

Research Article

Classification of Ordered Type Soliton Metric Lie Algebras by a Computational Approach

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We classify some soliton nilpotent Lie algebras and possible candidates in dimensions 8 and 9 up to isomorphism. We focus on $1 < 2 < \dots < n$ type of derivations, where n is the dimension of the Lie algebras. We present algorithms to generate possible algebraic structures.

1. Introduction

In this paper, we compute and classify n -dimensional ($n = 8, 9$) nilsoliton metric Lie algebras with eigenvalue type $1 < 2 < \dots < n$, which will be called “ordered type of Lie algebra” throughout this paper. We use MATLAB to achieve this goal. In the literature, *six*-dimensional nilpotent Lie algebras have been classified by algorithmic approaches [1]. In dimension *seven* and lower, nilsoliton metric Lie algebras have been classified [2–9]. Summary and details of some other classifications can be found in [10]. In our paper, we focus on dimensions eight and nine. We note that we have found that our algorithm gives consistent results with the literature in lower dimensions. We use a computational procedure that is similar to the one that we have used in our previous paper [4].

In our previous paper, we classified all the soliton and nonsoliton metric Lie algebras where the corresponding Gram matrix is invertible and of dimensions 7 and 8 up to isomorphism. If corresponding Gram matrix is invertible, then the soliton metric condition $U\nu = [1]$ has a unique solution. So in this case, it is easy to check if the algebra is soliton or not. But in noninvertible case, there is more than one solution. Therefore it is hard to guess if one of the solutions provides the soliton condition without solving Jacobi identity which is nonlinear. On the other hand, it may be easy if we can eliminate some algebras which admit a derivation D that does not have ordered eigenvalues without solving the

following soliton metric condition $U\nu = [1]$. For this, we prove that if the nilpotent Lie algebra admits a soliton metric with corresponding Gram matrix of η being noninvertible, all the solutions of $U\nu = [1]$ have a unique derivation. This theorem allows us to omit several cases that come from nonordered eigenvalues without considering Jacobi identity.

This paper is organized as follows. In Section 2, we provide some preliminaries that we use for our classifications. In Section 3, we give specific Jacobi identity conditions for Lie algebras up to dimension *nine*. This allows us to decide whether the Lie algebra has a soliton metric or not. In Section 4, we give details of our classifications with specific examples and provide algorithmic procedures. Section 5 contains our concluding remarks.

2. Preliminaries

Let (η_μ, Q) be a metric algebra, where $\mu \in \Lambda^2 \eta \otimes \eta^*$. Let $B = \{X_i\}_{i=1}^n$ be a Q -orthonormal basis of η_μ (we always assume that bases are ordered). The nil-Ricci endomorphism Ric_μ is defined as $\langle \text{Ric}_\mu X, Y \rangle = \text{ric}_\mu(X, Y)$, where

$$\begin{aligned} \text{ric}_\mu(X, Y) = & -\frac{1}{2} \sum_{i=1}^n \langle [X, X_i], [Y, X_i] \rangle \\ & + \frac{1}{4} \sum_{i=1}^n \langle [X_i, X_j], X \rangle \langle [X_i, X_j], Y \rangle \end{aligned} \quad (1)$$

for $X, Y \in \eta$ (we often write an inner product $Q(\cdot, \cdot)$ as $\langle \cdot, \cdot \rangle$). When η is a nilpotent Lie algebra, the nil-Ricci endomorphism is the Ricci endomorphism. If all elements of the basis are eigenvectors for the nil-Ricci endomorphism Ric_μ , we call the orthonormal basis a Ricci eigenvector basis.

Now we define some combinatorial objects associated to a set of integer triples $\Lambda \subset \{(i, j, k) \mid 1 \leq i, j, k \leq n\}$. For $1 \leq i, j, k \leq n$, define $1 \times n$ row vector y_{ij}^k to be $\epsilon_i^T + \epsilon_j^T - \epsilon_k^T$, where $\{\epsilon_i\}_{i=1}^n$ is the standard orthonormal basis for \mathbb{R}^n . We call the vectors in $\{y_{ij}^k \mid (i, j, k) \in \Lambda\}$ root vectors for Λ . Let y_1, y_2, \dots, y_m (where $m = |\Lambda|$) be an enumeration of the root vectors in dictionary order. We define root matrix Y_Λ for Λ to be the $m \times n$ matrix whose rows are the root vectors y_1, y_2, \dots, y_m . The Gram matrix U_Λ for Λ is the $m \times m$ matrix defined by $U_\Lambda = Y_\Lambda Y_\Lambda^T$; the (i, j) entry of U_Λ is the inner product of the i th and j th root vectors. It is easy to see that U is a symmetric matrix. It has the same rank as the root matrix; that is, $\text{Rank}(U_\Lambda) = \text{Rank}(Y_\Lambda)$. Diagonal elements of U are all three, and the off-diagonal entries of U are in the set $\{-2, -1, 0, 1, 2\}$. For more information, see [11]. Let D have distinct real positive eigenvalues, and let Λ index the structure constants for η with respect to eigenvector basis B . If $(i_1, j_1, k_1) \in \Lambda$ and $(i_2, j_2, k_2) \in \Lambda$, then $\langle y_{i_1, j_1}^{k_1}, y_{i_2, j_2}^{k_2} \rangle \neq 2$. Thus U does not contain two as an entry [4].

Lemma 1. *Let (η, Q) be an n -dimensional inner product space, and let μ be an element of $\Lambda^2 \eta^* \otimes \eta$. Suppose that η_μ admits a symmetric derivation D having n distinct eigenvalues $0 < \lambda_1 < \lambda_2 < \dots < \lambda_n$ with corresponding orthonormal eigenvectors X_1, X_2, \dots, X_n . Let α_{ij}^k denote the structure constants for η with respect to the ordered basis $B = \{X_i\}_{i=1}^n$. Let $1 \leq i < j \leq n$. Then*

- (1) *if there is some $k \in \{1, 2, \dots, n\}$ such that $\lambda_k = \lambda_i + \lambda_j$, then $[X_i, X_j]$ is a scalar multiple of X_k ; otherwise X_i and X_j commute;*
- (2) $\alpha_{ij}^k \neq 0$ *if and only if* $X_k \in [\eta_\mu, \eta_\mu]$.

Theorem 2 (see [11]). *Let η be a vector space, and let $B = X_{i=1}^n$ be a basis for η . Suppose that a set of nonzero structure constants $\alpha_{i,j}^k$ relative to B , indexed by Λ , defines a skew symmetric product on η . Assume that if $(i, j, k) \in \Lambda$, then $i < j < k$. Then the algebra is a Lie algebra if and only if whenever there exists m so that the inner product of root vectors $\langle y_{ij}^l, y_{lk}^m \rangle = -1$ for triples (i, j, l) and (l, k, m) or (k, l, m) in Λ , the equation*

$$\sum_{s < m} \alpha_{i,j}^s \alpha_{s,k}^m + \alpha_{j,k}^s \alpha_{s,i}^m + \alpha_{k,i}^s \alpha_{s,j}^m = 0 \tag{2}$$

holds. Furthermore, a term of form $\alpha_{i,j}^l \alpha_{l,k}^m$ is nonzero if and only if $\langle y_{i,j}^l, y_{l,k}^m \rangle = -1$

Theorem 3 (see [11]). *Let (η_μ, Q) be a metric algebra and $B = \{X_i\}_{i=1}^n$ a Ricci eigenvector basis for η_μ . Let Y be the root matrix*

for η_μ . Then the eigenvalues of the nil-Ricci endomorphism are given by

$$\text{Ric}_\mu^B = -\frac{1}{2} Y^T v, \tag{3}$$

where $v = [\alpha^2]$.

Theorem 4 (see [4, 11]). *Let (η, Q) be a nonabelian metric algebra with Ricci eigenvector basis B . The following are equivalent.*

- (1) (η_μ, Q) *satisfies the nilsoliton condition with nilsoliton constant β .*
- (2) *The eigenvalue vector V_D for $D = \text{Ric} - \beta \text{Id}$ with respect to B lies in the kernel of the root matrix for (η_μ, Q) with respect to B .*
- (3) *For noncommuting eigenvectors X and Y for the nil-Ricci endomorphism with eigenvalues κ_X and κ_Y , the bracket $[X, Y]$ is an eigenvector for the nil-Ricci endomorphism with eigenvalue $\kappa_X + \kappa_Y - \beta$.*
- (4) $\beta = y_{ij}^k \text{Ric}$ *for all (i, j, k) in $\Lambda(\eta_\mu, B)$.*

Theorem 5 (see [4]). *Let η be an n -dimensional nonabelian nilpotent Lie algebra which admits a derivation D having distinct real positive eigenvalues. Let B be a basis consisting of eigenvectors for the derivation D , and let Λ index the nonzero structure constants with respect to B . Let U be the $m \times m$ Gram matrix. If U is invertible, then the following hold:*

- (i) $|\Lambda| \leq n - 1$;
- (ii) *if $(i_1, j_1, k_1) \in \Lambda$ and $(i_2, j_2, k_2) \in \Lambda$, then $\langle y_{i_1, j_1}^{k_1}, y_{i_2, j_2}^{k_2} \rangle \neq -1$.*

3. Theory

This section provides some theorems and their proofs that allow us to consider fewer cases for our algorithm. The following theorem gives a pruning method while Gram matrix is noninvertible.

Theorem 6. *Let η be an n -dimensional nilsoliton metric Lie algebra, and U the corresponding Gram matrix which is noninvertible. Then $\text{Ker}(Y^T) = \text{Ker}(U)$. Furthermore all of the solutions of $Uv = [1]$ correspond to a unique derivation.*

Proof. Since rank of a matrix is equal to the rank of its Gram matrix, then $p = \text{Rank}(Y) = \text{Rank}(U)$. Let $U : \mathbb{R}^p \rightarrow \mathbb{R}^p$ and $Y^T : \mathbb{R}^p \rightarrow \mathbb{R}^n$ denote the linear functions (with respect to the standard basis) that correspond to the Gram matrix U and the transpose of the root matrix Y , respectively. Since $\text{Rank}(U) = \text{Rank}(Y) = \text{Rank}(Y^T)$ and by rank-nullity theorem, we have

$$\text{Ker}(U) = \text{Ker}(Y^T). \tag{4}$$

Let v be a particular solution and v_0 the last column of reduced row echelon matrix $[U, [1]]$. Then v_0 is also a solution

of $Uv_0 = [1]$. Therefore $U(v-v_0) = 0$; that is, $(v-v_0) \in \text{Ker}(U)$. Using (4), then $(v-v_0) \in \text{Ker}(Y^T)$. For the solution v_0 , suppose that we denote D_0 for the Nikolayevsky derivation, Ric_0 for the Ricci tensor, and β_0 for the soliton constant. Then using (3), we have

$$(\text{Ric} - \text{Ric}_0) = \frac{1}{2}Y^t(v-v_0) = 0. \tag{5}$$

Then $\text{Ric} = \text{Ric}_0$. Using Theorem 4, we have $\beta = \beta_0$, which implies that $D = D_0$. \square

Lemma 7. *If nilsoliton metric Lie algebra η has ordered type of derivations $1 < 2 < \dots < n$, then its index set Λ consists of triples $(i, j, i + j)$.*

Proof. If $V_D = (\lambda_1, \lambda_2, \dots, \lambda_n)^T$ is the eigenvalue vector of D with eigenvector basis $B = \{X_i\}_{i=1}^n$ for η , then by Theorem 4, V_D lies in the kernel of Y . Thus for each element $(i, j, k) \in \Lambda$, $\lambda_i + \lambda_j - \lambda_k = 0$; that is, $\lambda_i + \lambda_j = \lambda_k$. By Lemma 1, $[X_i, X_j] = \lambda X_k$ for some $\lambda \in \mathbb{R}$. Since $\lambda_i = i$ for all $i \in \{1, 2, \dots, n\}$, $k = i + j$ and $[X_i, X_j] = \lambda X_{i+j}$. Hence, the index set for ordered type of derivations is of form $(i, j, i + j)$. \square

The next corollary describes the index triples (i, j, k) and the Jacobi identity for algebras with ordered type of derivations.

Corollary 8. *The algebra η is a Lie algebra if and only if for all pairs of form (i, j, l) and (l, k, m) or (i, j, l) and (k, l, m) in Λ_B with $k \notin \{i, j\}$ and for all $m \geq \max\{i + 3, j + 2, 5\}$, the following equation holds:*

$$\sum_{3 \leq s < m, s \notin \{i, j, k\}} \alpha_{i,j}^s \alpha_{s,k}^m + \alpha_{j,k}^s \alpha_{s,i}^m + \alpha_{k,i}^s \alpha_{s,j}^m = 0. \tag{6}$$

If in addition $\lambda_i = i$ for $i = 1, \dots, n$, then the algebra η is a Lie algebra if and only if for all pairs of form $(i, j, i + j)$ and $(i + j, k, i + j + k)$ or $(i, j, i + j)$ and $(k, i + j, i + j + k)$ in Λ_B with $k \notin \{i, j, i + j\}$ and for all $m = i + j + k \geq \max\{2i + 2, j + 2, 6\}$, the equation

$$\sum_{4 \leq s < m, s \notin \{i, j\}} \alpha_{i,j}^s \alpha_{s,k}^m + \alpha_{j,k}^s \alpha_{s,i}^m + \alpha_{k,i}^s \alpha_{s,j}^m = 0 \tag{7}$$

holds.

Proof. By Theorem 7 of [11], the algebra η_μ defined by μ is a Lie algebra if and only if whenever there exists m so that $\langle y_{ij}^l, y_{lk}^m \rangle = -1$ for triples (i, j, l) and (l, k, m) or (k, l, m) in Λ_B , (2)

$$\sum_{s < m} \alpha_{i,j}^s \alpha_{s,k}^m + \alpha_{j,k}^s \alpha_{s,i}^m + \alpha_{k,i}^s \alpha_{s,j}^m = 0 \tag{8}$$

holds. Furthermore, if i, j , and k are distinct, the product $\alpha_{i,j}^l \alpha_{lk}^m$ is nonzero if and only if $\langle y_{ij}^l, y_{lk}^m \rangle = -1$.

Suppose that $\langle y_{ij}^l, y_{lk}^m \rangle = -1$ for $(i, j, l) \in \Lambda_B$ and (l, k, m) or (k, l, m) in Λ_B . By definition of Λ_B , we have $i < j$. By Lemma 1, $j < l, l < m$, and $k < m$. Since $i < j < l < m$,

we know that $m \geq i + 3$. Similarly, $j < l < m$ implies that $j + 2 \leq m$. If $i = k$ or $j = k$, then $\langle y_{ij}^l, y_{lk}^m \rangle = 0$, and so i, j , and k must be distinct. Since i, j, k , and l are all distinct and less than m , we know that $m \geq 5$. Thus an expression of form

$$\alpha_{i,j}^s \alpha_{s,k}^m + \alpha_{j,k}^s \alpha_{s,i}^m + \alpha_{k,i}^s \alpha_{s,j}^m \tag{9}$$

is nonzero only if $m \geq \max\{i + 3, j + 2, 5\}$, and $k \notin \{i, j\}$.

Suppose that $\lambda_i = i$, and (i, j, l) and (l, k, m) are in the index set. Then from Lemma 7, $l = i + j$ which implies that $m = i + j + k$. We know that $1 \leq i < j < l < m$. Then, since $j \geq i + 1$ and $k \geq 1$, we have $2i + 2 \leq i + j + k = m$. Since $2i + 2 \leq m, m = 5$ implies that $i = 1$. So there is no possible (i, j, k) , where all i, j, k are distinct and $i < j < m$ with $i + j + k$. Thus if $m = 5$, then $\langle y_{ij}^l, y_{lk}^m \rangle \neq -1$. Therefore, if $\lambda_i = i$, an expression of form

$$\alpha_{i,j}^s \alpha_{s,k}^m + \alpha_{j,k}^s \alpha_{s,i}^m + \alpha_{k,i}^s \alpha_{s,j}^m \tag{10}$$

is nonzero only if $m \geq \max\{2i + 2, j + 2, 6\}$, and $k \notin \{i, j\}$.

By Lemma 1, X_1 and X_2 are in $[\eta_\mu, \eta_\mu]^\perp$, and so $\alpha_{rt}^s \neq 0$ which implies that $s \geq 3$ and $r \neq s, t \neq s$. Therefore all expressions in (10) with $s < 3$ or $s \in \{i, j, k\}$ are identically zero and may be omitted from the summation for any m . \square

The next corollary describes some equations in the structure constants of a nilpotent metric Lie algebra that are equivalent to the Jacobi identity. Each of the terms $\alpha_{ij}^s \alpha_{sk}^m$ in the following equations corresponds to each of -1 entry in the Gram matrix U . Therefore, the following equations are useful for noninvertible case since there is no -1 entry in the Gram matrix for the invertible case.

Corollary 9. *Let $(\eta_\mu, \langle \dots, \dots \rangle)$ be an n -dimensional inner product space where $n \leq 9$, and, μ be an element of $\Lambda^2 \eta^* \otimes \eta$. Suppose that the algebra η_μ defined by μ admits a symmetric derivation D having n eigenvalues $1 < 2 < \dots < n$ with corresponding orthonormal eigenvectors X_1, X_2, \dots, X_n . $\alpha_{i,j}^k$ denote the structure constants for η with respect to the ordered basis $B = \{X_i\}_{i=1}^n$, and let λ_B index the nonzero structure constants as defined in (2). The algebra η_μ is a Lie algebra if and only if*

$$\alpha_{13}^4 \alpha_{42}^6 + \alpha_{23}^5 \alpha_{51}^6 = 0, \tag{11}$$

$$\alpha_{12}^3 \alpha_{3,4}^7 - \alpha_{14}^5 \alpha_{52}^7 - \alpha_{24}^6 \alpha_{61}^7 = 0, \tag{12}$$

$$\alpha_{12}^3 \alpha_{35}^8 + \alpha_{14}^5 \alpha_{53}^8 + \alpha_{15}^6 \alpha_{62}^8 + \alpha_{25}^7 \alpha_{71}^8 + \alpha_{34}^7 \alpha_{71}^8 = 0, \tag{13}$$

$$\alpha_{12}^3 \alpha_{36}^9 + \alpha_{13}^4 \alpha_{45}^9 - \alpha_{23}^5 \alpha_{45}^9 - \alpha_{24}^6 \alpha_{36}^9 - \alpha_{34}^7 \alpha_{27}^9 - \alpha_{35}^8 \alpha_{18}^9 = 0 \tag{14}$$

holds.

TABLE 1: Possible (i, j, k) for $m = 6$.

Case	i	j	k	PT
(a)	1	2	3	—
(b)	1	3	2	✓
(c)	1	4	1	—
(d)	2	3	1	✓
(e)	2	4	0	—

Proof. Following Lemma 7, the index set consists of elements of form $(i, j, i + j)$. Therefore the number s equals to $i + j$ in the expression $\alpha_{i,j}^s \alpha_{s,k}^m$. From the previous corollary,

$$\alpha_{i,j}^s \alpha_{s,k}^m + \alpha_{j,k}^s \alpha_{s,i}^m + \alpha_{k,i}^s \alpha_{s,j}^m \tag{15}$$

is nonzero only if $m \geq 6$, and $k \notin \{i, j\}$. Also $k \neq i + j$; otherwise $\alpha_{s,k}^m = 0$.

If $m = 6$, $2i + 2 \leq m$ implies that $i \leq 2$; that is, possible numbers for “ i ” are 1 and 2. Possible and not possible (i, j, k) triples, which are being used in

$$\sum_{3 \leq s < m, s \notin \{i, j, k\}} \alpha_{i,j}^s \alpha_{s,k}^m + \alpha_{j,k}^s \alpha_{s,i}^m + \alpha_{k,i}^s \alpha_{s,j}^m = 0 \tag{16}$$

and where $i + j + k = 6$, are illustrated in Table 1. The notations in the table are as follows ✓ := yes; — = no; PT = possible triple.

In the case (a), $i + j = k$, and then it is not a possible triple. In the case (b), $i < j$, i, j, k are distinct, and $k \neq i + j$. So it is a possible triple. In the case (c), $i = k$, and so it is not a possible triple. In the case (d), $i < j$, and all i, j, k are distinct as $i + j \neq k$. Thus it is a possible triple. In the case (e), k is not a natural number, and so it is not a possible triple. Therefore only possible (i, j, k) triples are (1, 3, 2) and (2, 3, 1). Triples (1, 3, 2) and (2, 3, 1) correspond to nonzero products $\alpha_{13}^4 \alpha_{42}^6$ and $\alpha_{23}^5 \alpha_{5,1}^6$, respectively. Using the skew-symmetry, (2) turns into the following equation:

$$\alpha_{13}^4 \alpha_{24}^6 + \alpha_{23}^5 \alpha_{15}^6 = 0, \tag{17}$$

which gives (11). Using the same procedure for $m = 7$, possible (i, j, k) triples are (1, 2, 4), (1, 4, 2) and (2, 4, 1), which correspond to nonzero $\alpha_{12}^3 \alpha_{34}^7$, $\alpha_{14}^5 \alpha_{52}^7$, and $\alpha_{24}^6 \alpha_{61}^7$ respectively. Therefore (12) is obtained. Equations (13) and (14) can be obtained by the same way. □

As an illustration, we show how to use the results of this section in the following example.

Example 10. Let η be an 8-dimensional algebra with nonzero structure constants relative to eigenvector basis B indexed by

$$\Lambda = \{(1, 2, 3), (1, 3, 4), (1, 4, 5), (1, 6, 7), (2, 3, 5), (2, 6, 8), (3, 4, 7), (3, 5, 8)\}. \tag{18}$$

Computation shows that the structure vector $[\alpha^2]$ is a solution to $Uv = [1]_{8 \times 1}$ if and only if it is of form

$$[\alpha^2] = \begin{pmatrix} (\alpha_{12}^3)^2 \\ (\alpha_{13}^4)^2 \\ (\alpha_{14}^5)^2 \\ (\alpha_{16}^7)^2 \\ (\alpha_{23}^5)^2 \\ (\alpha_{26}^8)^2 \\ (\alpha_{34}^7)^2 \\ (\alpha_{35}^8)^2 \end{pmatrix} = \begin{pmatrix} -\frac{3}{17} \\ \frac{10}{17} \\ \frac{15}{17} \\ -\frac{8}{17} \\ -\frac{5}{17} \\ \frac{12}{17} \\ \frac{7}{17} \\ 0 \\ 0 \end{pmatrix} \tag{19}$$

$$+ X \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + Y \cdot \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \\ -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

Equation (13) from the previous corollary leads

$$\left(-\frac{3}{17} + X + Y\right) \cdot Y = \left(\frac{15}{17} - X - Y\right) \cdot Y \tag{20}$$

$$\implies X + Y = \frac{9}{17}.$$

Moreover, using (12), we find that $X = 3/17$ and $Y = 6/17$, which means that $(\alpha_{16}^7)^2 = -2/17$. Thus η is not a Lie algebra.

4. Algorithm and Classifications

In this section, we describe our computational procedure and give the results in dimensions 8 and 9.

4.1. Algorithm. Now we describe the algorithm. The following algorithm can be used for both invertible and noninvertible cases.

Input. The input is the integer n which represents the dimension.

Output. The output is two 0 – 1 matrices W_{soliton} and U_{inv} listing characteristic vectors for index sets Λ of Θ_n . The matrix W_{soliton} has as its rows all possible characteristic vectors for canonical index sets Λ for nilpotent Lie algebras of dimension n with ordered type nonsingular nilsoliton

TABLE 2: 8-dimensional nilsoliton metric Lie algebras.

	Lie bracket	Index	Nullity
1	$(0, 0, \sqrt{274/2223} \cdot 12, \sqrt{99/764} \cdot 13, \sqrt{527/8179} \cdot 14 + \sqrt{1532/9311} \cdot 23, \sqrt{101/2154} \cdot 15 + \sqrt{250/4199} \cdot 24, \sqrt{150/3151} \cdot 16 + \sqrt{110/367} \cdot 25 + \sqrt{110/367} \cdot 34, \sqrt{7/17} \cdot 17)$	6	3
2	$(0, 0, \sqrt{82/7253} \cdot 12, \sqrt{4/34} \cdot 13, \sqrt{97/299} \cdot 14 + \sqrt{6/34} \cdot 23, \sqrt{1/34} \cdot 15 + \sqrt{9/34} \cdot 24, \sqrt{10/34} \cdot 16 + \sqrt{1998/12097} \cdot 25 + \sqrt{397/13917} \cdot 34, \sqrt{728/2883} \cdot 17 + \sqrt{637/4000} \cdot 35)$	6	4
3	$(0, 0, \sqrt{163/702} \cdot 12, \sqrt{263/2572} \cdot 13, \sqrt{343/1334} \cdot 14 + \sqrt{547/2871} \cdot 23, \sqrt{175/2358} \cdot 15 + \sqrt{160/1149} \cdot 24, \sqrt{388/2509} \cdot 16 + \sqrt{518/6185} \cdot 25, \sqrt{43/13870} \cdot 17 + \sqrt{915/2239} \cdot 35)$	6	3

TABLE 3: The 9-dimensional nilsoliton metric Lie algebras of nullities 1 and 2.

	Lie bracket	Index	Nullity
1	$(0, 0, 0, 0, \sqrt{45} \cdot 23, \sqrt{14} \cdot 24, \sqrt{91} \cdot 25 + \sqrt{91} \cdot 34, \sqrt{136} \cdot 17 + \sqrt{29} \cdot 26 + \sqrt{14} \cdot 35, \sqrt{104} \cdot 18 + 4.27)$	4	1
2	$(0, 0, 1.12, 0, 1.14 + \sqrt{3} \cdot 23, 0, \sqrt{6} \cdot 25 + \sqrt{6} \cdot 34, \sqrt{8} \cdot 17 + 1.26 + \sqrt{2} \cdot 35, \sqrt{6} \cdot 18 + \sqrt{2} \cdot 27)$	5	2
3	$(0, 0, \sqrt{21} \cdot 12, 0, \sqrt{21} \cdot 14 + \sqrt{39} \cdot 23, 0, \sqrt{13} \cdot 16 + \sqrt{70} \cdot 25 - \sqrt{70} \cdot 34, \sqrt{88} \cdot 17 + \sqrt{42} \cdot 35, \sqrt{65} \cdot 18 + \sqrt{39} \cdot 27)$	5	2

derivation whose canonical Gram matrix U is invertible. The matrix U_{inv} has as its rows all possible characteristic vectors for canonical index sets Λ for nilpotent Lie algebras of dimension n with ordered type nonsingular nilsoliton derivation whose canonical Gram matrix U is noninvertible. In the dimensions 8 and 9, there is no example for invertible case. Thus $W_{soliton}$ is an empty matrix. Therefore we give the algorithm for the noninvertible case.

Algorithm for the Noninvertible Case. Consider the following.

- (i) Enter the dimension n .
- (ii) Compute the matrix Z_n .
- (iii) Compute the matrix W .
- (iv) Delete all rows of W containing abelian factor which is the row that represents direct sums of Lie algebras.
- (v) Remove all rows of W such that the canonical Gram matrix U associated to the index set Λ is invertible.
- (vi) Define eigenvalue vector $v_D = (1, 2, 3, \dots, n)^T$ in dimension n .
- (vii) Remove all rows of W if $v(i) = v_0(i) \leq 0$ where v is the general solution of $U_\Lambda v = [1]_{m \times 1}$ and v_0 is the vector that we have defined in the proof of Theorem 6.
- (viii) Remove all the rows of W such that the corresponding algebra does not have a derivation of eigenvalue type $1 < 2 < \dots < n$.
- (ix) Remove all the rows of W such that the corresponding algebra does not satisfy Jacobi identity condition, which is obtained in Corollary 9.

After this process, we solve nonlinear systems which follow from Jacobi identity. In order to see how the algorithm works, we give the following example for $n = 6$.

Example 11. Let $n = 6$. Then

$$\Theta_6 = \{(1, 2, 3), (1, 3, 4), (1, 4, 5), (1, 5, 6), (2, 3, 5), (2, 4, 6)\}. \tag{21}$$

TABLE 4: The 8-dimensional nilsoliton metric Lie algebra candidates.

	Lie bracket	Index	Nullity
1	$(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.34, 1.17 + 1.26 + 1.35)$	6	4
2	$(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.25, 1.17 + 1.26 + 1.35)$	6	4
3	$(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.25 + 1.34, 1.17 + 1.26)$	6	4
4	$(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.25 + 1.34, 1.17 + 1.26 + 1.35)$	6	5

TABLE 5: Number of 9-dimensional nilsoliton metric Lie algebra candidates.

Nullity	Number of Lie algebras
3	98
4	81
5	45
6	22
7	7
8	1

So, matrix Z_6 is 6×3 of form

$$Z_6 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 5 \\ 1 & 5 & 6 \\ 2 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}. \tag{22}$$

Since $|\Theta_6| = 6$, the matrix W is of size $2^6 \times 6$ as follows

$$W_\Theta = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}. \tag{23}$$

TABLE 6: The 9-dimensional candidates of nullities 6 and 8.

	Lie bracket	Index	Nullity
1	(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.27 + 1.36 + 1.45)	6	6
2	(0, 0, 1.12, 1.13, 1.14, 1.15, 1.16 + 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.27 + 1.36 + 1.45)	7	6
3	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.34, 1.17 + 1.35, 1.18 + 1.27 + 1.36 + 1.45)	7	6
4	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.36 + 1.45)	7	6
5	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.34, 1.17 + 1.26, 1.18 + 1.27 + 1.36 + 1.45)	7	6
6	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.27 + 1.45)	7	6
7	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.27 + 1.36)	7	6
8	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.25 + 1.34, 1.17 + 1.35, 1.18 + 1.36 + 1.45)	7	6
9	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.25, 1.17 + 1.35, 1.18 + 1.27 + 1.36 + 1.45)	7	6
10	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.25 + 1.34, 1.17, 1.18 + 1.27 + 1.36 + 1.45)	7	6
11	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.25 + 1.34, 1.17 + 1.35, 1.18 + 1.27 + 1.45)	7	6
12	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.25 + 1.34, 1.17 + 1.35, 1.18 + 1.27 + 1.36)	7	6
13	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.25, 1.17 + 1.26 + 1.35, 1.18 + 1.36 + 1.45)	7	6
14	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.25 + 1.34, 1.17 + 1.26, 1.18 + 1.36 + 1.45)	7	6
15	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.45)	7	6
16	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.36)	7	6
17	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.25, 1.17 + 1.26, 1.18 + 1.27 + 1.36 + 1.45)	7	6
18	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.25, 1.17 + 1.26 + 1.35, 1.18 + 1.27 + 1.45)	7	6
19	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.25, 1.17 + 1.26 + 1.35, 1.18 + 1.27 + 1.36)	7	6
20	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.25 + 1.34, 1.17 + 1.26, 1.18 + 1.27 + 1.45)	7	6
21	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.25 + 1.34, 1.17 + 1.26, 1.18 + 1.27 + 1.36)	7	6
22	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.27)	7	6
23	(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.27 + 1.36 + 1.45)	7	8

The first row of W_θ represents empty matrix, row two represents the subset $\{(2, 4, 6)\}$ of Θ_6 , and so forth. Eliminating rows that represent direct sums, we have 33 rows in W matrix. Therefore none of the rows of W corresponds to Lie algebras that can be written as direct sums. These algebras correspond both to invertible and noninvertible Gram matrices. There is no example for the invertible case. For the noninvertible case, there is one ordered type nilsoliton metric Lie algebra η . Let B be the eigenvector basis for η , whose nonzero structure constants are indexed by

$$\Lambda_\eta = \{(1, 2, 3), (1, 3, 4), (1, 4, 5), (1, 5, 6), (2, 3, 5), (2, 4, 6)\}. \tag{24}$$

Computation shows that the structure vector $[\alpha^2]$ is a solution to $Uv = [1]_{6 \times 1}$ if and only if it is of form

$$[\alpha^2] = \begin{pmatrix} (\alpha_{12}^3)^2 \\ (\alpha_{13}^4)^2 \\ (\alpha_{14}^5)^2 \\ (\alpha_{15}^6)^2 \\ (\alpha_{23}^5)^2 \\ (\alpha_{24}^6)^2 \end{pmatrix} = \frac{1}{143} \begin{pmatrix} 2 \\ 1 \\ 2 \\ 5 \\ 5 \\ 0 \end{pmatrix} + \frac{t}{143} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ -1 \\ 1 \end{pmatrix}. \tag{25}$$

By Corollary 9, η satisfies (11). Solving the equation for t , we find that

$$\begin{aligned} (\alpha_{13}^4 \alpha_{24}^6)^2 &= (\alpha_{23}^5 \alpha_{15}^6)^2, \\ (11 + t)t &= (55 - t)^2, \\ t &= 25. \end{aligned} \tag{26}$$

After rescaling and solving for structure constants from $[\alpha^2]$, we see that letting

$$\begin{aligned} [X_1, X_2] &= \sqrt{22}X_3, & [X_1, X_3] &= 6X_4, \\ [X_1, X_4] &= \sqrt{22}X_5, & [X_1, X_5] &= \sqrt{30}X_6, \\ [X_2, X_3] &= \sqrt{30}X_5, & [X_2, X_4] &= 5X_6 \end{aligned} \tag{27}$$

defines a nilsoliton metric Lie algebra, previously found in [3].

4.2. *Classifications.* Classification results for dimensions 8 and 9 appear in Tables 2 and 3, respectively. We use vector notations to represent Lie algebra structures. For example, the list

$$\begin{aligned} (0, 0, 0, 0, \sqrt{45} \cdot 23, \sqrt{14} \cdot 24, \sqrt{91} \cdot 25 + \sqrt{91} \cdot 34, \sqrt{136} \cdot 17 \\ + \sqrt{29} \cdot 26 + \sqrt{14} \cdot 35, \sqrt{104} \cdot 18 + 4 \cdot 27) \end{aligned} \tag{28}$$

TABLE 7: The 9-dimensional nilsoliton metric Lie algebra candidates of nullity 3.

	Lie bracket	Index	Nullity
1	(0, 0, 0, 0, 1.23, 1.24, 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.18, 1.27 + 1.36 + 1.45)	4	3
2	(0, 0, 1.12, 0, 1.23, 1.24, 1.16 + 1.25 + 1.34, 1.17, 1.18 + 1.27 + 1.36 + 1.45)	5	3
3	(0, 0, 1.12, 0, 1.23, 1.24, 1.16 + 1.25 + 1.34, 1.17 + 1.35, 1.18 + 1.27 + 1.45)	5	3
4	(0, 0, 1.12, 0, 1.23, 1.24, 1.16 + 1.25 + 1.34, 1.17 + 1.35, 1.18 + 1.27 + 1.36)	5	3
5	(0, 0, 1.12, 0, 1.23, 1.24, 1.16 + 1.25 + 1.34, 1.17 + 1.26, 1.18 + 1.36 + 1.45)	5	3
6	(0, 0, 1.12, 0, 1.23, 1.24, 1.16 + 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.45)	5	3
7	(0, 0, 1.12, 0, 1.23, 1.24, 1.16 + 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.36)	5	3
8	(0, 0, 1.12, 0, 1.23, 1.24, 1.16 + 1.25 + 1.34, 1.17 + 1.26, 1.18 + 1.27 + 1.45)	5	3
9	(0, 0, 1.12, 0, 1.23, 1.24, 1.16 + 1.25 + 1.34, 1.17 + 1.26, 1.18 + 1.27 + 1.36)	5	3
10	(0, 0, 1.12, 0, 1.23, 1.24, 1.16 + 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.27)	5	3
11	(0, 0, 1.12, 0, 1.14 + 1.23, 0, 1.25 + 1.34, 1.17 + 1.26, 1.18 + 1.27 + 1.36 + 1.45)	5	3
12	(0, 0, 1.12, 0, 1.14 + 1.23, 0, 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.27 + 1.45)	5	3
13	(0, 0, 1.12, 0, 1.14 + 1.23, 0, 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.27 + 1.36)	5	3
14	(0, 0, 1.12, 0, 1.14 + 1.23, 1.24, 1.25 + 1.34, 1.17, 1.18 + 1.27 + 1.36 + 1.45)	5	3
15	(0, 0, 1.12, 0, 1.14 + 1.23, 1.24, 1.25 + 1.34, 1.17 + 1.35, 1.18 + 1.27 + 1.36)	5	3
16	(0, 0, 1.12, 0, 1.14 + 1.23, 1.24, 1.25 + 1.34, 1.17 + 1.26, 1.18 + 1.36 + 1.45)	5	3
17	(0, 0, 1.12, 0, 1.14 + 1.23, 1.24, 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.45)	5	3
18	(0, 0, 1.12, 0, 1.14 + 1.23, 1.24, 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.45)	5	3
19	(0, 0, 1.12, 0, 1.14 + 1.23, 1.24, 1.25 + 1.34, 1.17 + 1.26, 1.18 + 1.27 + 1.45)	5	3
20	(0, 0, 1.12, 0, 1.14 + 1.23, 1.24, 1.25 + 1.34, 1.17 + 1.26, 1.18 + 1.27 + 1.36)	5	3
21	(0, 0, 1.12, 0, 1.14 + 1.23, 1.24, 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.27)	5	3
22	(0, 0, 1.12, 0, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.17, 1.18 + 1.27 + 1.36 + 1.45)	5	3
23	(0, 0, 1.12, 0, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.17 + 1.35, 1.18 + 1.27 + 1.45)	5	3
24	(0, 0, 1.12, 0, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.17 + 1.35, 1.18 + 1.27 + 1.36)	5	3
25	(0, 0, 1.12, 0, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.17 + 1.26, 1.18 + 1.36 + 1.45)	5	3
26	(0, 0, 1.12, 0, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.45)	5	3
27	(0, 0, 1.12, 0, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.17 + 1.26, 1.18 + 1.27 + 1.45)	5	3
28	(0, 0, 1.12, 0, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.17 + 1.26, 1.18 + 1.27 + 1.36)	5	3
29	(0, 0, 1.12, 0, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.27)	5	3
30	(0, 0, 1.12, 0, 1.14 + 1.23, 1.24, 1.16 + 1.34, 1.17 + 1.35, 1.18 + 1.27 + 1.45)	4	3
31	(0, 0, 1.12, 0, 1.14 + 1.23, 1.24, 1.16 + 1.34, 1.17 + 1.35, 1.18 + 1.27 + 1.36)	4	3
32	(0, 0, 1.12, 0, 1.14 + 1.23, 1.24, 1.16 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.45)	4	3
33	(0, 0, 1.12, 0, 1.14 + 1.23, 1.24, 1.16 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.36)	4	3
34	(0, 0, 1.12, 0, 1.14 + 1.23, 1.24, 1.16 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.27)	4	3
35	(0, 0, 1.12, 1.13, 1.23, 0, 1.16 + 1.25, 1.17 + 1.26 + 1.35, 1.27 + 1.36 + 1.45)	4	3
36	(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.25 + 1.34, 1.26 + 1.35, 1.18 + 1.27 + 1.45)	5	3
37	(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.25 + 1.34, 1.26 + 1.35, 1.18 + 1.27 + 1.36)	5	3
38	(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.25 + 1.34, 1.17 + 1.26, 1.27 + 1.36 + 1.45)	5	3
39	(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.27 + 1.45)	5	3
40	(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.27 + 1.36)	5	3
41	(0, 0, 1.12, 1.13, 1.14, 0, 1.25 + 1.34, 1.17 + 1.26, 1.18 + 1.27 + 1.36 + 1.45)	5	3
42	(0, 0, 1.12, 1.13, 1.14, 0, 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.27 + 1.45)	6	3
43	(0, 0, 1.12, 1.13, 1.14, 0, 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.27 + 1.36)	6	3
44	(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.25 + 1.34, 1.17 + 1.26, 1.18 + 1.36 + 1.45)	6	3
45	(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.45)	6	3
46	(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.25 + 1.34, 1.17 + 1.26, 1.18 + 1.27 + 1.45)	6	3
47	(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.25 + 1.34, 1.17 + 1.26, 1.18 + 1.27 + 1.36)	6	3
48	(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.27)	6	3
49	(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.26 + 1.35, 1.27 + 1.45)	5	3
50	(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.26 + 1.35, 1.27 + 1.36)	5	3
51	(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.35, 1.18 + 1.27 + 1.45)	5	3
52	(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.35, 1.18 + 1.27 + 1.36)	5	3

TABLE 7: Continued.

	Lie bracket	Index	Nullity
53	$(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.26 + 1.35, 1.18 + 1.45)$	5	3
54	$(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.26 + 1.35, 1.18 + 1.36)$	5	3
55	$(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.26 + 1.35, 1.18 + 1.27)$	5	3
56	$(0, 0, 1.12, 1.13, 1.14, 0, 1.16 + 1.25 + 1.34, 1.17 + 1.26, 1.27 + 1.36 + 1.45)$	5	3
57	$(0, 0, 1.12, 1.13, 1.14, 0, 1.16 + 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.27 + 1.45)$	5	3
58	$(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.17, 1.27 + 1.36 + 1.45)$	5	3
59	$(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.17 + 1.35, 1.27 + 1.45)$	5	3
60	$(0, 0, 1.12, 1.13, 1.14, 0, 1.16 + 1.25 + 1.34, 1.17, 1.18 + 1.27 + 1.36 + 1.45)$	6	3
61	$(0, 0, 1.12, 1.13, 1.14, 0, 1.16 + 1.25 + 1.34, 1.17 + 1.35, 1.18 + 1.27 + 1.45)$	6	3
62	$(0, 0, 1.12, 1.13, 1.14, 0, 1.16 + 1.25 + 1.34, 1.17 + 1.35, 1.18 + 1.27 + 1.36)$	6	3
63	$(0, 0, 1.12, 1.13, 1.14, 0, 1.16 + 1.25 + 1.34, 1.17 + 1.26, 1.18 + 1.27 + 1.45)$	6	3
64	$(0, 0, 1.12, 1.13, 1.14, 0, 1.16 + 1.25 + 1.34, 1.17 + 1.26, 1.18 + 1.27 + 1.36)$	6	3
65	$(0, 0, 1.12, 1.13, 1.14, 0, 1.16 + 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.18 + 1.27)$	6	3
66	$(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.17, 1.18 + 1.36 + 1.45)$	6	3
67	$(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.17 + 1.35, 1.18 + 1.45)$	6	3
68	$(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.17 + 1.35, 1.18 + 1.45)$	6	3

TABLE 8: The 9-dimensional nilsoliton metric Lie algebra candidates of nullity 3.

	Lie bracket	Index	Nullity
69	$(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.17, 1.18 + 1.27 + 1.45)$	6	3
70	$(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.17, 1.18 + 1.27 + 1.36)$	6	3
71	$(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.17 + 1.35, 1.18 + 1.27)$	6	3
72	$(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.17 + 1.26, 1.18 + 1.45)$	6	3
73	$(0, 0, 1.12, 1.13, 1.14 + 1.23, 0, 1.16 + 1.25 + 1.34, 1.17 + 1.26 + 1.35, 1.18)$	6	3
74	$(0, 0, 1.12, 1.13, 1.14, 1.15, 1.25 + 1.34, 1.26, 1.18 + 1.27 + 1.36 + 1.45)$	6	3
75	$(0, 0, 1.12, 1.13, 1.14, 1.15, 1.25 + 1.34, 1.26 + 1.35, 1.18 + 1.27 + 1.45)$	6	3
76	$(0, 0, 1.12, 1.13, 1.14, 1.15, 1.25 + 1.34, 1.26 + 1.35, 1.18 + 1.27 + 1.36)$	6	3
77	$(0, 0, 1.12, 1.13, 1.14, 1.15, 1.25 + 1.34, 1.17, 1.18 + 1.27 + 1.36 + 1.45)$	6	3
78	$(0, 0, 1.12, 1.13, 1.14, 1.15, 1.25 + 1.34, 1.17 + 1.35, 1.18 + 1.27 + 1.36)$	6	3
79	$(0, 0, 1.12, 1.13, 1.14, 1.15, 1.16 + 1.25 + 1.34, 1.26, 1.27 + 1.36 + 1.45)$	7	3
80	$(0, 0, 1.12, 1.13, 1.14, 1.15, 1.16 + 1.25 + 1.34, 1.26 + 1.35, 1.27 + 1.45)$	7	3
81	$(0, 0, 1.12, 1.13, 1.14, 1.15, 1.16 + 1.25 + 1.34, 1.26 + 1.35, 1.27 + 1.36)$	7	3
82	$(0, 0, 1.12, 1.13, 1.14, 1.15, 1.16 + 1.25 + 1.34, 1.35, 1.18 + 1.27 + 1.45)$	7	3
83	$(0, 0, 1.12, 1.13, 1.14, 1.15, 1.16 + 1.25 + 1.34, 1.35, 1.18 + 1.27 + 1.36)$	7	3
84	$(0, 0, 1.12, 1.13, 1.14, 1.15, 1.16 + 1.25 + 1.34, 1.26, 1.18 + 1.27 + 1.45)$	7	3
85	$(0, 0, 1.12, 1.13, 1.14, 1.15, 1.16 + 1.25 + 1.34, 1.26, 1.18 + 1.27 + 1.36)$	7	3
86	$(0, 0, 1.12, 1.13, 1.14, 1.15, 1.16 + 1.25 + 1.34, 1.26 + 1.35, 1.18 + 1.27)$	7	3
87	$(0, 0, 1.12, 1.13, 1.14, 1.15, 1.16 + 1.25 + 1.34, 1.17, 1.27 + 1.36 + 1.45)$	6	3
88	$(0, 0, 1.12, 1.13, 1.14, 1.15, 1.16 + 1.25 + 1.34, 1.17 + 1.35, 1.27 + 1.45)$	6	3
89	$(0, 0, 1.12, 1.13, 1.14, 1.15, 1.16 + 1.25 + 1.34, 1.17 + 1.35, 1.27 + 1.36)$	6	3
90	$(0, 0, 1.12, 1.13, 1.14, 1.15, 1.16 + 1.25 + 1.34, 1.17, 1.18 + 1.27 + 1.45)$	7	3
91	$(0, 0, 1.12, 1.13, 1.14, 1.15, 1.16 + 1.25 + 1.34, 1.17, 1.18 + 1.27 + 1.36)$	7	3
92	$(0, 0, 1.12, 1.13, 1.14, 1.15, 1.16 + 1.25 + 1.34, 1.17 + 1.35, 1.18 + 1.27)$	7	3
93	$(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.34, 1.17 + 1.35, 1.18)$	7	3
94	$(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.34, 1.17 + 1.26, 1.18)$	7	3
95	$(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.25, 1.17, 1.18 + 1.45)$	7	3
96	$(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.25, 1.17, 1.18 + 1.36)$	7	3
97	$(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.25, 1.17 + 1.35, 1.18)$	7	3
98	$(0, 0, 1.12, 1.13, 1.14 + 1.23, 1.15 + 1.24, 1.16 + 1.25 + 1.34, 1.17, 1.18)$	7	3

in the first row of Table 3 is meant to encode the metric Lie algebra $(n, \langle \cdot, \cdot \rangle)$ with orthonormal basis $B = \{X_i\}_{i=1}^8$ and bracket relations

$$\begin{aligned} [X_2, X_3] &= \sqrt{45}X_5, & [X_2, X_4] &= \sqrt{14}X_6, \\ [X_2, X_5] &= \sqrt{91}X_7, & [X_3, X_4] &= \sqrt{91}X_7, \\ [X_1, X_7] &= \sqrt{136}X_8, & [X_2, X_6] &= \sqrt{29}X_8, \\ [X_3, X_5] &= \sqrt{14}X_8, & [X_1, X_8] &= \sqrt{104}X_9, \\ [X_2, X_7] &= 4X_9. \end{aligned} \quad (29)$$

4.2.1. Candidates of Nilsoliton Metrics. Table 5 illustrates how many possible candidates of Lie algebras appear in dimension 9 up to the nullity of its Gram matrix. The algebras illustrated in Table 4 are possible candidates of nilsoliton metric Lie algebras with ordered type of derivations in dimension 8. Here, as an example we give potential Lie algebra structures when the nullity of their corresponding Gram matrices are 3, 6 and 8 in Tables 6, 7, and 8 respectively for dimension *nine*.

5. Conclusion

In this work, we have focused on nilpotent metric Lie algebras of dimensions eight and nine with ordered type of derivations. We have given specific Jacobi identity conditions for Lie algebras which allowed us to simplify the Jacobi identity condition. We have classified nilsoliton metric Lie algebras for the corresponding Gram matrix U being invertible and noninvertible. For dimension 8, we have focused on nilsoliton metric Lie algebras with noninvertible Gram matrix which leads to more than one solution for $U\nu = [1]$. We have proved that if the nilpotent Lie algebra admits a soliton metric with corresponding Gram matrix being noninvertible, all the solutions of $U\nu = [1]$ correspond to a unique derivation. This theorem has allowed us to omit several cases that come from nonordered eigenvalues without considering Jacobi condition. Moreover, we have classified some nilsoliton metric Lie algebras with derivation types $1 < 2 < \dots < n$ and provided some candidates that may be classified. We are currently working on an algorithm that provides a full list of classifications for dimensions *eight* and *nine*.

Appendix

See Tables 5, 6, 7, and 8.

Conflict of Interest

Author declares that she has no competing interest.

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