is an epimorphism.

Suppose that $y \in \ker\widetilde{\sigma}$; then $y \in \mathfrak{S}$. So we may suppose that $y = \sum_{i=-1}^{n-2} y_i$ and $y_i \in \mathfrak{S}_i$. Since $\mathfrak{S}_i = S_{(i)}/S_{(i+1)}$, let $y_i = z_i + S_{(i+1)}$, where $z_i \in S_{(i)}$. Hence $\widetilde{\sigma}(y_i) = \sigma(z_i) + S'_{(i+1)}$. It follows from $\widetilde{\sigma}(y) = 0$ that $\sum_{i=-1}^{n-2} \widetilde{\sigma}(y_i) = 0$. Thus $\widetilde{\sigma}(y_i) = 0$; that is, $\sigma(z_i) + S'_{(i+1)} = 0$. It follows that $\sigma(z_i) \in S'_{(i+1)}$. By Proposition 11, we have $z_i \in \sigma^{-1}(S'_{(i+1)}) = S_{(i+1)}$. Then $y_i = z_i + S_{(i+1)} = 0$ for $-1 \leq i \leq n-2$. Therefore, $y = 0$ and $\ker\widetilde{\sigma} = 0$. Consequently, $\widetilde{\sigma}$ is an isomorphism induced by σ such that $\widetilde{\sigma}(\mathfrak{S}_i) = \mathfrak{S}'_i$ for all $i \geq -1$. □

Theorem 13. *$S \cong S'$ if and only if $m = m'$ and $n = n'$.*

Proof. Because the sufficiency is obvious, it suffices to prove the necessity. Suppose that $\phi : S \to \mathfrak{S}$ is the isomorphism given in the proof of Lemma 10. Similarly, there also exists the $\phi' : S' \to \mathfrak{S}'$. According to the equality (28) and Lemma 12, we have

$$\phi(S_i) = \mathfrak{S}_i, \qquad \phi'\left(S'_i\right) = \mathfrak{S}'_i, \qquad \widetilde{\sigma}(\mathfrak{S}_i) = \mathfrak{S}'_i \tag{31}$$

for $-1 \leq i \leq n-2$. Let $\psi = (\phi')^{-1}\widetilde{\sigma}\phi$. Then

$$\psi(S_i) = \left(\phi'\right)^{-1}\widetilde{\sigma}\phi(S_i) = \left(\phi'\right)^{-1}\widetilde{\sigma}(\mathfrak{S}_i) = \left(\phi'\right)^{-1}\left(\mathfrak{S}'_i\right) = S'_i. \tag{32}$$

In particular, $\psi(S_{-1}) = S'_{-1}$. It follows from $\dim S_{-1} = \dim S'_{-1}$ that $np^m = n'p^{m'}$. By virtue of the definition of S_i, we have

$$S_0 = \operatorname{span}_{\mathbb{F}}\left\{D_{ij}\left(x_k x_l y^{\lambda}\right) \in S \mid i, j, k, l \in Y, \lambda \in G\right\}. \tag{33}$$

Thus $\dim S_0 = (n^2 - 1)p^m$. Similarly, $\dim S'_0 = (n'^2 - 1)p^{m'}$. According to $\dim S_0 = \dim S'_0$ and $np^m = n'p^{m'}$, we have $n = n'$. In conclusion, the proof is completed. □

Acknowledgments

The authors thank Yang Jiang for the helpful comments and suggestions. They also give their special thanks to the referees for many helpful suggestions. This work was supported by the National Natural Science Foundation of China (Grant no. 11171055) and the Fundamental Research Funds for the Central Universities (no. 12SSXT139).

References

[1] V. G. Kac, "Lie superalgebras," *Advances in Mathematics*, vol. 26, no. 1, pp. 8–96, 1977.

[2] M. Scheunert, *The Theory of Lie Superalgebras*, vol. 716 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1979, An introduction.

[3] Y. C. Su, "Classification of finite-dimensional modules of the Lie superalgebra $sl(2/1)$," *Communications in Algebra*, vol. 20, no. 11, pp. 3259–3277, 1992.

[4] Y. Kochetkov and D. Leites, "Simple Lie algebras in characteristic 2 recovered from superalgebras and on the notion of a simple finite group," in *Proceedings of the International Conference on Algebra*, vol. 131 of *Contemporary Mathematics*, pp. 59–67, American Mathematical Society, Providence, RI, USA, 1992.

[5] V. M. Petrogradski, "Identities in the enveloping algebras for modular Lie superalgebras," *Journal of Algebra*, vol. 145, no. 1, pp. 1–21, 1992.

[6] Y. Z. Zhang, "Finite-dimensional Lie superalgebras of Cartan type over fields of prime characteristic," *Chinese Science Bulletin*, vol. 42, no. 9, pp. 720–724, 1997.

[7] H. Strade, *Simple Lie Algebras Over Fields of Positive Characteristic*, vol. 38, Walter de Gruyter & Co., Berlin, Germany, 2004.

[8] V. G. Kac, "Classification of infinite-dimensional simple linearly compact Lie superalgebras," *Advances in Mathematics*, vol. 139, no. 1, pp. 1–55, 1998.

[9] Y. Chen and W. Liu, "Finite-dimensional odd contact superalgebras over a field of prime characteristic," *Journal of Lie Theory*, vol. 21, no. 3, pp. 729–754, 2011.

[10] W. D. Liu and J. X. Yuan, "Special odd Lie superalgebras in prime characteristic," *Science China*, vol. 55, no. 3, pp. 567–576, 2012.

[11] B. Shu and Y. F. Yao, "Character formulas for restricted simple modules of the special superalgebras," *Mathematische Nachrichten*, vol. 285, no. 8-9, pp. 1107–1116, 2012.

[12] L. M. Tang and W. D. Liu, "Automorphism groups of finite-dimensional special odd Hamiltonian superalgebras in prime characteristic," *Frontiers of Mathematics in China*, vol. 7, no. 5, pp. 907–918, 2012.

[13] J. X. Yuan, Y. Chen, and W. D. Liu, "Automorphisms and p-characters of restricted Lie superalgebras of Cartan type," *Algebra Colloquium*, vol. 18, no. 3, pp. 397–410, 2011.

[14] R. E. Block and R. L. Wilson, "Classification of the restricted simple Lie algebras," *Journal of Algebra*, vol. 114, no. 1, pp. 115–259, 1988.

[15] H. Strade, "The classification of the simple modular Lie algebras—4. Determining the associated graded algebra," *Annals of Mathematics*, vol. 138, no. 1, pp. 1–59, 1993.

[16] V. G. Kac, "Description of filtered Lie algebras with which graded Lie algebras of Cartan type are associated," *Mathematics of the USSR-Izvestiya*, vol. 8, pp. 801–835, 1974.

[17] A. I. Kostrikin and I. R. Shafarevic, “Graded Lie algebras of finite characteristic,” *Mathematics of the USSR-Izvestiya*, vol. 3, pp. 237–304, 1969.

[18] Y. Z. Zhang and H. C. Fu, “Finite-dimensional Hamiltonian Lie superalgebra,” *Communications in Algebra*, vol. 30, no. 6, pp. 2651–2673, 2002.

[19] Y. Z. Zhang and J. Z. Nan, “Finite-dimensional Lie superalgebras $W(m,n,t)$ and $S(m,n,t)$ of Cartan type,” *Advances in Mathematics*, vol. 27, no. 3, pp. 240–246, 1998.

[20] S. Awuti and Y. Z. Zhang, “Modular Lie superalgebra $\overline{W}(n,m)$,” *Journal of Northeast Normal University*, vol. 40, no. 2, pp. 7–11, 2008 (Chinese).

[21] L. Ren, Q. Mu, and Y. Z. Zhang, “A class of finite-dimensional Lie superalgebra of Hamiltonian type,” *Algebra Colloquium*, vol. 18, no. 2, pp. 347–360, 2011.

[22] N. Jin, “ad-nilpotent elements, quasi-nilpotent elements and invariant filtrations of infinite-dimensional Lie algebras of Cartan type,” *Science in China*, vol. 35, no. 10, pp. 1191–1200, 1992 (Chinese).

[23] N. Jacobson, *Lie Algebras*, Interscience Publishers, New York, NY, USA, 1962.