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Research Article

Construction of Nodal Bubbling Solutions for the Weighted Sinh-Poisson Equation

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We consider the weighted sinh-Poisson equation $\Delta u+2\varepsilon^2|x|^{2\alpha}\sinh u=0$ in $B_1(0)$, $u=0$ on $\partial B_1(0)$, where $\varepsilon>0$ is a small parameter, $\alpha\in(-1,+\infty)\setminus\{0\}$, and $B_1(0)$ is a unit ball in $\mathbb{R}^2$. By a constructive way, we prove that for any positive integer m, there exists a nodal bubbling solution u_ε which concentrates at the origin and the other m-points $\tilde{q}_l=(\lambda\cos(2\pi(l-1)/m),\lambda\sin(2\pi(l-1)/m))$, $l=2,\dots,m+1$, such that as $\varepsilon\to 0$, $2\varepsilon^2|x|^{2\alpha}\sinh u_\varepsilon\rightharpoonup 8\pi(1+\alpha)\delta_0+\sum_{l=2}^{m+1}8\pi(-1)^{l-1}\delta_{\tilde{q}_l}$, where $\lambda\in(0,1)$ and m is an odd integer with $(1+\alpha)(m+2)-1>0$, or m is an even integer. The same techniques lead also to a more general result on general domains.

1. Introduction

We are concerned with stationary Euler equations for an incompressible, homogeneous, and inviscid fluid on a bounded, smooth planar domain Ω, consider

$$\begin{aligned}(\mathbf{w}\cdot\nabla)\,\mathbf{w}+\nabla p=0,&\quad\text{in }\Omega,\\ \operatorname{div}\mathbf{w}:=\nabla\cdot\mathbf{w}=0,&\quad\text{in }\Omega,\\ \mathbf{w}\cdot\nu=0,&\quad\text{on }\partial\Omega,\end{aligned}\tag{1}$$

where $\mathbf{w}$ is the velocity field, p is the pressure, and ν is the unit outer normal vector to $\partial\Omega$. Let us introduce the vorticity $\omega=\operatorname{curl}\mathbf{w}$. By applying the curl operator to the first equation in (1), we have

$$\mathbf{w}\cdot\nabla\omega=0,\quad\text{in }\Omega.\tag{2}$$

On the other hand, the second equation is equivalent to rewriting the velocity field $\mathbf{w}$ as

$$\mathbf{w}=(\nabla\psi)^{\perp}:=\left(-\partial_{x_2}\psi,\partial_{x_1}\psi\right).\tag{3}$$

In return, the vorticity ω is expressed as $\omega=-\Delta\psi$ in term of ψ, the so-called stream function. Now, the ansatz $\omega=\omega(\psi)$ guarantees that (2) is also automatically satisfied, and then the Euler equations reduce to solving the Dirichlet elliptic problem as follows

$$\begin{aligned}-\Delta\psi=\omega(\psi),&\quad\text{in }\Omega,\\ \psi=0,&\quad\text{on }\partial\Omega.\end{aligned}\tag{4}$$

Over the past decades, some vortex-type configurations for planar stationary turbulent Euler flows have aroused wide concern among the people (see [1–3]). Many functions $\omega(\psi)$ have been chosen in the physical perspective to describe turbulent Euler flows with vorticity ω concentrated in small "blobs". For example, on the basis of the statistical mechanics approach, Joyce and Montgomery proposed the Stuart vortex pattern $\omega(\psi)=\varepsilon^2e^{\psi}$ with a small positive parameter ε to describe positive vortices (see [4–8]). Meanwhile, they also proposed the Mallier-Maslowe vortex pattern $\omega(\psi)=2\varepsilon^2\sinh\psi$ to describe coexisting positive and negative vortices (see [9, 10]). Recently, Tur and Yanovsky in [11] have used the singular ansatz $\omega(\psi)=\varepsilon^2e^{\psi}-4\pi N\delta_q$ to describe vortex patterns of necklace type with $N+1$-fold symmetry in rational shear flow, where $N \in \mathbb{N}$ and δ_q denotes the Dirac mass at $q \in \Omega$. Now, we adopt another new singular ansatz in [12] $\omega(\psi) = 2\varepsilon^2|x|^{2\alpha}\sinh\psi$ with $\alpha \in (-1,+\infty)\setminus\{0\}$ to study the corresponding vortices with concentrated vorticity. To do it, we hope to investigate the effect of the presence of the weight $|x|^{2\alpha}$ on the existence of nodal bubbling solutions for the weighted sinh-Poisson equation as follows:

$$\Delta u + 2\varepsilon^2|x|^{2\alpha}\sinh u = 0, \quad \text{in } B_1(0), \qquad u = 0, \quad \text{on } \partial B_1(0), \tag{5}$$

where $\varepsilon > 0$ is a small parameter, $\alpha \in (-1,+\infty)\setminus\{0\}$, and $B_1(0)$ is a unit ball in $\mathbb{R}^2$.

Let us first recall the two-dimensional sinh-Poisson equation as follows:

$$\Delta u + 2\varepsilon^2 \sinh u = 0, \tag{6}$$

which relates to various dynamics of vorticity with respect to geophysical flows, rotating and stratified fluids, and fluid layers excited by electromagnetic forces (see [13–15] and the references therein) and the geometry of constant mean curvature surfaces studied by many works (see [16–20] and the references therein). Recently, the asymptotic behavior of solutions to (6) has been studied on a closed Riemann surface in [21, 22], and the authors applied the so-called "Pohozaev identity" and "symmetrization method," respectively, to show that there possibly exist two different types of blow up for a family of solutions to (6). Furthermore, Grossi and Pistoia in [23] exhibited sign-changing multiple blow-up phenomena for the Dirichlet problem (6), more precisely, if $0 \in \Omega$ and Ω is symmetric with respect to the origin, for any integer k if ε is small enough, there exists a family of solutions to (6), which blows up at the origin, whose positive mass is $4\pi k(k-1)$ and negative mass is $4\pi k(k+1)$. This gives a complete answer to an open problem in [21]. Besides, for a similar equation, precisely the Neumann sinh-Gordon equation on a unit ball, Esposito and Wei in [24] also constructed a family of solutions with a multiple blow up at the origin. On the other hand, Bartolucci and Pistoia in [25] tried to construct blow-up solutions of (6) with Dirichlet boundary condition, and proved that for $\varepsilon > 0$ small enough, there exist at least two pairs of solutions, which change sign exactly once, concentrate in the domain, and whose nodal lines intersect the boundary. Furthermore, Bartsch et al. in [26] obtained the existence of changing sign solutions for this equation on an arbitrary bounded domain Ω, which have three and four alternate-sign concentration points. In particular, when Ω has an axial symmetry they proved for each $N \in \mathbb{N}$ there exists a nodal bubbling solution, which changes sign N-times and whose alternate-sign concentration points align on the symmetric axis of Ω. For (6) with Neumann boundary condition, Wei et al. in [27] constructed a family of solutions concentrating positively and negatively in the domain and its boundary. As for the presence of the weight $|x|^{2\alpha}$, the authors in [12] showed that there exists a family of nodal bubbling solutions u_ε to (5) only involving $0 < \alpha \notin \mathbb{N}$, such that $2\varepsilon^2|x|^{2\alpha}\sinh u_\varepsilon$ not only develops many positive and negative Dirac deltas with weight 8π and -8π, respectively, but also a Dirac data with weight $8\pi(1+\alpha)$ at the origin.

We mention that an analogous blow-up analysis can be applied to multiple blow-up solutions for the Liouville equation with or without singular data as follows:

$$-\Delta u = \varepsilon^2 e^u - 4\pi\sum_{i=1}^{M}\alpha_i\delta_{q_i}, \quad \text{in } \Omega, \qquad u = 0, \quad \text{on } \partial\Omega, \tag{7}$$

where Ω is a smooth bounded domain in $\mathbb{R}^2$, $M \geq 0$, $\alpha_i > -1$, $q_i \in \Omega$, δ_{q_i} defines the Dirac mass at q_i, and $\varepsilon > 0$ is a small parameter. For the past decades, the asymptotic analysis for blow up solutions of problem (7) has been deeply studied in the vast literature (see [28–33] and the reference therein), which exhibits the quantiaztion properties of the weak limit of $\varepsilon^2 e^u$ as $\varepsilon \to 0$ if $\varepsilon^2\int_\Omega e^u$ remains uniformly bounded, and characterizes the location of concentration points as critical points of a functional in terms of Green's function. Reciprocally, an obvious problem is how to construct solutions of (7) with these properties. In [34, 35], the authors use the asymptotic analysis to construct solutions with multiple interior concentration points for (7) with $M \geq 0$ and $0 < \alpha_i \notin \mathbb{N}$. More generally, by a constructive way, similar results related to $-1 < \alpha_i$ can also be obtained in [36–40] under some milder notions of stability of critical points. In particular, when $M \geq 1$ and α_is are positive numbers, D'Aprile in [37] recently established a family of solutions to (7) consisting of $\widetilde{N} := \sum_{i=1}^{M} N_i$ blow up points in $\Omega\setminus\{q_1,\ldots,q_M\}$ as long as $N_i < 1+\alpha_i$ for any i, provided that the weights α_i avoid the integers $1,2,\ldots,\widetilde{N}-1$, and so the result of del Pino et al. in [39] can be extended to the case of several singular sources.

In this paper, we will continue the study of the existence of solutions to (5). We prove that there exists a family of solutions u_ε concentrating positively and negatively at the origin and outside the origin as long as $\alpha \in (-1,+\infty)\setminus\{0\}$. Concerning the sign-changing concentration at the origin and outside the origin, if we introduce the function $f(\lambda) : (0,1) \to \mathbb{R}$ defined as

$$f(\lambda) = -16\pi\left\{\sum_{l,j=2,l\neq j}^{m+1}(-1)^{l+j}\times\left[\frac{1}{2}\log\left(\lambda^4+1-2\lambda^2\times\cos\left(2\pi\frac{l-j}{m}\right)\right) - \log\left|2\sin\left(\pi\frac{l-j}{m}\right)\right|\right] + \left[(1+\alpha)(m-2A)-A^2\right]\log\lambda + m\log\left(1-\lambda^2\right)\right\}, \tag{8}$$

with $A = \sum_{l=2}^{m+1}(-1)^{l+1}$, our main result for problem (5) can be stated as follows.

Theorem 1. *For any positive integer m, there exists a nodal solution u_ε to problem (5) which concentrates at the origin and the other m-points $\tilde{q}_l = (\lambda_0\cos(2\pi(l-1)/m), \lambda_0\sin(2\pi(l-1)/m))$, $l = 2,\dots,m+1$, such that as $\varepsilon \to 0$,*

$$2\varepsilon^2|x|^{2\alpha}\sinh u_\varepsilon \rightharpoonup 8\pi(1+\alpha)\delta_0 + \sum_{l=2}^{m+1}8\pi(-1)^{l-1}\delta_{\tilde{q}_l}, \quad (9)$$

weakly in the sense of measures in $\overline{B_1(0)}$, where λ_0 is an absolute minimum point of $f(\lambda)$ in $(0,1)$, m is an odd integer with $(1+\alpha)(m+2)-1>0$, or m is an even integer. Moreover, for any $\delta>0$, u_ε remains uniformly bounded on $B_1(0)\setminus(B_\delta(0)\cup\bigcup_{l=2}^{m+1}B_\delta(\tilde{q}_l))$, and as $\varepsilon \to 0$,

$$\sup_{B_\delta(0)} u_\varepsilon \longrightarrow +\infty,$$
$$\sup_{B_\delta(\tilde{q}_l)} (-1)^{l-1}u_\varepsilon \longrightarrow +\infty \quad \forall l = 2,\dots,m+1. \quad (10)$$

Theorem 1 is based on a constructive way which also works for the more generally weighted sinh-Poisson equation as follows:

$$\Delta u + 2\varepsilon^2 c(x)\prod_{i=1}^{M}|x-q_i|^{2\alpha_i}\sinh u = 0, \quad \text{in } \Omega,$$
$$u = 0, \quad \text{on } \partial\Omega, \quad (11)$$

for $\varepsilon > 0$ small, where Ω is a bounded smooth domain in $\mathbb{R}^2$, $M = n+k$ with $M \geq 0$, $\{q_1,\dots,q_M\}$ are different singular sources in Ω, $\{\alpha_1,\dots,\alpha_n\} \subset (-1,+\infty)\setminus(\mathbb{N}\cup\{0\})$, $\{\alpha_{n+1},\dots,\alpha_M\}\subset\mathbb{N}$, and $c:\Omega\to\mathbb{R}$ is a continuous function such that $c(q_i)>0$ for any $i=1,\dots,M$.

To further state our results, we need to introduce some notations. Let $G(x,y)$ be Green's function of Δ_x such that for $y\in\Omega$, $-\Delta_x G(x,y) = \delta_y(x)$ in Ω and $G(x,y)=0$ on $\partial\Omega$, and let $H(x,y)$ be its regular part defined as $H(x,y) = G(x,y) + (1/2\pi)\log|x-y|$. Besides, let us denote $S(x) = \prod_{i=1}^{M}|x-q_i|^{2\alpha_i}$, $\Omega' = \{x\in\Omega : c(x)>0\}$, $\Gamma = \{q_1,\dots,q_n\}$, $J_1 = \{1,\dots,n\}$, $J_2 = \{n+1,\dots,n+k\}$, $J_3 = \{n+k+1,\dots,n+k+m\}$, and $\Delta_{k+m} = \{(q_{n+1},\dots,q_{n+k+m}) : q_i = q_j \quad \text{for some } i\neq j\}$, where $m\geq 0$ is an integer. In what follows, we fix n different points q_i, $i\in J_1$, and define

$$\varphi_{k,m}^n(q) = -\sum_{i\in J_2\cup J_3} d_i\left\{\sum_{j\in J_1}a_ia_jd_jG(q_i,q_j) + \frac{1}{2}d_iH(q_i,q_i) + \sum_{j\in J_2\cup J_3, j\neq i}\frac{1}{2}a_ia_jd_jG(q_i,q_j)\right\} - \sum_{i\in\bigcup_{l=1}^{3}J_3} d_i\log c_i(q_i), \quad (12)$$

which is well defined on the following domain:

$$\Lambda_{k,m} = \frac{(\Omega'\setminus\Gamma)^{k+m}}{\Delta_{k+m}}, \quad (13)$$

where $q = (q_{n+1},\dots,q_{M+m})$, $\alpha_i = 0$ for $i\in J_3$, $c_i(x) = c(x)S(x)/|x-q_i|^{2\alpha_i}$ for $i\in\bigcup_{l=1}^{3}J_l$, $d_i = 8\pi(1+\alpha_i)$ for $i\in\bigcup_{l=1}^{3}J_l$, and $a_i=\pm1$ for $i\in\bigcup_{l=1}^{3}J_l$.

Definition 2 (see [41]). We say that q^* is a C^0-stable critical point of $\varphi_{k,m}^n$ in $\Lambda_{k,m}$ if for any sequence of functions ψ_j such that $\psi_j\to\varphi_{k,m}^n$ uniformly on the compact subsets of $\Lambda_{k,m}$, ψ_j has a critical point ξ_j such that $\psi_j(\xi_j)\to\varphi_{k,m}^n(q^*)$.

In particular, if q^* is a strict local maximum or minimum point of $\varphi_{k,m}^n$, q^* is a C^0-stable critical point of $\varphi_{k,m}^n$.

Theorem 3. *Assume that $\{n,k,m\}\subset\mathbb{N}\cup\{0\}$ and $q^* = (q_{n+1}^*,\dots,q_{M+m}^*)$ is a C^0-stable critical point of $\varphi_{k,m}^n$ in $\Lambda_{k,m}$ with $k+m\geq 1$. Then, for any sufficiently small $\varepsilon>0$, there exists different points $q_{\varepsilon,l}\in\Omega'\setminus\Gamma$, $l\in J_2\cup J_3$, away from $\partial\Omega\cup\Gamma$, so that problem (11), for $q_l = q_{\varepsilon,l}$, $l\in J_2$, has a nodal solution u_ε such that as $\varepsilon\to 0$,*

$$2\varepsilon^2 c(x)\prod_{i=1}^{M}|x-q_i|^{2\alpha_i}\sinh u_\varepsilon - \sum_{J_2\cup J_3}a_ld_l\delta_{q_{\varepsilon,l}} \rightharpoonup \sum_{J_1}a_ld_l\delta_{q_l}, \quad (14)$$

weakly in the sense of measures in $\overline{\Omega}$. Moreover, up to a subsequence, there exists $\tilde{q} = (\tilde{q}_{n+1},\dots,\tilde{q}_{M+m})\in\Lambda_{k,m}$ such that

$$\varphi_{k,m}^n(\tilde{q}) = \varphi_{k,m}^n(q^*),$$
$$d(q_{\varepsilon,l},\tilde{q}_l)\longrightarrow 0, \quad \forall l\in J_2\cup J_3, \text{ as } \varepsilon\longrightarrow 0. \quad (15)$$

Besides, u_ε remains uniformly bounded on $\overline{\Omega}\setminus(\bigcup_{i\in J_1}B_\delta(q_i))\cup(\bigcup_{i\in J_2\cup J_3}B_\delta(\tilde{q}_i))$ for any $\delta>0$, and for any points q_l, $l\in J_1$, and $\tilde{q}_l$, $l\in J_2\cup J_3$, as $\varepsilon\to 0$,

$$\sup_{\overline{\Omega}\cap B_\delta(q_l)} a_lu_\varepsilon \longrightarrow +\infty, \qquad \sup_{\overline{\Omega}\cap B_\delta(\tilde{q}_l)} a_lu_\varepsilon \longrightarrow +\infty. \quad (16)$$

Note that for the case $n=1$ and $k=0$ (or $n=k=0$), Theorem 3 was partly proved in [12] (or [25]) only when $c(x)=1$ and $\alpha_l\in(0,+\infty)\setminus\mathbb{N}$, $l\in J_1$. In contrast with the results of [12, 25], this theorem provides a more complex concentration phenomenon involving the existence of changing-sign solutions for problem (11) with both positive and negative bubbles near the singular sources q_l, $l\in J_1\cup J_2$, and some other discrete points. Unlike the concentration set in [12] only contains singular sources q_l with $\alpha_l\in(0,+\infty)\setminus\mathbb{N}$ and $l\in J_1$, and no singular source points in the domain, our concentration set also contains some singular sources q_l with $\alpha_l\in(-1,0)$ and $l\in J_1$, except for singular sources q_l, $l\in J_2$, where concentration points and singular sources coincide at the limit. As for the latter exception, which till now is a similar but very simple concentration phenomenon it appears only in [38] for the study of the Liouville equation with a singular source of integer multiplicity.

In order to obtain multiple sign-changing blow up solutions of problem (11), we use a Lyapunov-Schmidt finite-dimensional reduction scheme and convert the problem into a finite-dimensional one, for a suitable asymptotic reduced energy, related to $\varphi^n_{k,m}$ in (12). Thus, a stable critical point of $\varphi^n_{k,m}$ leads to the existence of multiple sign-changing blow up solutions to (11). However, in view of different signs of Green's functions in (12), it seems very difficult to find out a stable critical point of $\varphi^n_{k,m}$ for a general bounded domain Ω. A simple approach can help us to overcome this difficulty by imposing the very strong symmetry condition on the domain of the problem, namely, we use the symmetry of the unit ball $B_1(0)$ to reduce the problem of finding solutions of (5) to that of finding an absolute minimum point of $f(\lambda)$ defined in (8), and so we get the existence of nodal bubbling solutions for (5) in Theorem 1. On the other hand, motivated by the obtained results in [23], we believe that Theorem 1 should be valid for a general domain than a unit ball. More precisely, we suspect that, if $0 \in \Omega$ and Ω is symmetric with respect to the origin, it is possible to construct a family of sign-changing blow up solutions whose maxima and minima are located alternately at the origin and the vertices of a regular polygon, and so Theorem 1 will be a consequence of this general result.

It is important to point out that to prove the above results, we need to use classification solutions of the Liouville-type equation to construct approximate solutions of problem (5) (or (11)) as follows:

$$\Delta u + |x|^{2\alpha} e^u = 0, \quad \text{in } \mathbb{R}^2, \qquad \int_{\mathbb{R}^2} |x|^{2\alpha} e^u < +\infty, \quad \alpha \in (-1, +\infty). \tag{17}$$

In complex notations, a complete classification of the solutions of (17) takes the following form:

$$u(z) = \log \frac{8(1+\alpha)^2 \tilde{\delta}^2}{\left(\tilde{\delta}^2 + |z^{\alpha+1} - \xi|^2\right)^2}, \tag{18}$$

where $\tilde{\delta} > 0$, $\xi \in \mathbb{C}$ if $\alpha \in \mathbb{N} \cup \{0\}$ and $\xi = 0$ if $\alpha \in (-1, +\infty) \setminus (\mathbb{N} \cup \{0\})$ (see [33, 42–44]). Using classification solutions scaled up and projected to satisfy the Dirichlet boundary condition up to a right order, the approximate solutions can be built up as a summation of these initial approximations with some suitable signs. Thus, the nodal bubbling solutions can be constructed as a small additive perturbation of these approximations through the so-called "localized energy method," which combined the Lyapunov-Schmidt finite dimensional reduction and variational techniques. Here, we follow [12, 25], but we will overcome some of the difficulties that the nodal concentration phenomenon brings by delicate analysis.

2. Construction of the Approximate Solution

In this section, we will provide a first approximation for the solutions of problem (11). Given a sufficiently small but fixed number $\delta > 0$, let us first fix n different points q_i, $i \in J_1$, and assume that points $q = (q_{n+1}, \ldots, q_{n+k+m}) \in \Lambda_{k,m}(\delta)$, where

$$\Lambda_{k,m}(\delta) := \left\{ q \in \Lambda_{k,m} : \min_{i \in J_2 \cup J_3} d\left(q_i, \partial\left(\Omega' \setminus \Gamma\right)\right) \geq 2\delta, \min_{i,j \in J_2 \cup J_3, i \neq j} d\left(q_i, q_j\right) \geq 2\delta \right\}. \tag{19}$$

Suppose that μ_i, $i \in \bigcup_{l=1}^3 J_l$, are positive numbers to be chosen later, we define

$$\rho_i = \varepsilon^{(1/(\alpha_i+1))}, \qquad \nu_i = \mu_i^{(1/(\alpha_i+1))},$$
$$u_i(x) = \log \frac{8\mu_i^2 (1+\alpha_i)^2}{c_i(q_i)\left[\mu_i^2 \varepsilon^2 + |x - q_i|^{2(1+\alpha_i)}\right]^2}. \tag{20}$$

The ansatz is

$$U_q(x) = \sum_{i \in \bigcup_{l=1}^3 J_l} a_i P u_i(x), \tag{21}$$

where $a_i = \pm 1$, $P : H^1(\Omega) \to H_0^1(\Omega)$ is a linear operator such that for any $u \in H^1(\Omega)$, $\Delta P u = \Delta u$ in Ω, and $Pu = 0$ on $\partial\Omega$. By harmonicity, we easily get that

$$Pu_i(x) = u_i(x) + d_i H(x, q_i) - \log \frac{8\mu_i^2 (1+\alpha_i)^2}{c_i(q_i)} + O(\rho), \quad \text{in } C^1(\overline{\Omega}), \tag{22}$$

$$Pu_i(x) = d_i G(x, q_i) + O(\rho), \quad \text{in } C_{\text{loc}}\left(\overline{\Omega} \setminus \{q_i\}\right), \tag{23}$$

where $\rho := \max\{\rho_i : i \in \bigcup_{l=1}^3 J_l\}$.

Consider that the scaling of solution to problem (11) is as follows:

$$v(y) = u(\varepsilon y), \quad \forall y \in \overline{\Omega}_\varepsilon, \tag{24}$$

where $\Omega_\varepsilon = (1/\varepsilon)\Omega$, then v satisfies

$$\Delta v + 2\varepsilon^4 c(\varepsilon y) S(\varepsilon y) \sinh v = 0, \quad \text{in } \Omega_\varepsilon, \qquad v = 0, \quad \text{on } \partial\Omega_\varepsilon. \tag{25}$$

We will seek solutions of problem (25) of the form $v = V_q + \phi$, where

$$V_q(y) = U_q(\varepsilon y) = \sum_{i \in \bigcup_{l=1}^3 J_l} a_i P u_i(\varepsilon y), \quad \forall y \in \overline{\Omega}_\varepsilon. \tag{26}$$

In terms of ϕ, problem (11) (or (25)) becomes

$$L(\phi) := \Delta\phi + W_q\phi = -\left[R_q + N(\phi)\right], \quad \text{in } \Omega_\varepsilon, \qquad \phi = 0, \quad \text{on } \partial\Omega_\varepsilon, \tag{27}$$

where

$$W_q = 2\varepsilon^4 c(\varepsilon y) S(\varepsilon y) \cosh V_q, \tag{28}$$

R_q is the "error term":

$$R_q = \Delta V_q + 2\varepsilon^4 c(\varepsilon y) S(\varepsilon y) \sinh V_q, \tag{29}$$

and $N(\phi)$ denotes the following "nonlinear term":

$$N(\phi) = 2\varepsilon^4 c(\varepsilon y) S(\varepsilon y) \times \left[\sinh\left(V_q + \phi\right) - \sinh V_q - \phi \cosh V_q\right]. \tag{30}$$

Finally, in order to make R_q sufficiently small, we choose the parameters $\mu_i, i \in \bigcup_{l=1}^3 J_l$, as

$$\log \frac{8\mu_i^2(1+\alpha_i)^2}{c_i(q_i)} = d_i H(q_i, q_i) + \sum_{j\neq i} a_i a_j d_j G(q_i, q_j), \tag{31}$$

so that from Appendix, we have

$$W_q = \begin{cases} \left(\dfrac{\varepsilon}{\nu_i\rho_i}\right)^2 |z_i|^{2\alpha_i} \left\{ \dfrac{8(1+\alpha_i)^2 \left[1 + O(\rho_i|z_i|) + O(\rho)\right]}{\left[1+|z_i|^{2(1+\alpha_i)}\right]^2} + O(\varepsilon^4) \right\}, & \text{if } |z_i| \le \delta(\nu_i\rho_i)^{-1}, \\ O(\varepsilon^4), & \text{otherwise}, \end{cases} \tag{32}$$

$$R_q = \begin{cases} \left(\dfrac{\varepsilon}{\nu_i\rho_i}\right)^2 |z_i|^{2\alpha_i} \left\{ \dfrac{8a_i(1+\alpha_i)^2 \left[O(\rho_i|z_i|) + O(\rho)\right]}{\left[1+|z_i|^{2(1+\alpha_i)}\right]^2} + O(\varepsilon^4) \right\}, & \text{if } |z_i| \le \delta(\nu_i\rho_i)^{-1}, \\ O(\varepsilon^4), & \text{otherwise}, \end{cases} \tag{33}$$

$z_i := (1/\nu_i\rho_i)(\varepsilon y - q_i)$.

3. The Linearized Problem and the Nonlinear Problem

In this section, we will first prove the bounded invertibility of the linearized operator L under suitable orthogonality conditions. Let us define

$$\begin{aligned} L_i(\phi) &= \Delta\phi + \frac{8(1+\alpha_i)^2|z|^{2\alpha_i}}{\left[1+|z|^{2(1+\alpha_i)}\right]^2}\phi \quad \forall i \in \bigcup_{l=1}^3 J_l, \\ z_{i0} &= \frac{1}{\mu_i}\frac{|z|^{2(1+\alpha_i)}-1}{|z|^{2(1+\alpha_i)}+1} \quad \forall i \in \bigcup_{l=1}^3 J_l, \\ z_{i1} &= \frac{4\,\mathrm{Re}\left(z^{1+\alpha_i}\right)}{|z|^{2(1+\alpha_i)}+1} \quad \forall i \in \bigcup_{l=2}^3 J_l, \\ z_{i2} &= \frac{4\,\mathrm{Im}\left(z^{1+\alpha_i}\right)}{|z|^{2(1+\alpha_i)}+1} \quad \forall i \in \bigcup_{l=2}^3 J_l. \end{aligned} \tag{34}$$

The key fact to develop a satisfactory solvability theory for the operator L is that L formally approaches L_i under dilations and translations, and any bounded solution of $L_i(\phi) = 0$ in $\mathbb{R}^2$ is a linear combination of z_{i0}, z_{i1}, and z_{i2} for $i \in J_2 \cup J_3$ (see [34, 45, 46]), or proportional to z_{i0} for $i \in J_1$ (see [35, 36, 41]). Let us denote

$$\begin{aligned} Z_{i0}(y) &:= z_{i0}(z_i), \quad \forall i \in \bigcup_{l=1}^3 J_l, \\ Z_{ij}(y) &:= z_{ij}(z_i), \quad \forall i \in \bigcup_{l=2}^3 J_l, \ j = 1, 2, \\ \theta_i(y) &:= \arg(z_i), \quad \forall i \in J_2, \\ \chi_i(y) &:= \chi(|z_i|), \quad \forall i \in \bigcup_{l=1}^3 J_l, \end{aligned} \tag{35}$$

where $\chi(r)$ is a smooth, nonincreasing cut-off function such that for a large but fixed number $R_0 > 0$, $\chi(r) = 1$ if $r \le R_0$, and $\chi(r) = 0$ if $r \ge R_0 + 1$.

Additionally, we consider the following Banach space:

$$\mathscr{C}_* = \left\{h : \|h\|_* < +\infty\right\}, \tag{36}$$

with the norm

$$\|h\|_* = \sup_{y\in\overline{\Omega}_\varepsilon} \left(|h(y)| \times \left(\varepsilon^2 + \sum_{\alpha_i<0,\, i\in J_1} \left(\frac{\varepsilon}{\nu_i\rho_i}\right)^2\right.\right.$$

$$\times \frac{|z_i|^{2\alpha_i}}{(1+|z_i|)^{4+2\alpha+2\alpha_i}} + \sum_{\alpha_i\geq 0,\, i\in J_1\cup J_2\cup J_3} \left(\frac{\varepsilon}{\nu_i\rho_i}\right)^2 \times \frac{1}{(1+|z_i|)^{4+2\alpha}}\Bigg)^{-1}\Bigg), \tag{37}$$

where $\alpha \in (-1, \alpha_0)$ and $\alpha_0 := \min\{\alpha_i : i \in \bigcup_{i=1}^{3} J_i\}$.

Given that $h \in \mathscr{C}_*$ and $q \in \Lambda_{k,m}(\delta)$, we consider the linear problem of finding a function ϕ and scalars c_{ij}, $i \in J_2 \cup J_3$, $j = 1, 2$, such that

$$\begin{gathered} L(\phi) = h + \sum_{j=1}^{2}\sum_{i=n+1}^{n+k+m} c_{ij}\chi_i \cos^2\left(\frac{1}{2}\alpha_i\theta_i\right) Z_{ij}, \quad \text{in } \Omega_\varepsilon, \\ \phi = 0, \quad \text{on } \partial\Omega_\varepsilon, \\ \int_{\Omega_\varepsilon} \chi_i \cos^2\left(\frac{1}{2}\alpha_i\theta_i\right) Z_{ij}\phi\, dy = 0, \quad \forall i \in J_2 \cup J_3,\ j = 1, 2. \end{gathered} \tag{38}$$

Our main interest in this problem is its bounded solvability for any $h \in \mathscr{C}_*$, uniform in small ε and points $q \in \Lambda_{k,m}(\delta)$, as the following result states.

Proposition 4. *There exist positive numbers ε_0 and C such that for any $h \in \mathscr{C}_*$, there is a unique solution $\phi \in L^\infty(\Omega_\varepsilon)$, scalars $c_{ij} \in \mathbb{R}$, $i \in J_2 \cup J_3$, $j = 1, 2$, to problem (38) for all $\varepsilon < \varepsilon_0$ and points $q \in \Lambda_{k,m}(\delta)$, which satisfies*

$$\|\phi\|_\infty \leq C\left(\log\frac{1}{\varepsilon}\right)\|h\|_*. \tag{39}$$

Moreover, the map $q' \mapsto \phi$ is C^1, precisely for $l \in J_2 \cup J_3$, $s = 1, 2$,

$$\left\|\partial_{q'_{l,s}}\phi\right\|_\infty \leq C\frac{\varepsilon}{\nu_l\rho_l}\left(\log\frac{1}{\varepsilon}\right)^2\|h\|_*, \tag{40}$$

where $q' := (1/\varepsilon)q = ((1/\varepsilon)q_{n+1}, \ldots, (1/\varepsilon)q_{n+k+m})$.

We begin by stating a priori estimate for solutions of (38) satisfying orthogonality conditions with respect to Z_{i0}, $i \in J_1$, and Z_{ij}, $i \in J_2 \cup J_3$, $j = 0, 1, 2$.

Lemma 5. *There exist positive numbers ε_0 and C such that for any points $q \in \Lambda_{k,m}(\delta)$, $h \in \mathscr{C}_*$ and any solution ϕ to the following equation:*

$$\begin{gathered} L(\phi) = h, \quad \text{in } \Omega_\varepsilon, \\ \phi = 0, \quad \text{on } \partial\Omega_\varepsilon, \end{gathered} \tag{41}$$

with the orthogonality conditions

$$\begin{gathered} \int_{\Omega_\varepsilon} \chi_i Z_{i0}\phi\, dy = 0, \quad \forall i \in J_1, \\ \int_{\Omega_\varepsilon} \chi_i \cos^2\left(\frac{1}{2}\alpha_i\theta_i\right) Z_{ij}\phi\, dy = 0, \\ \forall i \in J_2 \cup J_3,\ j = 0, 1, 2, \end{gathered} \tag{42}$$

one has

$$\|\phi\|_\infty \leq C\|h\|_*, \tag{43}$$

for all $\varepsilon < \varepsilon_0$.

Proof. We have the following steps.

Step 1. We first construct a suitable barrier. To realize it, we claim that for $\varepsilon > 0$ small enough, there exist $R_d > 0$ and

$$\psi : \overline{\Omega_\varepsilon \setminus \bigcup_{i=1}^{n+k+m} B_{i,R_d}} \longmapsto \mathbb{R} \tag{44}$$

positive and uniformly bounded so that

$$L(\psi) \leq -\sum_{i=1}^{n+k+m}\left(\frac{\varepsilon}{\nu_i\rho_i}\right)^2 \frac{1}{|z_i|^{2\alpha+4}} - \varepsilon^2, \tag{45}$$

where $-1 < \alpha < \alpha_0$ and $B_{i,R_d} = \{y \in \Omega_\varepsilon : |z_i| < R_d\}$. Take

$$\begin{gathered} g_0(z) = \frac{|z|^{2(1+\alpha)} - 1}{|z|^{2(1+\alpha)} + 1}, \qquad R_d = \frac{1}{d}3^{(1/2(1+\alpha))}, \\ g_{1i}(y) = g_0(d|z_i|), \quad \text{where } |z_i| \geq R_d. \end{gathered} \tag{46}$$

Thus, we have $1/2 \leq g_{1i}(y) \leq 1$, and

$$\begin{aligned} \Delta g_{1i}(y) &= -\left(\frac{\varepsilon}{\nu_i\rho_i}\right)^2 d^2 \frac{8(1+\alpha)^2|dz_i|^{2\alpha}}{\left[1+|dz_i|^{2(1+\alpha)}\right]^2} \times g_0(d|z_i|) \\ &< -\left(\frac{\varepsilon}{\nu_i\rho_i}\right)^2 \frac{d^{-2(1+\alpha)}(1+\alpha)^2}{|z_i|^{2\alpha+4}}. \end{aligned} \tag{47}$$

By (32),

$$\begin{gathered} W_q \leq C_1\sum_{i=1}^{n+k+m}\left(\frac{\varepsilon}{\nu_i\rho_i}\right)^2 \frac{1}{|z_i|^{2\alpha_i+4}}, \\ \forall y \in \overline{\Omega_\varepsilon \setminus \bigcup_{i=1}^{n+k+m} B_{i,R_d}}. \end{gathered} \tag{48}$$

So, if d is taken small and fixed, by (47)-(48), it follows that for $|z_i| \geq R_d$,

$$L(g_{1i})(y) < -\sum_{i=1}^{n+k+m}\left(\frac{\varepsilon}{\nu_i\rho_i}\right)^2 \frac{1}{|z_i|^{2\alpha+4}} < 0. \tag{49}$$

Let $\tilde{g}_2$ be a positive, bounded solution of $-\Delta\tilde{g}_2 = 1$ in Ω and $\tilde{g}_2 = 0$ on $\partial\Omega$. Set $g_2(y) = \tilde{g}_2(\varepsilon y)$. Then, g_2 is a positive, uniformly bounded function in Ω_ε such that

$$-\Delta g_2(y) = \varepsilon^2, \quad \text{in } \Omega_\varepsilon, \qquad g_2 = 0, \quad \text{on } \partial\Omega_\varepsilon. \tag{50}$$

Consider

$$\psi(y) = C_2 \sum_{i=1}^{n+k+m} g_{1i}(y) + g_2(y), \quad \forall y \in \overline{\Omega_\varepsilon \setminus \bigcup_{i=1}^{n+k+m} B_{i,R_d}}, \tag{51}$$

where $C_2 > 0$ is a sufficiently large constant. Obviously, ψ is a positive and uniformly bounded function. Moreover, in view of (48)–(50), it is easy to check that ψ satisfies the estimate (45), and the claim follows.

Step 2. Consider the following "inner norm":

$$\|\phi\|_l = \sup_{\overline{\bigcup_{i=1}^{n+k+m} B_{i,R_d}}} |\phi|. \tag{52}$$

Let us claim that there exists a constant $C_3 > 0$, such that

$$\|\phi\|_\infty \le C_3 \left(\|\phi\|_l + \|h\|_*\right). \tag{53}$$

Let us take

$$\tilde{\phi}(y) = C_4 \psi(y) \left(\|\phi\|_l + \|h\|_*\right), \quad \forall y \in \overline{\Omega_\varepsilon \setminus \bigcup_{i=1}^{n+k+m} B_{i,R_d}}, \tag{54}$$

where $C_4 > 0$ is chosen larger if necessary. Then, for $y \in \Omega_\varepsilon \setminus \bigcup_{i=1}^{n+k+m} B_{i,R_d}$,

$$\begin{aligned} L\left(\tilde{\phi} \pm \phi\right)(y) &\le -C_4 \left(\|\phi\|_l + \|h\|_*\right) \times \left[\sum_{i=1}^{n+k+m} \left(\frac{\varepsilon}{\nu_i\rho_i}\right)^2 \frac{1}{|z_i|^{2\alpha+4}} + \varepsilon^2\right] \pm h(y) \\ &\le -|h(y)| \pm h(y) \le 0, \end{aligned} \tag{55}$$

for $y \in \partial\Omega_\varepsilon$,

$$\left(\tilde{\phi} \pm \phi\right)(y) = \tilde{\phi}(y) > 0, \tag{56}$$

and for $y \in \bigcup_{i=1}^{n+k+m} \partial B_{i,R_d}$,

$$\left(\tilde{\phi} \pm \phi\right)(y) > \|\phi\|_l \pm \phi(y) \ge 0. \tag{57}$$

From the maximum principle, it follows that $-\phi(y) \le \tilde{\phi}(y) \le \phi(y)$ on $\overline{\Omega_\varepsilon \setminus \bigcup_{i=1}^{n+k+m} B_{i,R_d}}$, which provides the estimate (53).

Step 3. We prove the priori estimate (43) by contradiction. Let us assume the existence of a sequence $\varepsilon_j \to 0$, and points $q^j = (q^j_{n+1}, \ldots, q^j_{n+k+m}) \in \Lambda_{k,m}(\delta)$, functions h_j with $\|h_j\|_* \to 0$, solutions ϕ_j with $\|\phi_j\|_\infty = 1$, such that (41)-(5) hold. From the estimate (53), $\|\phi_j\|_l \ge \kappa$ for some $\kappa > 0$, namely, $\sup_{B_{i,R_d}} |\phi_j| \ge \kappa$ for some i. To simplify the notation, let us denote $\varepsilon = \varepsilon_j$ and $q_i = q^j_i$. Set $\hat{\phi}_j(z) = \phi_j((\nu_i\rho_i z + q_i)/\varepsilon)$ and $\hat{h}_j(z) = h_j((\nu_i\rho_i z + q_i)/\varepsilon)$. By (32), $\hat{\phi}_j$ satisfies

$$\Delta\hat{\phi}_j + \frac{8(1+\alpha_i)^2 |z|^{2\alpha_i}}{\left[1 + |z|^{2(1+\alpha_i)}\right]^2} \left[1 + O(\rho_i |z|) + O(\rho)\right] \hat{\phi}_j = \left(\frac{\nu_i\rho_i}{\varepsilon}\right)^2 \hat{h}_j, \tag{58}$$

for $z \in B_{R_d}(0)$. Obviously, for $\beta \in [1, -(1/\alpha)]$, $(\nu_i\rho_i/\varepsilon)^2 \hat{h}_j \to 0$ in $L^\beta(B_{R_d}(0))$. Since $8(1+\alpha_i)^2|z|^{2\alpha_i}/[1 + |z|^{2(1+\alpha_i)}]^2$ is bounded in $L^\beta(B_{R_d}(0))$ and $\|\hat{\phi}_j\|_\infty = 1$, the elliptic regularity theory implies that $\hat{\phi}_j$ converges uniformly over compact sets near the origin to a bounded nontrivial solution of $L_i(\hat{\phi}) = 0$ in $\mathbb{R}^2$. Then, $\hat{\phi}$ is proportional to z_{i0} for $i \in J_1$ (see [35, 36, 41]), or a linear combination of z_{i0}, z_{i1}, and z_{i2} for $i \in J_2 \cup J_3$ (see [34, 45, 46]). However, our assumed conditions (5) on i and ϕ_j pass to the limit and yield $\int \chi z_{i0}\hat{\phi} dz = 0$ for $i \in J_1$, or $\int \chi \cos^2((1/2)\alpha_i\theta) z_{il}\hat{\phi} dz = 0$ for $i \in J_2 \cup J_3$, $l = 0, 1, 2$, which implies that $\hat{\phi} \equiv 0$. This is absurd because $\hat{\phi}$ is nontrivial. □

We will give next a priori estimate for the solutions of (41) satisfying orthogonality conditions with respect to Z_{ij}, $i \in J_2 \cup J_3$, $j = 1, 2$.

Lemma 6. *There exist positive numbers ε_0 and C such that for any points $q \in \Lambda_{k,m}(\delta)$, $h \in \mathscr{C}_*$ and any solution ϕ to (41) with the following orthogonality conditions:*

$$\int_{\Omega_\varepsilon} \chi_i \cos^2\left(\frac{1}{2}\alpha_i\theta_i\right) Z_{ij}\phi\, dy = 0, \quad \forall i \in J_2 \cup J_3, \; j = 1, 2, \tag{59}$$

one has

$$\|\phi\|_\infty \le C \left(\log\frac{1}{\varepsilon}\right) \|h\|_*, \tag{60}$$

for all $\varepsilon < \varepsilon_0$.

Proof. Consider the radial solution ψ_i, $i \in \bigcup_{l=1}^3 J_l$, for the following equation:

$$\begin{aligned} \Delta\psi_i(r) &= 0, \quad \text{in } R < r < \delta(\nu_i\rho_i)^{-1}, \\ \psi_i(r) &= 1, \quad \text{on } r = R, \\ \psi_i(r) &= 0, \quad \text{on } r = \delta(\nu_i\rho_i)^{-1}, \end{aligned} \tag{61}$$

where $R > R_0 + 1$ is a large number. Then, $\psi_i(r)$ is explicitly given by

$$\psi_i(r) = \frac{\log\left[\delta(\nu_i\rho_i)^{-1}\right] - \log r}{\log\left[\delta(\nu_i\rho_i)^{-1}\right] - \log R}. \tag{62}$$

Let η_1 and η_2 be radial smooth cut-off functions on $\mathbb{R}^2$ so that

$$\begin{aligned} &0 \le \eta_1 \le 1; \quad |\nabla\eta_1| \le C \quad \text{in } \mathbb{R}^2; \\ &\eta_1 \equiv 1 \quad \text{in } B_R(0); \\ &\eta_1 \equiv 0 \quad \text{in } \mathbb{R}^2 \setminus B_{R+1}(0); \\ &0 \le \eta_2 \le 1; \quad |\nabla\eta_2| \le C \quad \text{in } \mathbb{R}^2; \\ &\eta_2 \equiv 1 \quad \text{in } B_{\delta/4}(0); \\ &\eta_2 \equiv 0 \quad \text{in } \mathbb{R}^2 \setminus B_\delta(0). \end{aligned} \tag{63}$$

Set

$$\begin{aligned} \tilde{\psi}_i(y) &= \psi_i(z_i), \\ \eta_{1i}(y) &= \eta_1(z_i), \\ \eta_{2i}(y) &= \eta_2(\nu_i\rho_i z_i). \end{aligned} \tag{64}$$

Besides, let us define

$$\tilde{\phi} = \phi + \sum_{i=1}^{n+k+m} d_i \widetilde{Z}_{i0}, \tag{65}$$

where

$$\widetilde{Z}_{i0}(y) = \eta_{1i} Z_{i0} + (1-\eta_{1i})\eta_{2i}\tilde{\psi}_i Z_{i0}, \tag{66}$$

and d_i is chosen such that for $i \in J_1$,

$$d_i \int_{\Omega_\varepsilon} \chi_i |Z_{i0}|^2 + \int_{\Omega_\varepsilon} \chi_i Z_{i0}\phi = 0, \tag{67}$$

and for $i \in J_2 \cup J_3$,

$$\begin{aligned} d_i &\int_{\Omega_\varepsilon} \chi_i \cos^2\left(\frac{1}{2}\alpha_i\theta_i\right) Z_{i0}^2 \\ &+ \int_{\Omega_\varepsilon} \chi_i \cos^2\left(\frac{1}{2}\alpha_i\theta_i\right) Z_{i0}\phi = 0. \end{aligned} \tag{68}$$

Thus,

$$\begin{aligned} L(\tilde{\phi}) &= h + \sum_{i=1}^{n+k+m} d_i L\widetilde{Z}_{i0}, \quad \text{in } \Omega_\varepsilon, \\ \tilde{\phi} &= 0, \quad \text{on } \partial\Omega_\varepsilon, \end{aligned} \tag{69}$$

and $\tilde{\phi}$ satisfies the orthogonality conditions (5). By (43),

$$\|\tilde{\phi}\|_\infty \le C\left\{\|h\|_* + \sum_{i=1}^{n+k+m} |d_i| \cdot \|L\widetilde{Z}_{i0}\|_*\right\}. \tag{70}$$

Multiplying (69) by $\widetilde{Z}_{i0}, i \in \bigcup_{l=1}^3 J_l$, and integrating by parts, it follows that

$$\left\langle L(\widetilde{Z}_{i0}), \tilde{\phi}\right\rangle = \left\langle \widetilde{Z}_{i0}, h\right\rangle + d_i \left\langle L(\widetilde{Z}_{i0}), \widetilde{Z}_{i0}\right\rangle, \tag{71}$$

where $\langle f, g\rangle = \int_{\Omega_\varepsilon} fg$. Then, for $i \in \bigcup_{l=1}^3 J_l$, by (70) and (71),

$$\begin{aligned} \left|d_i \left\langle L(\widetilde{Z}_{i0}), \widetilde{Z}_{i0}\right\rangle\right| &\le C\|h\|_* \left\{1 + \left\|L(\widetilde{Z}_{i0})\right\|_*\right\} \\ &+ C\left\|L(\widetilde{Z}_{i0})\right\|_* \sum_{l=1}^{n+k+m} |d_l| \cdot \left\|L(\widetilde{Z}_{l0})\right\|_*. \end{aligned} \tag{72}$$

Let us claim that for $i \in \bigcup_{l=1}^3 J_l$, there exists some constant $C > 0$ independent of ε, such that

$$\left\|L(\widetilde{Z}_{i0})\right\|_* = O\left(\frac{1}{|\log\varepsilon|}\right), \tag{73}$$

$$\left\langle L(\widetilde{Z}_{i0}), \widetilde{Z}_{i0}\right\rangle < -\frac{C}{|\log\varepsilon|}\left\{1 + O\left(\frac{1}{|\log\varepsilon|}\right)\right\}. \tag{74}$$

Once these estimates (73)-(74) are proven, it easily follows that

$$|d_i| \le C\left(\log\frac{1}{\varepsilon}\right)\|h\|_*. \tag{75}$$

This, together with (65), (70), and (73), easily gives the estimate (60) of ϕ. □

Proofs of (73) *and* (74). Let us first define that

$$\begin{aligned} W_{i0} &= \frac{8(1+\alpha_i)^2 |z_i|^{2\alpha_i}}{\left[1 + |z_i|^{2(1+\alpha_i)}\right]^2}, \\ \tilde{z}_{i0}(z_i) &= \eta_1 z_{i0} + (1-\eta_1)\eta_2(\nu_i\rho_i z_i)\psi_i z_{i0}. \end{aligned} \tag{76}$$

For $r_i := |z_i| \le R$, by (32),

$$\begin{aligned} L(\widetilde{Z}_{i0}) &= \left(\frac{\varepsilon}{\nu_i\rho_i}\right)^2 W_{i0} z_{i0}\left\{O(\rho_i|z_i|) + O(\rho)\right\}, \\ I_1 &= \int_{r_i \le R} \widetilde{Z}_{i0} \cdot L(\widetilde{Z}_{i0}) \\ &= \int_{|z|\le R} W_{i0} z_{i0}^2 \left[O(\rho_i|z|) + O(\rho)\right] = O(\rho). \end{aligned} \tag{77}$$

For $R < r_i \le R+1$, $\widetilde{Z}_{i0}(y) = \eta_1 z_{i0} + (1-\eta_1)\psi_i z_{i0}$, $1 - \psi_i = O(1/|\log\varepsilon|)$, and $|\nabla\psi_i| = O(1/|\log\varepsilon|)$). Then,

$$\begin{aligned} L(\widetilde{Z}_{i0}) &= \left(\frac{\varepsilon}{\nu_i\rho_i}\right)^2 \{2(1-\psi_i)\nabla\eta_1\nabla z_{i0} \\ &\quad + 2(1-\eta_1)\nabla\psi_i\nabla z_{i0} - 2z_{i0}\nabla\eta_1\nabla\psi_i \\ &\quad + (1-\psi_i) z_{i0}\Delta\eta_1 \\ &\quad + W_{i0}\tilde{z}_{i0}\left[O(\rho_i|z_i|) + O(\rho)\right]\}(z_i) \\ &= \left(\frac{\varepsilon}{\nu_i\rho_i}\right)^2 O\left(\frac{1}{|\log\varepsilon|}\right). \end{aligned} \tag{78}$$

Furthermore,

$$\begin{aligned} I_2 &= \int_{R\le r_i\le R+1} \widetilde{Z}_{i0}\cdot L(\widetilde{Z}_{i0}) \\ &= \int_{R\le|z|\le R+1} \widetilde{z}_{i0}\{(1-\psi_i) z_{i0}\Delta\eta_1 \\ &\quad + 2(1-\psi_i)\nabla\eta_1\nabla z_{i0} - 2z_{i0}\nabla\eta_1\nabla\psi_i \\ &\quad + 2(1-\eta_1)\nabla\psi_i\nabla z_{i0}\}\,dz + O(\rho). \end{aligned} \tag{79}$$

Integrating by parts the first term of I_2, it follows that

$$\begin{aligned} I_2 &= \int_{R\le|z|\le R+1} 2(1-\eta_1)\widetilde{z}_{i0}\nabla\psi_i\nabla z_{i0} \\ &\quad - z_{i0}^2\nabla\psi_i\nabla\eta_1 - z_{i0}^2(1-\psi_i)^2|\nabla\eta_1|^2 + O(\rho). \end{aligned} \tag{80}$$

Integrating by parts the first term again, it also follows that

$$\begin{aligned} I_2 &= \int_{|z|=R+1} \psi_{i'}(|z|)\,\psi_i z_{i0}^2 \\ &\quad - \int_{R\le|z|\le R+1} z_{i0}^2 |(1-\eta_1)\nabla\psi_i \\ &\quad + (1-\psi_i)\nabla\eta_1|^2 + O(\rho). \end{aligned} \tag{81}$$

Then,

$$I_2 = \int_{|z|=R+1} \psi_{i'}(|z|)\,\psi_i z_{i0}^2 + O\left(\frac{1}{|\log\varepsilon|^2}\right). \tag{82}$$

For $R+1 < r_i \le \delta(4\nu_i\rho_i)^{-1}$, $\widetilde{Z}_{ij}(y) = \psi_i z_{ij}(z_i)$. Then,

$$\begin{aligned} L(\widetilde{Z}_{i0}) &= \left(\frac{\varepsilon}{\nu_i\rho_i}\right)^2 \{2\nabla\psi_i\nabla z_{i0} + \psi_i\Delta z_{i0} \\ &\quad + W_{i0}\psi_i z_{i0}[1 + O(\rho_i|z_i|) \\ &\quad + O(\rho)]\}(z_i). \end{aligned} \tag{83}$$

Note that $|\nabla z_{i0}| \le C/|z_i|^{3+2\alpha_i}$, $|\nabla\psi_i| \le C/(|z_i|\log(1/\varepsilon))$, and $|W_{i0}| \le C/|z_i|^{4+2\alpha_i}$. Then,

$$\begin{aligned} L(\widetilde{Z}_{i0}) &= \left(\frac{\varepsilon}{\nu_i\rho_i}\right)^2 \frac{1}{|z_i|^{4+2\alpha_i}} \\ &\quad \times \left\{O\left(\frac{1}{|\log\varepsilon|}\right) + O(\rho_i|z_i|) + O(\rho)\right\}. \end{aligned} \tag{84}$$

Besides,

$$\begin{aligned} I_3 &= \int_{R+1\le r_i\le\delta(4\nu_i\rho_i)^{-1}} \widetilde{Z}_{i0}\cdot L(\widetilde{Z}_{i0}) \\ &= \int_{R+1\le|z|\le\delta(4\nu_i\rho_i)^{-1}} \psi_i z_{i0} \\ &\quad \times\{2\nabla\psi_i\nabla z_{i0} + W_{i0}\psi_i z_{i0} \\ &\quad \times[O(\rho_i|z|) + O(\rho)]\} \\ &= \left(\int_{|z|=\delta(4\nu_i\rho_i)^{-1}} - \int_{|z|=R+1}\right)\psi_i'(|z|) \\ &\quad \times\psi_i z_{i0}^2 - \int_{R+1\le|z|\le\delta(4\nu_i\rho_i)^{-1}} z_{i0}^2|\nabla\psi_i|^2 + O(\rho). \end{aligned} \tag{85}$$

Some simple computations show that

$$\int_{|z|=\delta(4\nu_i\rho_i)^{-1}} \psi_i'(|z|)\,\psi_i z_{i0}^2 = O\left(\frac{1}{|\log\varepsilon|^2}\right), \tag{86}$$

$$\begin{aligned} &\int_{R+1\le|z|\le\delta(4\nu_i\rho_i)^{-1}} z_{i0}^2|\nabla\psi_i|^2 \\ &\quad \ge \frac{\pi}{2}\frac{\log(\delta/4\nu_i\rho_i) - \log(R+1)}{(\log(\delta/\nu_i\rho_i) - \log R)^2}. \end{aligned} \tag{87}$$

Then, there exists some constant $C > 0$ independent of ε and R such that

$$\begin{aligned} I_3 &< -\int_{|z|=R+1} \psi_i'(|z|)\,\psi_i z_{i0}^2 \\ &\quad - \frac{C}{|\log\varepsilon|}\left\{1 + O\left(\frac{1}{|\log\varepsilon|}\right)\right\}. \end{aligned} \tag{88}$$

For $\delta(4\nu_i\rho_i)^{-1} < r_i \le \delta(\nu_i\rho_i)^{-1}$, $\widetilde{Z}_{i0}(y) = \eta_2(\nu_i\rho_i z_i)\psi_i z_{i0}$. Then,

$$\begin{aligned} L(\widetilde{Z}_{i0}) &= \left(\frac{\varepsilon}{\nu_i\rho_i}\right)^2 \{\psi_i z_{i0}\Delta\eta_2(\nu_i\rho_i z_i) \\ &\quad + 2\nabla\eta_2(\nu_i\rho_i z_i)\nabla\psi_i z_{i0} \\ &\quad + 2\psi_i\nabla\eta_2(\nu_i\rho_i z_i)\nabla z_{i0} \\ &\quad + 2\eta_2(\nu_i\rho_i z_i)\nabla\psi_i\nabla z_{i0} \\ &\quad + W_{i0}\eta_2(\nu_i\rho_i z_i)\psi_i z_{i0} \\ &\quad \times[O(\rho_i|z_i|) + O(\rho)]\}. \end{aligned} \tag{89}$$

Note that $\psi_i = O(1/|\log\varepsilon|)$, $|\nabla\psi_i| = O(\rho_i/|\log\varepsilon|)$, and $|\nabla z_{i0}| = O(\rho_i^{3+2\alpha_i})$. Then,

$$L(\widetilde{Z}_{i0}) = \left(\frac{\varepsilon}{\nu_i\rho_i}\right)^2 O\left(\frac{\rho_i^2}{|\log\varepsilon|}\right). \tag{90}$$

Furthermore,

$$I_4 = \int_{\delta(4\nu_i\rho_i)^{-1}<r_i\le\delta(\nu_i\rho_i)^{-1}} \widetilde{Z}_{i0}\cdot L(\widetilde{Z}_{i0}) = O\left(\frac{1}{|\log\varepsilon|^2}\right). \tag{91}$$

As a result, the estimates (73)-(74) can be easily derived from (77)–(91). □

Proof of Proposition 4. We first establish the a priori estimate (39). Testing (38) against $\eta_{2i}Z_{ij}$, $i \in J_2 \cup J_3$, $j = 1, 2$, and integrating by parts, it follows that

$$\left|c_{ij}\right| \le C\left(\frac{\varepsilon}{\nu_i\rho_i}\right)^2 \left(\left|\left\langle\phi, L\left(\eta_{2i}Z_{ij}\right)\right\rangle\right| + \left|\left\langle h, \eta_{2i}Z_{ij}\right\rangle\right|\right). \quad (92)$$

Observe that

$$\begin{aligned} L\left(\eta_{2i}Z_{ij}\right) = \left(\frac{\varepsilon}{\nu_i\rho_i}\right)^2 \Bigg\{ & O\left(\frac{\rho_i}{\left(1+\left|z_i\right|\right)^{2+\alpha_i}}\right) \\ & + O\left(\frac{\rho_i^2}{\left(1+\left|z_i\right|\right)^{1+\alpha_i}}\right)\Bigg\} \\ & + \eta_2\left(\nu_i\rho_i z_i\right) z_{ij}\left(W_q - W_{i0}\right). \end{aligned} \quad (93)$$

Then,

$$\left\langle\phi, L\left(\eta_{2i}Z_{ij}\right)\right\rangle = O\left(\rho\|\phi\|_\infty\right). \quad (94)$$

Note that $\langle h, \eta_{2i}Z_{ij}\rangle = O(\|h\|_*)$. This, together with (92)-(94), implies that

$$\left|c_{ij}\right| \le C\left(\frac{\varepsilon}{\nu_i\rho_i}\right)^2 \left(\|h\|_* + \rho\|\phi\|_\infty\right). \quad (95)$$

By (60),

$$\|\phi\|_\infty \le C\left(\log\frac{1}{\varepsilon}\right)\left\{\|h\|_* + \sum_{j=1}^{2}\sum_{i=n+1}^{n+k+m}\left(\frac{\nu_i\rho_i}{\varepsilon}\right)^2\left|c_{ij}\right|\right\}. \quad (96)$$

Combining this with (95), we find that the a priori estimate (39) holds. Furthermore,

$$\left|c_{ij}\right| \le C\left(\frac{\varepsilon}{\nu_i\rho_i}\right)^2 \|h\|_*, \quad (97)$$

which implies that there exists a unique trivial solution to problem (38) with $h \equiv 0$.

Next, we prove the solvability of problem (38). Consider the following Hilbert space:

$$\mathbb{H} = \left\{\phi \in H_0^1\left(\Omega_\varepsilon\right) : \phi \text{ satisfies the orthogonality conditions (59)}\right\}, \quad (98)$$

endowed with the usual norm $\|\psi\|_{H_0^1} = \int_{\Omega_\varepsilon}|\nabla\psi|^2$. Problem (38) is equivalent to that of finding $\phi \in \mathbb{H}$ such that

$$\int_{\Omega_\varepsilon} \nabla\phi\nabla\psi - W_q\phi\psi + h\psi = 0, \quad \forall\psi \in \mathbb{H}. \quad (99)$$

By Fredholm's alternative, this is equivalent to the uniqueness of solutions to this problem, which is guaranteed by (39). As a consequence, there exists a unique solution $\phi = T(h)$, scalars c_{ij}, $i \in J_2 \cup J_3$, $j = 1, 2$, for problem (38) with $h \in \mathscr{C}_*$, where $T : \mathscr{C}_* \mapsto L^\infty(\Omega_\varepsilon)$ is a continuous linear map satisfying $\|T(h)\|_\infty \le C(\log(1/\varepsilon))\|h\|_*$.

Finally, we give the a priori estimate (40). Let us Differentiate (38) with respect to the parameters $q'_{l,s}$, $l \in J_2 \cup J_3$, $s = 1, 2$. Formally, $Z = \partial_{q'_{l,s}}\phi$ should satisfy

$$\begin{aligned} L(Z) = & -\phi\,\partial_{q'_{l,s}} W_q \\ & + \sum_{j=1}^{2}\sum_{i=n+1}^{n+k+m}\Bigg\{c_{ij}\partial_{q'_{l,s}} \times\left[\chi_i\cos^2\left(\frac{1}{2}\alpha_i\theta_i\right)Z_{ij}\right] \\ & + d_{ij}\chi_i\cos^2\left(\frac{1}{2}\alpha_i\theta_i\right)Z_{ij}\Bigg\}, \end{aligned} \quad (100)$$

where (still formally) $d_{ij} = \partial_{q'_{l,s}} c_{ij}$. The orthogonality conditions now become

$$\begin{aligned} & \int_{\Omega_\varepsilon}\chi_i\cos^2\left(\frac{1}{2}\alpha_i\theta_i\right)Z_{ij}Z \\ & = -\int_{\Omega_\varepsilon}\phi\partial_{q'_{l,s}}\left[\chi_i\cos^2\left(\frac{1}{2}\alpha_i\theta_i\right)Z_{ij}\right], \\ & \qquad \forall i \in J_2 \cup J_3, \ j = 1, 2. \end{aligned} \quad (101)$$

We consider the constants b_{ij}, $i \in J_2 \cup J_3$, $j = 1, 2$, defined as

$$\begin{aligned} & b_{ij}\int_{\Omega_\varepsilon}\chi_i\cos^2\left(\frac{1}{2}\alpha_i\theta_i\right)\left|Z_{ij}\right|^2 \\ & = -\int_{\Omega_\varepsilon}\phi\partial_{q'_{l,s}}\left[\chi_i\cos^2\left(\frac{1}{2}\alpha_i\theta_i\right)Z_{ij}\right]. \end{aligned} \quad (102)$$

By (31), it follows that $|\partial_{q'_{l,s}}\log\mu_i| = O(\varepsilon)$ uniformly holds on $\Lambda_{k,m}(\delta)$, which implies that for $i \in J_2 \cup J_3$, $j = 1, 2$,

$$\left|b_{ij}\right| \le \begin{cases} C\dfrac{\varepsilon}{\nu_l\rho_l}\|\phi\|_\infty, & \text{if } i = l, \\ C\varepsilon\|\phi\|_\infty, & \text{if } i \ne l. \end{cases} \quad (103)$$

Define

$$\widetilde{Z} = Z - \sum_{j=1}^{2}\sum_{i=n+1}^{n+k+m} b_{ij}\eta_{2i}Z_{ij},$$

$$\begin{aligned} f = & \sum_{j=1}^{2}\sum_{i=n+1}^{n+k+m}\Bigg\{c_{ij}\partial_{q'_{l,s}} \times\left[\chi_i\cos^2\left(\frac{1}{2}\alpha_i\theta_i\right)Z_{ij}\right] \\ & - b_{ij}L\left(\eta_{2i}Z_{ij}\right)\Bigg\} \\ & - \phi\partial_{q'_{l,s}}W_q. \end{aligned} \quad (104)$$

We then have

$$L\left(\widetilde{Z}\right) = f + \sum_{j=1}^{2}\sum_{i=n+1}^{n+k+m} d_{ij}\chi_i \cos^2\left(\frac{1}{2}\alpha_i\theta_i\right), \quad \text{in } \Omega_\varepsilon,$$
$$\phi = 0, \quad \text{on } \partial\Omega_\varepsilon, \tag{105}$$
$$\int_{\Omega_\varepsilon} \chi_i \cos^2\left(\frac{1}{2}\alpha_i\theta_i\right) Z_{ij}\widetilde{Z} = 0, \quad \forall i \in J_2 \cup J_3, \ j = 1, 2.$$

Thus, $Z = \partial_{q'_{l,s}}\phi$ can be uniquely expressed as

$$Z = T(f) + \sum_{j=1}^{2}\sum_{i=n+1}^{n+k+m} b_{ij}\eta_{2i}Z_{ij}. \tag{106}$$

Furthermore, elliptic regularity theory implies that $\phi = T(h)$ is differentiable with respect to $q'_{l,s}$, $l \in J_2 \cup J_3$, $s = 1, 2$. Note that $\|L(\eta_{2i}Z_{ij})\|_* = O(\rho)$, $\|\partial_{q'_{l,s}} W_q\|_* = O(\rho_l^{\alpha_l})$, $\|\partial_{q'_{l,s}}[\chi_l \cos^2((1/2)\alpha_l\theta_l)Z_{lj}]\|_* = O(\rho_l^{-\alpha_l})$, and $\|\partial_{q'_{l,s}}[\chi_i \cos^2((1/2)\alpha_i\theta_i)Z_{ij}]\|_* = O(\rho_i^{1-\alpha_i})$ for $i \neq l$. This, together with (39) and (97)–(106), implies that

$$\begin{aligned}
\|Z\|_\infty &\leq C\left(\log\frac{1}{\varepsilon}\right)\|f\|_* + C\sum_{j=1}^{2}\sum_{i=n+1}^{n+k+m}\left|b_{ij}\right| \\
&\leq C\left(\log\frac{1}{\varepsilon}\right) \\
&\quad\times \sum_{j=1}^{2}\sum_{i=n+1}^{n+k+m}\left\{\left(\frac{\varepsilon}{\nu_i\rho_i}\right)^2\|h\|_* \right.\\
&\quad\quad\times \left\|\partial_{q'_{l,s}}\left[\chi_i\cos^2\left(\frac{1}{2}\alpha_i\theta_i\right)Z_{ij}\right]\right\|_* \\
&\quad\quad\left. + \rho\left|b_{ij}\right|\right\} \\
&\quad + C\left(\log\frac{1}{\varepsilon}\right)\|\phi\|_\infty\left\|\partial_{q'_{l,s}}W_q\right\|_* \\
&\quad + C\frac{\varepsilon}{\nu_l\rho_l}\|\phi\|_\infty \\
&\leq C\frac{\varepsilon}{\nu_l\rho_l}\left(\log\frac{1}{\varepsilon}\right)^2\|h\|_*,
\end{aligned} \tag{107}$$

which implies that the estimate (40) holds. □

Now, we solve the auxiliary nonlinear problem: for $q \in \Lambda_{k,m}(\delta)$, we find a function ϕ and scalars c_{ij}, $i \in J_2 \cup J_3$, $j = 1, 2$, such that

$$L(\phi) = -\left[R_q + N(\phi)\right] + \sum_{j=1}^{2}\sum_{i=n+1}^{n+k+m} c_{ij}\chi_i\cos^2\left(\frac{1}{2}\alpha_i\theta_i\right)Z_{ij}, \quad \text{in } \Omega_\varepsilon,$$
$$\phi = 0, \quad \text{on } \partial\Omega_\varepsilon,$$
$$\int_{\Omega_\varepsilon}\chi_i\cos^2\left(\frac{1}{2}\alpha_i\theta_i\right)Z_{ij}\phi\,dy = 0, \quad \forall i \in J_2 \cup J_3, \ j = 1, 2. \tag{108}$$

The following result can be proved through arguments similar to these of [39].

Proposition 7. *There exist positive numbers ε_0 and C such that for any $\varepsilon < \varepsilon_0$ and points $q \in \Lambda_{k,m}(\delta)$, there is a unique solution $\phi \in L^\infty(\Omega_\varepsilon)$, scalars $c_{ij} \in \mathbb{R}$, $i \in J_2 \cup J_3$, $j = 1, 2$, of problem* (3), *which satisfies*

$$\|\phi\|_\infty \leq C\rho\log\frac{1}{\varepsilon}. \tag{109}$$

Moreover, the map $q' \mapsto \phi$ is C^1, precisely for $l \in J_2 \cup J_3$, $s = 1, 2$,

$$\left\|\partial_{q'_{l,s}}\phi\right\|_\infty \leq C\rho\frac{\varepsilon}{\nu_l\rho_l}\left(\log\frac{1}{\varepsilon}\right)^2, \tag{110}$$

where $\rho = \max\{\rho_i : i \in \bigcup_{l=1}^{3} J_l\}$.

4. Variational Reduction

In what follows, we only need to find a solution of problem (27) (or (25)) with $k + m \geq 1$, and hence to problem (3) if points $q \in \Lambda_{k,m}(\delta)$ satisfy

$$c_{ij}\left(q'\right) = 0, \quad \forall i \in J_2 \cup J_3, \ j = 1, 2. \tag{111}$$

To realize it, we consider the energy functional J_ε associated with problem (25), namely,

$$J_\varepsilon(v) = \frac{1}{2}\int_{\Omega_\varepsilon}|\nabla v|^2 - 2\varepsilon^4\int_{\Omega_\varepsilon} c(\varepsilon y)S(\varepsilon y)\cosh v, \quad \forall u \in H_0^1(\Omega_\varepsilon). \tag{112}$$

We define

$$F_\varepsilon(q) = J_\varepsilon\left(V_q + \phi\right), \tag{113}$$

where V_q is defined in (26) and ϕ is the unique solution of problem (3). Critical points of F_ε correspond to solutions of (111) for small ε, as the following result states.

Lemma 8. *For any points $q \in \Lambda_{k,m}(\delta)$, the functional $F_\varepsilon(q)$ is of class C^1. Moreover, for $\varepsilon > 0$ small enough, if $D_qF_\varepsilon(q) = 0$, then points $q \in \Lambda_{k,m}(\delta)$ satisfy* (112).

Proof. Observe that for $l \in J_2 \cup J_3$, $s = 1, 2$,

$$\begin{aligned}
\partial_{q_{l,s}}F_\varepsilon(q) &= \varepsilon^{-1}DJ_\varepsilon\left(V_q + \phi\right) \\
&\quad\times\left[\partial_{q'_{l,s}}\left(V_q + \phi\right)\right] \\
&= \varepsilon^{-1}\sum_{j=1}^{2}\sum_{i=n+1}^{n+k+m} c_{ij}K_{il}(j, s),
\end{aligned} \tag{114}$$

where

$$K_{il}(j,s)=\int_{\Omega_\varepsilon}\chi_i\cos^2\left(\frac{1}{2}\alpha_i\theta_i\right)Z_{ij}\left(\partial_{q'_{l,s}}V_q+\partial_{q'_{l,s}}\phi\right)dy. \tag{115}$$

Then, if $D_qF_\varepsilon(q)=0$, we have

$$\sum_{j=1}^{2}\sum_{i=n+1}^{n+k+m}c_{ij}K_{il}(j,s)=0,\quad \forall l\in J_2\cup J_3,\ s=1,2. \tag{116}$$

Set

$$F_l=\int_0^{R_0+1}\frac{16(1+\alpha_l)r^{3(1+\alpha_l)}}{\left[1+r^{2(1+\alpha_l)}\right]^2}\chi(r)\,dr,\quad \forall l\in J_2\cup J_3. \tag{117}$$

A simple computation shows that

$$\partial_{q'_{l,s}}V_q=\frac{\varepsilon}{\nu_l\rho_l}\frac{4(1+\alpha_l)a_l|z_l|^{2\alpha_l}z_{l,s}}{1+|z_l|^{2(1+\alpha_l)}}+O(\varepsilon),\quad \forall l\in J_2\cup J_3,\ s=1,2. \tag{118}$$

This, together with the estimate (110) of $\partial_{q'_{l,s}}\phi$, implies that

$$K_{il}(j,s)=\begin{cases}\left(\dfrac{\nu_i\rho_i}{\varepsilon}\right)^2O\left(\rho\dfrac{\varepsilon}{\nu_l\rho_l}|\log\varepsilon|^2\right), & \forall i\neq l \text{ or } j\neq s,\\ \dfrac{\nu_l\rho_l}{\varepsilon}\left[A_l+O\left(\rho|\log\varepsilon|^2\right)\right], & \forall i=l \text{ and } j=s,\end{cases} \tag{119}$$

where

$$A_l=\begin{cases}\pi a_lF_l, & \forall l\in J_3,\\ \dfrac{1}{4}\pi a_lF_l, & \forall l\in J_2.\end{cases} \tag{120}$$

Set

$$\tilde{c}_{ij}=\left(\frac{\nu_i\rho_i}{\varepsilon}\right)^2c_{ij},\quad \forall i\in J_2\cup J_3,\ j=1,2. \tag{121}$$

Thus, (116) can be rewritten as: for any $l\in J_2\cup J_3$, $s=1,2$,

$$\tilde{c}_{ls}a_lF_l=\sum_{j=1}^{2}\sum_{i=n+1}^{n+k+m}\tilde{c}_{ij}a_iO\left(\rho|\log\varepsilon|^2\right),\quad \forall l\in J_2\cup J_3,\ s=1,2. \tag{122}$$

This is a diagonal dominant system and we thus get $c_{ij}=0$ for all i, j. □

Next, we give a precise asymptotic expansion of $F_\varepsilon(q)$ defined in (113).

Lemma 9. *The following precise asymptotic expansion holds*

$$\begin{aligned}F_\varepsilon(q)=&-\sum_{i\in J_1}d_i\left\{\frac{1}{2}d_iH(q_i,q_i)+\sum_{j\in J_1,i\neq j}\frac{1}{2}a_ia_jd_jG(q_j,q_i)\right\}\\&+\sum_{i\in\bigcup_{l=1}^3J_l}d_i\left\{\log 8(1+\alpha_i)^2-2(1+\log\varepsilon)\right\}\\&+\varphi_{k,m}^n(q)+O(\rho),\end{aligned} \tag{123}$$

uniformly for any points $q\in\Lambda_{k,m}(\delta)$, where $\varphi_{k,m}^n(q)$ is defined in (12).

Proof. Let us first give a priori estimate of $\theta_\varepsilon(q)$, where $\theta_\varepsilon(q)=F_\varepsilon(q)-J_\varepsilon(V_q)$. Using $DJ_\varepsilon(V_q+\phi)[\phi]=0$, a Taylor expansion and an integration by parts, it follows that

$$\begin{aligned}\theta_\varepsilon(q)&=-\int_0^1D^2J_\varepsilon\left(V_q+s\phi\right)[\phi]^2s\,ds\\&=-\int_0^1\left(\int_{\Omega_\varepsilon}\left[N(\phi)+R_q\right]\phi+2\varepsilon^4c(\varepsilon y)S(\varepsilon y)\times\left[\cosh V_q-\cosh\times\left(V_q+s\phi\right)\right]\phi^2\right)s\,ds.\end{aligned} \tag{124}$$

Note that $\|N(\phi)\|_*=O(\|\phi\|_\infty^2)$, $\|R_q\|_*=O(\rho)$, and $\|2\varepsilon^4c(\varepsilon y)S(\varepsilon y)[\cosh V_q-\cosh(V_q+s\phi)]\|_*=O(\|\phi\|_\infty)$. These, together with (109), imply that $\theta_\varepsilon(q)=O(\rho^2|\log\varepsilon|)$, and so $F_\varepsilon(q)=J_\varepsilon(V_q)+O(\rho^2|\log\varepsilon|)$.

Next, we only need to give an asymptotic expansion of $J_\varepsilon(V_q)$. Observe that

$$J_\varepsilon\left(V_q\right)=-\frac{1}{2}\sum_{i,j\in\bigcup_{l=1}^3J_l}a_ia_j\int_\Omega Pu_i\Delta u_j\,dx-\int_{\Omega_\varepsilon}W_q\,dy. \tag{125}$$

By (22),

$$\begin{aligned}-\int_\Omega Pu_i\Delta u_j&=\int_\Omega\varepsilon^2c_j\left(q_j\right)\left|x-q_j\right|^{2\alpha_j}e^{u_j}\times\left\{u_i+d_iH(x,q_i)-\log\frac{8\mu_i^2(1+\alpha_i)^2}{c_i(q_i)}+O(\rho)\right\}\\&=I_{ij}+J_{ij},\end{aligned} \tag{126}$$

where

$$I_{ij} = \int_{\Omega_{v_j\rho_j}} \frac{8(1+\alpha_j)^2 |z_j|^{2\alpha_j}}{\left[1+|z_j|^{2(1+\alpha_j)}\right]^2} \times \log \frac{1}{\left[\mu_i^2\varepsilon^2 + |v_j\rho_j z_j + q_j - q_i|^{2(1+\alpha_i)}\right]^2} dz_j, \tag{127}$$

$$J_{ij} = \int_{\Omega_{v_j\rho_j}} \frac{8(1+\alpha_j)^2 |z_j|^{2\alpha_j}}{\left[1+|z_j|^{2(1+\alpha_j)}\right]^2} \times \left\{d_i H(q_j, q_i) + O(\rho_j |z_j|) + O(\rho)\right\} dz_j.$$

Note that

$$\int_{\Omega_{v_j\rho_j}} \frac{8(1+\alpha_j)^2 |z|^{2\alpha_j}}{\left[1+|z|^{2(1+\alpha_j)}\right]^2} dz = d_j + O(\varepsilon^2), \tag{128}$$

$$\int_{\Omega_{v_j\rho_j}} \frac{8(1+\alpha_j)^2 |z|^{2\alpha_j}}{\left[1+|z|^{2(1+\alpha_j)}\right]^2} O(\rho_j |z|) dz = O(\rho_j) + O(\varepsilon^2), \tag{129}$$

and for $i \neq j$,

$$\begin{aligned} &\log\left[\mu_i^2\varepsilon^2 + |v_j\rho_j z_j + q_j - q_i|^{2(1+\alpha_i)}\right] \\ &\quad = 2(1+\alpha_i)\log|q_j - q_i| \\ &\qquad + O(\rho_j|z_j|) + O(\varepsilon^2). \end{aligned} \tag{130}$$

Then, for any $i \neq j$, by (128)–(130),

$$I_{ij} = -4(1+\alpha_i) d_j \log|q_j - q_i| + O(\rho). \tag{131}$$

By (128),

$$\begin{aligned} I_{jj} = {} & -2\int_{\Omega_{v_j\rho_j}} \frac{8(1+\alpha_j)^2 |z|^{2\alpha_j}}{\left[1+|z|^{2(1+\alpha_j)}\right]^2} \\ & \times \log\left[1+|z|^{2(1+\alpha_j)}\right] dz \\ & - 4\left[d_j + O(\varepsilon^2)\right]\log(\mu_j\varepsilon). \end{aligned} \tag{132}$$

Making the complex changes of variables $\omega = z^{1+\alpha_j}$ with $\omega = |\omega| e^{i\tilde{\theta}}$, we get

$$\begin{aligned} I_{jj} = {} & -2\int_0^{2\pi(1+\alpha_j)} d\tilde{\theta} \int_0^{(\mu_j\varepsilon)^{-1}} \frac{8|\omega|}{\left[1+|\omega|^2\right]^2} \\ & \times \log\left[1+|\omega|^2\right] d|\omega| \\ & - 4 d_j \log(\mu_j\varepsilon) + O(\rho) \\ = {} & -4 d_j \log(\mu_j\varepsilon) - 2d_j + O(\rho). \end{aligned} \tag{133}$$

On the other hand, by (128)-(129), we have

$$J_{ij} = d_i d_j H(q_i, q_j) + O(\rho). \tag{134}$$

Thus, by (131)–(134),

$$-\int_\Omega P u_i \Delta u_j = \begin{cases} d_i d_j G(q_i, q_j) + O(\rho), & \forall i \neq j, \\ d_j\left\{d_j H(q_j, q_j) - 4\log(\mu_j\varepsilon) - 2\right\} + O(\rho), & \forall i = j. \end{cases} \tag{135}$$

Besides, by (32),

$$\int_{\Omega_\varepsilon} W_q = \sum_{\bigcup_{l=1}^3 J_l} d_i + O(\rho). \tag{136}$$

Using the choice for μ_i's by (31), together with (125), (135), and (136), it follows that

$$\begin{aligned} J_\varepsilon(V_q) = {} & \sum_{i \in \bigcup_{l=1}^3 J_l} d_i \left\{ \log \frac{8(1+\alpha_i)^2}{c_i(q_i)} \right. \\ & - 2(1+\log\varepsilon) - \frac{1}{2} d_i H(q_i, q_i) \\ & \left. - \frac{1}{2} \sum_{j\neq i, j \in \bigcup_{l=1}^3 J_l} a_i a_j d_j G(q_j, q_i) \right\} \\ & + O(\rho). \end{aligned} \tag{137}$$

This, together with (12), easily gives the asymptotic expansion (123) of $F_\varepsilon(q)$. □

5. Proofs of Theorems

Proof of Theorem 3. According to Lemma 8, we only need to find a critical point of the function, consider

$$\begin{aligned} \widetilde{F}_\varepsilon(q) = {} & F_\varepsilon(q) \\ & - \sum_{i \in \bigcup_{l=1}^3 J_l} d_i \left\{\log 8(1+\alpha_i)^2 - 2(1+\log\varepsilon)\right\} \\ & + \sum_{i \in J_1} d_i \left\{ \frac{1}{2} d_i H(q_i, q_i) \right. \\ & \left. + \sum_{j \in J_1, i \neq j} \frac{1}{2} a_i a_j d_j G(q_j, q_i) \right\}. \end{aligned} \tag{138}$$

By (123), $\widetilde{F}_\varepsilon = \varphi_{k,m}^n + O(\rho) \to \varphi_{k,m}^n$ uniformly holds on $\Lambda_{k,m}(\delta)$. By Definition 2, there exists a critical point q_ε of F_ε such that $\varphi_{k,m}^n(q_\varepsilon) \to \varphi_{k,m}^n(q^*)$. Moreover, up to a subsequence, there exists points $\widetilde{q} = (\widetilde{q}_{n+1}, \ldots, \widetilde{q}_{M+m}) \in \Lambda_{k,m}$ such that $q_\varepsilon \to \widetilde{q} \in \Lambda_{k,m}$ as $\varepsilon \to 0$, and $\varphi_{k,m}^n(q^*) = \varphi_{k,m}^n(\widetilde{q})$. Thus,

$v_\varepsilon = V_{q_\varepsilon} + \phi(q_\varepsilon)$ is a family of solutions of problem (25) (or (27)). Set $u_\varepsilon(x) = V_{q_\varepsilon}((1/\varepsilon)x) + \phi(q_\varepsilon)((1/\varepsilon)x)$ for any $x \in \Omega$. As a result, u_ε is a family of solutions of problem (11) with the qualitative properties predicted by the theorem, as it can be easily shown. □

Proof of Theorem 1. Set $q_1 = (0,0)$, $c(x) = 1$, and $a_l = (-1)^{l-1}$ for $l = 1,\dots,m+1$. If $\alpha \in \mathbb{N}$,

$$\begin{aligned}\varphi^0_{1,m}(q) = {} & -32\pi^2 \\ & \times \left\{ \frac{\alpha}{2\pi}\sum_{l=2}^{m+1} \log|q_l| + (1+\alpha)^2 H(0,0) \right. \\ & + \sum_{l=2}^{m+1} H(q_l,q_l) + 2(1+\alpha) \\ & \times \sum_{l=2}^{m+1} (-1)^{l+1} G(0,q_l) \\ & \left. + \sum_{l,j=2,l\neq j}^{m+1} (-1)^{l+j} G(q_l,q_j) \right\},\end{aligned} \tag{139}$$

and if $\alpha \in (-1,+\infty) \setminus (\mathbb{N} \cup \{0\})$,

$$\begin{aligned}\varphi^1_{0,m}(q) = {} & -32\pi^2 \\ & \times \left\{ \frac{\alpha}{2\pi}\sum_{l=2}^{m+1} \log|q_l| + 2(1+\alpha) \right. \\ & \times \sum_{l=2}^{m+1} (-1)^{l+1} G(0,q_l) + \sum_{l=2}^{m+1} H(q_l,q_l) \\ & \left. + \sum_{l,j=2,l\neq j}^{m+1} (-1)^{l+j} G(q_l,q_j) \right\}.\end{aligned} \tag{140}$$

We will seek a nodal solution for problem (5) with the concentration points 0 and $\tilde{q}_l = (\lambda\cos(2\pi(l-1)/m), \lambda\sin(2\pi(l-1)/m))$, $l = 2,\dots,m+1$. Note that

$$\begin{aligned} G(x,y) &= -\frac{\log|x-y|}{2\pi} \\ &\quad + \frac{\log\left(|x|^2|y|^2 + 1 - 2\langle x,y\rangle\right)}{4\pi}, \\ H(x,x) &= \frac{\log\left(1-|x|^2\right)}{2\pi}. \end{aligned} \tag{141}$$

By Theorem 3, we can reduce the problem of finding solutions of (5) to that of finding a C^0-stable critical point of the function $f(\lambda) : (0,1) \to \mathbb{R}$ defined in (8). Obviously, $\lim_{\lambda\to 0+} f(\lambda) = \lim_{\lambda \to 1-} f(\lambda) = +\infty$, which implies that $f(\lambda)$ has an absolute minimum point λ_0 in $(0,1)$. Then, λ_0 is a C^0-stable critical point of $f(\lambda)$. □

Appendix

Proofs of (32) *and* (33). For $|z_i| \le \delta(\nu_i\rho_i)^{-1}$, by (22)-(23),

$$\begin{aligned} V_q &= a_i P u_i(\varepsilon y) + \sum_{j\neq i} a_j P u_j(\varepsilon y) \\ &= a_i \left[u_i(\varepsilon y) + d_i H(\varepsilon y, q_i) \right. \\ &\qquad \left. - \log\frac{8\mu_i^2(1+\alpha_i)^2}{c_i(q_i)} + O(\rho) \right] \\ &\quad + \sum_{j\neq i} a_j \left[d_j G(\varepsilon y, q_j) + O(\rho) \right] \\ &= a_i \{ u_i(\varepsilon y) + d_i H(q_i,q_i) \\ &\qquad - \log\frac{8\mu_i^2(1+\alpha_i)^2}{c_i(q_i)} \\ &\qquad + \sum_{j\neq i} a_i a_j d_j G(q_i,q_j) \\ &\qquad + O(|\varepsilon y - q_i|) + O(\rho) \}, \end{aligned} \tag{A.1}$$

which, together with the definition (31) of μ_i, implies that

$$V_q = a_i \{ u_i(\varepsilon y) + O(|\varepsilon y - q_i|) + O(\rho) \}. \tag{A.2}$$

Furthermore, for $|z_i| \le \delta(\nu_i\rho_i)^{-1}$, a direct computation shows that

$$\begin{aligned} W_q &= \varepsilon^4 c(\varepsilon y) S(\varepsilon y) \left[e^{V_q} + e^{-V_q} \right] \\ &= \varepsilon^4 c_i(\varepsilon y) |\varepsilon y - q_i|^{2\alpha_i} \left[e^{u_i(\varepsilon y)} + e^{-u_i(\varepsilon y)} \right] \\ &\quad \cdot \left[1 + O(|\varepsilon y - q_i|) + O(\rho) \right] \\ &= \varepsilon^4 c_i(q_i) |\varepsilon y - q_i|^{2\alpha_i} \\ &\quad \times \left[\frac{8\mu_i^2(1+\alpha_i)^2}{c_i(q_i)\left[\mu_i^2\varepsilon^2 + |x-q_i|^{2(1+\alpha_i)}\right]^2} + O(1) \right] \\ &\quad \cdot \left[1 + O(|\varepsilon y - q_i|) + O(\rho) \right] \\ &= \left(\frac{\varepsilon}{\nu_i\rho_i} \right)^2 |z_i|^{2\alpha_i} \\ &\quad \times \left\{ \frac{8(1+\alpha_i)^2\left[1 + O(\rho_i|z_i|) + O(\rho)\right]}{\left[1 + |z_i|^{2(1+\alpha_i)}\right]^2} + O\left(\varepsilon^4\right) \right\}. \end{aligned} \tag{A.3}$$

Similarly, for $|z_i| \leq \delta(\nu_i\rho_i)^{-1}$,

$$2\varepsilon^4 c(\varepsilon y) S(\varepsilon y) \sinh V_q = \left(\frac{\varepsilon}{\nu_i\rho_i}\right)^2 |z_i|^{2\alpha_i} \times \left\{ \frac{8a_i(1+\alpha_i)^2 \left[1 + O(\rho_i |z_i|) + O(\rho)\right]}{\left[1 + |z_i|^{2(1+\alpha_i)}\right]^2} + O(\varepsilon^4) \right\}. \quad (A.4)$$

On the other hand, if $|z_i| \geq \delta(\nu_i\rho_i)^{-1}$, for any $i \in \bigcup_{l=1}^3 J_l$, it is easy to check that

$$W_q = O(\varepsilon^4), \qquad 2\varepsilon^4 c(\varepsilon y) S(\varepsilon y) \sinh V_q = O(\varepsilon^4). \quad (A.5)$$

This, together with (A.3), implies (32).

Next, by our definitions,

$$\begin{aligned} \Delta V_q &= \sum_{i \in \bigcup_{l=1}^3 J_l} \varepsilon^2 a_i \Delta u_i(\varepsilon y) \\ &= \sum_{i \in \bigcup_{l=1}^3 J_l} \varepsilon^4 a_i c_i(q_i) |\varepsilon y - q_i|^{2\alpha_i} e^{u_i(\varepsilon y)} \\ &= \sum_{i \in \bigcup_{l=1}^3 J_l} \varepsilon^4 a_i c_i(q_i) |\varepsilon y - q_i|^{2\alpha_i} \times \frac{8\mu_i^2 (1+\alpha_i)^2}{c_i(q_i)\left[\mu_i^2 \varepsilon^2 + |\varepsilon y - q_i|^{2(1+\alpha_i)}\right]^2} \\ &= \sum_{i \in \bigcup_{l=1}^3 J_l} \left(\frac{\varepsilon}{\nu_i\rho_i}\right)^2 |z_i|^{2\alpha_i} \times \frac{8a_i(1+\alpha_i)^2}{\left[1 + |z_i|^{2(1+\alpha_i)}\right]^2}. \end{aligned} \quad (A.6)$$

So, if $|z_i| \geq \delta(\nu_i\rho_i)^{-1}$, for all i,

$$\Delta V_q = O(\varepsilon^4), \quad (A.7)$$

while if $|z_i| \leq \delta(\nu_i\rho_i)^{-1}$,

$$\Delta V_q = \left(\frac{\varepsilon}{\nu_i\rho_i}\right)^2 |z_i|^{2\alpha_i} \times \frac{8a_i(1+\alpha_i)^2}{\left[1 + |z_i|^{2(1+\alpha_i)}\right]^2} + O(\varepsilon^4). \quad (A.8)$$

As a result, combining (A.4)-(A.5) with (A.7)-(A.8), we get (33). □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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