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Research Article

Affine Differential Invariants of Functions on the Plane

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A differential invariant is a function defined on the jet space of functions that remains the same under a group action. It is an important concept to solve the equivalence problem. This paper presents an effective method to derive a special type of affine differential invariants. Given some functions defined on the plane and an affine group acting on the plane, there are induced actions of the group on the functions and on the derivative functions of the functions. Affine differential invariants of these functions are useful in many applications. However, there has been little systematic study of this problem at present. No clear and simple results are available for application users to use directly. We propose a direct and simple method to construct affine differential invariants in this situation. Some useful explicit formulas of affine differential invariants of 2D functions are presented.

1. Introduction

The concept of an invariant is ubiquitous in science. It is crucial to solve the equivalence problem. The fundamental equivalence problem is to determine whether two objects of a set of objects are equivalent with respect to a given equivalence relation. An invariant with respect to the equivalence relation is a function defined on the set of objects that is constant on the equivalence classes. Two objects are not equivalent if they have different invariants. The classification problem can be solved by deriving a system of invariants that separates any two equivalence classes. When a set of objects is acted on by a group, invariants can be used to classify the set of objects under the action of the group. In this case the equivalence classes are the orbits of the group.

Classical invariant theory originated in the 19th century with the study of constructing (relative) invariants of a system of forms [1–4]. In this case, the invariants are polynomial functions on the coefficients of a system of forms. Algebraic methods for deriving (relative integral) invariants were developed during this period. Classical invariant theory is closely related to projective geometry. The vanishing of a form in n variables corresponds to a hypersurface in an $n-1$-dimensional projective space. The invariants are then used to classify hypersurfaces into projective equivalence classes.

Differential invariants were introduced and studied by Lie and Tresse in the late 19th century [5, 6]. A differential invariant is a function defined on a *jet space* that is invariant under the action of a Lie group. Given q functions with p independent variables, a *jet space* is defined by considering the independent variables, the dependent variables, and the derivatives of the dependent variables as functionally independent coordinates of the space. Since differential equations are surfaces in the *jet space*, differential invariants serve to classify differential equations into equivalence classes. If one differential equation is solved in an equivalence class, then all differential equations in the equivalence class are solved.

The idea of an invariant is important in computer vision [7–9]. Two views of a 3D scene are geometrically related by the epipolar constraint. Two views of a planar object are geometrically related by a projective transformation. Invariants of objects are widely used to match two images that are related by a geometric transformation. Apart from their direct application in object recognition, differential invariants provide a mathematical foundation to design local feature detectors of images.

An elegant tool for deriving differential invariants is the moving frame method [10–22]. The concept of moving frames has a long history. Cartan derived some differential invariants based on his moving frame method [10, 11].

The method was generalized by Fels and Olver for general transformation groups [12, 13]. Differential invariants of one-dimensional manifolds are well studied. There are a few papers devoted to the differential invariants of surfaces [18–22].

This paper presents an effective method to derive a special type of affine differential invariants. Given some functions defined on the plane and an affine group acting on the plane, there are induced actions of the group on the functions and on the derivative functions of the functions. Affine differential invariants of these functions are useful in many applications. However, we have found little systematic study of this problem at present. No clear and simple results are available for application users to use directly. We propose a direct and simple method to construct affine differential invariants in this situation. Some useful formulas of affine differential invariants are presented in explicit form.

The rest of the paper is organized as follows. In Section 2, we define the basic concepts and notation. In Section 3, we present a few theorems on the properties of the affine differential invariants. In Section 4, we present the basic theorem for constructing the affine differential invariants. We give a few explicit formulas of affine differential invariants in Section 5 for ordinary people to user directly. We conclude in Section 6.

2. The Basic Concepts and Notation

In this section, we introduce the basic concepts and define the notation we will use. In what follows, we take the ground field to be $\mathbb{R}$, the field of real numbers. Given a group G and a space S, an action α of G on S is a map $\alpha : G \times S \rightarrow S$ such that $\alpha(e, x) = x$ and $\alpha(g, \alpha(h, x)) = \alpha(gh, x)$ for all $g, h \in G$ and $x \in S$, where e is the unit element of G.

A group G acting on a space S is called a transformation group if there is a group homomorphism $\rho : G \rightarrow \alpha^X$ mapping G to the group of invertible maps on X induced by the action α. That is, the action of a transformation group G preserves the structure endowed on S. The name of a transformation group reflects to a certain extent the behavior of the action and the structure endowed on S.

A G invariant of S under the action of α is a real valued function $\mathcal{I}$ defined on S such that

$$\mathcal{I}(\alpha(g, x)) = \mathcal{I}(x), \tag{1}$$

for all $g \in G$ and $x \in S$. That is, $\mathcal{I}$ is constant on the orbits of G.

Given an action α of a group G on a space S, there is an induced action μ on the set of smooth functions $\mathscr{C}^\infty(S, \mathbb{R})$ from S to $\mathbb{R}$. The action μ is usually defined as $\mu : (g, f) \mapsto \mu(g, f)$ for $g \in G$ and $f \in \mathscr{C}^\infty(S, \mathbb{R})$, where $\mu(g, f)(x) := f(\alpha(g^{-1}, x))$ for $x \in S$.

In this paper, we will denote a sequence by $[\![_{i=1}^{n} a_i$. That is,

$$[\![_{i=1}^{n} a_i = a_1, a_2, \ldots, a_n. \tag{2}$$

The nth order partial derivative of a smooth function $u(x, y)$ is denoted by $u_{i,n-i}$:

$$u_{i,n-i} = \frac{\partial^n u(x, y)}{\partial x^i \partial y^{n-i}}. \tag{3}$$

Given q smooth functions $[\![_{i=1}^{q} u^i$ that depend on p independent variables $[\![_{i=1}^{p} x^i$, the nth *jet space* $\mathcal{J}^n(X \times U^{(n)})$ is an Euclidean space of dimension $p + q(p+n)!p!n!$, where X is the space whose coordinates are the independent variables $[\![_{i=1}^{p} x^i$ and $U^{(n)}$ is the fiber space whose coordinates are the functions $[\![_{i=1}^{q} u^i$ and their derivatives up to the order n. These coordinates are assumed to be functionally independent.

In this paper, we study smooth functions with two variables x and y. In this case a point in the nth jet space $\mathcal{J}$ has the following form:

$$\Upsilon = \left(x, y, [\![_{i=1}^{q} u^i, [\![_{k=1}^{q} [\![_{j=1}^{n} [\![_{i=0}^{j} u^k_{i,j-i}\right). \tag{4}$$

After defining the action of a group G on functions, we can proceed to define the group action on derivatives of functions. This induced action is called the prolonged action. A differential invariant is a function defined on *jet space* $\mathcal{J}$, that is, invariant under the prolonged group action.

Let $GL(2, \mathbb{R})$ denote the general linear group of all invertible matrices of order 2 over $\mathbb{R}$. An affine transformation group $\mathbf{A}(2)$ is the group $GL(2, \mathbb{R}) \ltimes \mathbb{R}^2$ acting on $\mathbb{R}^2$ via the following:

$$\alpha(g, x) \mapsto Ax + b, \tag{5}$$

where $g = (A, b) \in \mathbf{A}(2)$, $A \in GL(2, \mathbb{R})$, and $b \in \mathbb{R}^2$. In order to study the specific properties of affine differential invariants, we would like to write an affine transformation in the following explicit form:

$$\begin{aligned} \tilde{x} &= ax + by + e, \\ \tilde{y} &= cx + dy + f, \end{aligned} \tag{6}$$

where $a, b, c, d, e, f \in \mathbb{R}$ and $(ad - bc) \neq 0$. The point $(\tilde{x}, \tilde{y})$ is called the transformed point. This convention is not obligatory. From the point of view of transformation, each one of (x, y) and $(\tilde{x}, \tilde{y})$ is the transformed point of the other.

Under the affine transformation (6), the relation between a smooth function $u(x, y)$ and the transformed function $\tilde{u}(\tilde{x}, \tilde{y})$ is as follows:

$$\tilde{u}(\tilde{x}, \tilde{y}) = \tilde{u}(ax + by + e, cx + dy + f) = u(x, y). \tag{7}$$

Likewise, which one of $u(x, y)$ and $\tilde{u}(\tilde{x}, \tilde{y})$ is called the transformed function is just a point of view. The nth order partial derivative of the transformed function $\tilde{u}(\tilde{x}, \tilde{y})$ is denoted by the following:

$$\tilde{u}_{i,n-i} = \frac{\partial^n \tilde{u}(\tilde{x}, \tilde{y})}{\partial \tilde{x}^i \partial \tilde{y}^{n-i}}. \tag{8}$$

A relative affine differential invariant with respect to transformation (6) is a polynomial $\mathscr{I}$ defined on the *jet space* $\mathscr{J}$ such that

$$\mathscr{I}\left(\llbracket_{k=1}^{q}\llbracket_{j=1}^{n}\llbracket_{i=0}^{j}\tilde{u}_{i,j-i}^{k}\right)=\frac{1}{(ad-bc)^{w}}\mathscr{I}\left(\llbracket_{k=1}^{q}\llbracket_{j=1}^{n}\llbracket_{i=0}^{j}u_{i,j-i}^{k}\right), \tag{9}$$

where w is called the *weight* of the differential invariant. The *degree* of a term in $\mathscr{I}$ is the sum of the exponents of the derivatives in the term. From the principles of invariant theory, it suffices to consider only homogeneous polynomials. In this case, all terms in $\mathscr{I}$ have the same degree, which will be called the *degree* of the differential invariant. The *order* of the differential invariant is the highest order of derivative occurring in $\mathscr{I}$. It is easy to see that an invariant in the sense of (1) can be constructed directly if two relative invariants are known. From now on, when we speak of invariants, we always mean relative invariants in the sense of (9).

The definition of affine differential invariant in (9) does not include x, y, and $u(x,y)$. The reason for not including $u(x,y)$ is that it is always invariant under affine transformation (6). The inclusion of $u(x,y)$ would be redundant. The reason for not including x and y is that any nontrivial polynomial of x and y cannot be an affine invariant. We will prove this in the next section.

3. The Properties of Affine Differential Invariants

In this section, we derive a few properties of the differential invariants of functions defined on the plane under the action of the affine group $\mathbf{A}(2)$. These properties are important for constructing affine differential invariants. We begin with a proof of the proposition we claimed in the last section.

Theorem 1. *An affine invariant polynomial $\mathscr{I}$ cannot contain x or y.*

Proof. It suffices to show that a translation invariant polynomial $\mathscr{I}$ cannot contain x or y. Let $\mathscr{I}(x, y, \llbracket_{k=1}^{q}\llbracket_{j=1}^{n}\llbracket_{i=0}^{j}u_{i,j-i}^{k})$ be a translation invariant polynomial. After a translation of the following form:

$$\tilde{x}=x+e, \qquad \tilde{y}=y+f, \tag{10}$$

we have the following identities:

$$\tilde{u}_{i,n-i}=u_{i,n-i}, \tag{11}$$

for all $n \geq 1$ and $0 \leq i \leq n$. According to the definition of affine differential invariant, using identities in (11), we have the following:

$$\mathscr{I}\left(x, y, \llbracket_{k=1}^{q}\llbracket_{j=1}^{n}\llbracket_{i=0}^{j}u_{i,j-i}^{k}\right)=\mathscr{I}\left(\tilde{x}, \tilde{y}, \llbracket_{k=1}^{q}\llbracket_{j=1}^{n}\llbracket_{i=0}^{j}u_{i,j-i}^{k}\right), \tag{12}$$

under translation transformation (10). Differentiating both sides of (12) with respect to e, we have the following:

$$0=\frac{\partial\mathscr{I}}{\partial\tilde{x}}\frac{\partial\tilde{x}}{\partial e}+\frac{\partial\mathscr{I}}{\partial\tilde{y}}\frac{\partial\tilde{y}}{\partial e}+\sum\frac{\partial\mathscr{I}}{\partial u_{i,j-i}^{k}}\frac{\partial u_{i,j-i}^{k}}{\partial e}. \tag{13}$$

Since

$$\frac{\partial\tilde{x}}{\partial e}=1, \qquad \frac{\partial\tilde{y}}{\partial e}=0, \qquad \frac{\partial u_{i,j-i}^{k}}{\partial e}=0, \tag{14}$$

for all i, j, and k under consideration, we have the following:

$$\frac{\partial\mathscr{I}}{\partial\tilde{x}}=0. \tag{15}$$

That is, $\mathscr{I}$ does not depend on x. Similarly, we can prove that $\mathscr{I}$ does not depend on y. □

Since $\tilde{u}_{i,n-i}=u_{i,n-i}$ for all $n \geq 1$ and $0 \leq i \leq n$ under translation transformation, we immediately have the following theorem.

Theorem 2. *Any polynomial function $\mathscr{I}(\llbracket_{k=1}^{q}\llbracket_{j=1}^{n}\llbracket_{i=0}^{j}u_{i,j-i}^{k})$ defined on a jet space of two dimensional functions is translation invariant.*

Now we give a theorem that is useful for deriving properties of the affine differential invariants.

Theorem 3. *Every affine transformation of the form* (6) *is equivalent to successively performing the following four types of transformations:*

$$\tilde{x}=ax, \quad \tilde{y}=dy, \quad (a, d \in \mathbb{R},\ ad \neq 0), \tag{16}$$

$$\tilde{x}=x+by, \quad \tilde{y}=y, \quad (b \in \mathbb{R}), \tag{17}$$

$$\tilde{x}=x, \quad \tilde{y}=cx+y, \quad (c \in \mathbb{R}), \tag{18}$$

$$\tilde{x}=x+e, \quad \tilde{y}=y+f, \quad (e, f \in \mathbb{R}). \tag{19}$$

Proof. An affine transformation of the form (6) is equivalent to a linear transformation followed by a translation. It is proved that every binary linear transformation is a product of linear transformations of the forms (16), (17), and (18) [1]. Thus, every affine transformation is composed of transformations of the forms (16), (17), and (18) plus a translation of the form (19). □

We now prove a theorem which describes relations between the original derivative functions and the transformed derivative functions under the 2D affine action.

Theorem 4. *Let $u(x,y)$ be a smooth function defined on the plane, under the action of affine group $\mathbf{A}(2)$ defined in* (6)*:*

$$u_{i,n-i}=\sum_{j=0}^{i}\sum_{k=0}^{n-i}\binom{i}{j}\binom{n-i}{k}a^{j}c^{i-j}b^{k}\times d^{n-i-k}\tilde{u}_{j+k,n-j-k}, \quad n \geq 1,\ 0 \leq i \leq n. \tag{20}$$

Proof. We prove the theorem by induction on n. For the base case $n=1$, since

$$\begin{aligned} u_{10}&=a\tilde{u}_{10}+c\tilde{u}_{01}, \\ u_{01}&=b\tilde{u}_{10}+d\tilde{u}_{01}, \end{aligned} \tag{21}$$

the statement is true. Now suppose that the theorem holds for $n = m > 1$. Then we have the following:

$$\begin{aligned}
u_{i,m+1-i} &= \frac{\partial}{\partial y}\left(u_{i,m-i}\right) \\
&= \sum_{j=0}^{i}\sum_{k=0}^{m-i}\binom{i}{j}\binom{m-i}{k} a^j c^{i-j} b^{k+1} \\
&\qquad \times d^{m+1-i-(k+1)} \tilde{u}_{j+k+1,m+1-j-(k+1)} \\
&\quad + \sum_{j=0}^{i}\sum_{k=0}^{m-i}\binom{i}{j}\binom{m-i}{k} a^j c^{i-j} b^{k} \\
&\qquad \times d^{m+1-i-k} \tilde{u}_{j+k,m+1-j-k} \\
&= \sum_{j=0}^{i}\sum_{k=1}^{m+1-i}\binom{i}{j}\binom{m-i}{k-1} a^j c^{i-j} b^{k} \\
&\qquad \times d^{m+1-i-k} \tilde{u}_{j+k,m+1-j-k} \\
&\quad + \sum_{j=0}^{i}\sum_{k=0}^{m-i}\binom{i}{j}\binom{m-i}{k} a^j c^{i-j} b^{k} \\
&\qquad \times d^{m+1-i-k} \tilde{u}_{j+k,m+1-j-k} \\
&= \sum_{j=0}^{i}\sum_{k=0}^{m+1-i}\binom{i}{j}\binom{m+1-i}{k} a^j c^{i-j} b^{k} \\
&\qquad \times d^{m+1-i-k} \tilde{u}_{j+k,m+1-j-k}, \\
u_{i+1,m-i} &= \frac{\partial}{\partial x}\left(u_{i,m-i}\right) \\
&= \sum_{j=0}^{i}\sum_{k=0}^{mi}\binom{i}{j}\binom{m-i}{k} a^{j+1} c^{i+1-(j+1)} b^{k} \\
&\qquad \times d^{m+1-(i+1)-k} \tilde{u}_{j+k+1,m+1-(j+1)-k} \\
&\quad + \sum_{j=0}^{i}\sum_{k=0}^{m-i}\binom{i}{j}\binom{m-i}{k} a^j c^{i+1-j} b^{k} \\
&\qquad \times d^{m+1-(i+1)-k} \tilde{u}_{j+k,m+1-j-k} \\
&= \sum_{j=1}^{i+1}\sum_{k=0}^{m-i}\binom{i}{j-1}\binom{m-i}{k} a^j c^{i+1-j} b^{k} \\
&\qquad \times d^{m+1-(i+1)-k} \tilde{u}_{j+k,m+1-j-k} \\
&\quad + \sum_{j=0}^{i}\sum_{k=0}^{m-i}\binom{i}{j}\binom{m-i}{k} a^j c^{i+1-j} b^{k} \\
&\qquad \times d^{m+1-(i+1)-k} \tilde{u}_{j+k,m+1-j-k} \\
&= \sum_{j=0}^{i+1}\sum_{k=0}^{m-i}\binom{i+1}{j}\binom{m-i}{k} a^j c^{i+1-j} b^{k} \\
&\qquad \times d^{m+1-(i+1)-k} \tilde{u}_{j+k,m+1-j-k}.
\end{aligned} \tag{22}$$

So, the theorem is true for $n = m + 1$. We conclude from the principle of mathematical induction that the theorem is true for all positive integer n. □

Next, we proceed to prove several theorems that describe properties of the affine differential invariants.

Theorem 5. *Let*

$$\mathscr{I} = \sum \mathscr{C}_I \prod_{i=1}^{q}\prod_{j=1}^{n}\prod_{k=0}^{j}\left(u^i_{k,j-k}\right)^{l_{Iijk}} \tag{23}$$

be a homogeneous and isobaric polynomial function defined on the jet space, where $\mathscr{C}_I$ are coefficients indexed by I. A necessary and sufficient condition for the function $\mathscr{I}$ to be an invariant with respect to the transformation (16) *is*

$$\sum_{i=1}^{q}\sum_{j=1}^{n}\sum_{k=0}^{j} k l_{Iijk} = \sum_{i=1}^{q}\sum_{j=1}^{n}\sum_{k=0}^{j}(j-k)\, l_{Iijk}. \tag{24}$$

Proof. Let

$$\mathscr{I} = \sum \mathscr{C}_I \prod_{i=1}^{q}\prod_{j=1}^{n}\prod_{k=0}^{j}\left(u^i_{k,j-k}\right)^{l_{Iijk}} \tag{25}$$

be an affine differential invariant with respect to the linear transformation (16). From (20), we have the following:

$$u_{i,n-i} = a^i d^{n-i} \tilde{u}_{i,n-i}. \tag{26}$$

Substituting (26) into (25), we have the following:

$$\begin{aligned}
\mathscr{I} &= \sum \mathscr{C}_I \prod_{i=1}^{q}\prod_{j=1}^{n}\prod_{k=0}^{j}\left(u^i_{k,j-k}\right)^{l_{Iijk}} \\
&= \sum \mathscr{C}_I \prod_{i=1}^{q}\prod_{j=1}^{n}\prod_{k=0}^{j}\left(a^k d^{j-k} \tilde{u}^i_{k,j-k}\right)^{l_{Iijk}}.
\end{aligned} \tag{27}$$

From the definition in (9), for $\mathscr{I}$ to be an affine differential invariant, there must be an identity of the following form:

$$\begin{aligned}
&\sum \mathscr{C}_I \prod_{i=1}^{q}\prod_{j=1}^{n}\prod_{k=0}^{j}\left(\tilde{u}^i_{k,j-k}\right)^{l_{Iijk}} \\
&\quad = \frac{1}{(ad)^{\lambda}} \sum \mathscr{C}_I \prod_{i=1}^{q}\prod_{j=1}^{n}\prod_{k=0}^{j}\left(u^i_{k,j-k}\right)^{l_{Iijk}}.
\end{aligned} \tag{28}$$

From (27) and (28), we have

$$\begin{aligned}
&\sum \mathscr{C}_I \prod_{i=1}^{q}\prod_{j=1}^{n}\prod_{k=0}^{j}\left(a^k d^{j-k} \tilde{u}^i_{k,j-k}\right)^{l_{Iijk}} \\
&\quad = (ad)^{\lambda} \sum \mathscr{C}_I \prod_{i=1}^{q}\prod_{j=1}^{n}\prod_{k=0}^{j}\left(\tilde{u}^i_{k,j-k}\right)^{l_{Iijk}}.
\end{aligned} \tag{29}$$

Comparing the degrees of a and d on both sides of (29), we conclude that

$$\lambda = \sum_{i=1}^{q}\sum_{j=1}^{n}\sum_{k=0}^{j}(j-k)\,l_{Iijk} = \sum_{i=1}^{q}\sum_{j=1}^{n}\sum_{k=0}^{j}k l_{Iijk}. \tag{30}$$

This ends the proof. □

Theorem 6. *A necessary and sufficient condition for a polynomial function $\mathcal{I}$ to be an affine differential invariant with respect to the linear transformation* (17) *is*

$$\mathcal{O}\mathcal{I} = 0, \tag{31}$$

where

$$\mathcal{O} = \sum_{i=1}^{q}\sum_{j=1}^{n}\sum_{k=0}^{j-1}(j-k)\,u^{i}_{k+1,j-k-1}\frac{\partial}{\partial u^{i}_{k,j-k}}. \tag{32}$$

Proof. With respect to the linear transformation (17), we have the following:

$$\tilde{u}_{i,n-i} = \sum_{k=0}^{n-i}\binom{n-i}{k}\beta^{k}u_{i+k,n-i-k}, \tag{33}$$

where $\beta = -b$. Differentiating both sides of (33) with respect to β, we have the following:

$$\begin{aligned}\frac{d}{d\beta}\tilde{u}_{i,n-i} &= \sum_{k=1}^{n-i}\binom{n-i}{k}k\beta^{k-1}u_{i+k,n-i-k}\\ &= \sum_{k=0}^{n-i-1}\binom{n-i}{k+1}(k+1)\,\beta^{k}u_{i+k+1,n-i-k-1}\\ &= \sum_{k=0}^{n-i-1}\frac{(n-i)!}{(k+1)!\,(n-i-k-1)!}\\ &\quad\times(k+1)\,\beta^{k}u_{i+k+1,n-i-k-1}\\ &= (n-i)\sum_{k=0}^{n-i-1}\binom{n-i-1}{k}\beta^{k}u_{i+k+1,n-i-k-1}\\ &= (n-i)\,\tilde{u}_{i+1,n-i-1}.\end{aligned} \tag{34}$$

With respect to the linear transformation (17), an affine differential invariant has to make the following equation:

$$\mathcal{I}\left(\llbracket_{i=1}^{q}\llbracket_{j=1}^{n}\llbracket_{k=0}^{j}\tilde{u}^{i}_{k,j-k}\right) = \mathcal{I}\left(\llbracket_{i=1}^{q}\llbracket_{j=1}^{n}\llbracket_{k=0}^{j}u^{i}_{k,j-k}\right). \tag{35}$$

Holding in the β. Differentiating both sides of (35) with respect to β, we have the following:

$$\sum_{i=1}^{q}\sum_{j=1}^{n}\sum_{k=0}^{j}\frac{\partial\mathcal{I}\left(\llbracket_{i=1}^{q}\llbracket_{j=1}^{n}\llbracket_{k=0}^{j}\tilde{u}^{i}_{k,j-k}\right)}{\partial\tilde{u}^{i}_{k,j-k}}\frac{d\tilde{u}^{i}_{k,j-k}}{d\beta} = 0. \tag{36}$$

Substituting (34) into (36), we have the following:

$$\sum_{i=1}^{q}\sum_{j=1}^{n}\sum_{k=0}^{j}(j-k)\,\tilde{u}^{i}_{k+1,j-k-1}\frac{\partial\mathcal{I}\left(\llbracket_{i=1}^{q}\llbracket_{j=1}^{n}\llbracket_{k=0}^{j}\tilde{u}^{i}_{k,j-k}\right)}{\partial\tilde{u}^{i}_{k,j-k}} = 0. \tag{37}$$

This ends the proof. □

Similarly, we can prove the following theorem.

Theorem 7. *A necessary and sufficient condition for a polynomial function $\mathcal{I}$ to be an affine differential invariant with respect to the linear transformation* (18) *is*

$$\mathcal{D}\mathcal{I} = 0, \tag{38}$$

where

$$\mathcal{D} = \sum_{i=1}^{q}\sum_{j=1}^{n}\sum_{k=1}^{j}k u^{i}_{k-1,j-k+1}\frac{\partial}{\partial u^{i}_{k,j-k}}. \tag{39}$$

Combining Theorems 2, 5, 6, and 7, we have the following.

Theorem 8. *The necessary and sufficient conditions for a homogeneous and isobaric polynomial function $\mathcal{I}$ to be an affine differential invariant with respect to the affine transformation* (6) *is*

$$\begin{aligned}&\mathcal{O}\mathcal{I} = 0,\\ &\mathcal{D}\mathcal{I} = 0,\\ &\sum_{i=1}^{q}\sum_{j=1}^{n}\sum_{k=0}^{j}k l_{Iijk} = \sum_{i=1}^{q}\sum_{j=1}^{n}\sum_{k=0}^{j}(j-k)\,l_{Iijk}.\end{aligned} \tag{40}$$

4. The Generating Operator of Affine Differential Invariants

We present direct and simple methods to construct the differential invariants in this section. We begin with two special cases.

Let $u^1(x,y)$ and $u^2(x,y)$ be two smooth functions in variables x and y. The *Jacobian*

$$J = \det\begin{pmatrix}u^1_{10} & u^1_{01}\\ u^2_{10} & u^2_{01}\end{pmatrix} \tag{41}$$

of the two functions is invariant with respect to the affine transformation (6). This proposition can be proved through direct calculation:

$$\begin{aligned}&\det\begin{pmatrix}u^1_{10} & u^1_{01}\\ u^2_{10} & u^2_{01}\end{pmatrix}\\ &= \det\begin{pmatrix}a\tilde{u}^1_{10} + c\tilde{u}^1_{01} & b\tilde{u}^1_{10} + d\tilde{u}^1_{01}\\ a\tilde{u}^2_{10} + c\tilde{u}^2_{01} & b\tilde{u}^2_{10} + d\tilde{u}^2_{01}\end{pmatrix}\\ &= (ad-bc)\det\begin{pmatrix}\tilde{u}^1_{10} & \tilde{u}^1_{01}\\ \tilde{u}^2_{10} & \tilde{u}^2_{01}\end{pmatrix}.\end{aligned} \tag{42}$$

Let $u(x,y)$ be a smooth function in variables x and y. The *Hessian*

$$H = \det\begin{pmatrix} u_{20} & u_{11} \\ u_{11} & u_{02} \end{pmatrix} \tag{43}$$

of the function is invariant with respect to the affine transformation (6). This can be verified by direct calculation also:

$$\begin{aligned}
&\det\begin{pmatrix} u_{20} & u_{11} \\ u_{11} & u_{02} \end{pmatrix} \\
&= \det\begin{pmatrix} c^2\tilde{u}_{02} + 2ac\tilde{u}_{11} + a^2\tilde{u}_{20} & cd\tilde{u}_{02} + cb\tilde{u}_{11} + ad\tilde{u}_{11} + ab\tilde{u}_{20} \\ cd\tilde{u}_{02} + cb\tilde{u}_{11} + ad\tilde{u}_{11} + ab\tilde{u}_{20} & d^2\tilde{u}_{02} + 2bd\tilde{u}_{11} + b^2\tilde{u}_{20} \end{pmatrix} \\
&= \det\begin{pmatrix} a & c \\ b & d \end{pmatrix} \det\begin{pmatrix} \tilde{u}_{20} & \tilde{u}_{11} \\ \tilde{u}_{11} & \tilde{u}_{02} \end{pmatrix} \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \\
&= (ad-bc)^2 \det\begin{pmatrix} \tilde{u}_{20} & \tilde{u}_{11} \\ \tilde{u}_{11} & \tilde{u}_{02} \end{pmatrix}.
\end{aligned} \tag{44}$$

The *Jacobian* and *Hessian* are two special cases of a general method for constructing affine differential invariants from two smooth functions. The method is presented in the following theorem.

Theorem 9. *Let $u^1(x,y)$ and $u^2(x,y)$ be two smooth functions in variables x and y. The following function*

$$\sum_{s=0}^{r} (-1)^s \binom{r}{s} u^1_{r-s,s} u^2_{s,r-s} \tag{45}$$

is invariant with respect to the affine transformation (6)*, where r is a positive integer.*

Proof. From Theorem 2, (45) is invariant under translation transformation. Now we have to show that (45) is also invariant under linear transformations (16), (17), and (18). Under the linear transformation (16), we have the following:

$$\begin{aligned}
&\sum_{s=0}^{r} (-1)^s \binom{r}{s} u^1_{r-s,s} u^2_{s,r-s} \\
&\quad = (ad)^r \sum_{s=0}^{r} (-1)^s \binom{r}{s} \tilde{u}^1_{r-s,s} \tilde{u}^2_{s,r-s}.
\end{aligned} \tag{46}$$

The theorem is true in this case. With respect to the linear transformation (17), since

$$u^{\tau}_{i,n-i} = \sum_{k=0}^{n-i} \binom{n-i}{k} b^k \tilde{u}^{\tau}_{i+k,n-i-k}, \quad \tau = 1,2, \tag{47}$$

we have the following:

$$\begin{aligned}
&\sum_{s=0}^{r} (-1)^s \binom{r}{s} u^1_{r-s,s} u^2_{s,r-s} \\
&= \sum_{s=0}^{r} \sum_{j=0}^{s} \sum_{k=0}^{r-s} (-1)^s b^{j+k} \binom{r}{s} \binom{s}{j} \\
&\qquad \times \binom{r-s}{k} \tilde{u}^1_{r-s+j,s-j} \tilde{u}^2_{s+k,r-s-k} \\
&= \sum_{j=0}^{r} \sum_{s=j}^{r} \sum_{k=0}^{r-s} (-1)^s b^{j+k} \binom{r}{s} \binom{s}{j} \\
&\qquad \times \binom{r-s}{k} \tilde{u}^1_{r-s+j,s-j} \tilde{u}^2_{s+k,r-s-k} \\
&= \sum_{j=0}^{r} \sum_{s=0}^{r-j} \sum_{k=0}^{r-j-s} (-1)^{s+j} b^{j+k} \binom{r}{s+j} \binom{s+j}{j} \\
&\qquad \times \binom{r-s-j}{k} \tilde{u}^1_{r-s,s} \tilde{u}^2_{s+j+k,r-s-j-k} \\
&= \sum_{s=0}^{r} \sum_{j=0}^{r-s} \sum_{k=0}^{r-j-s} (-1)^{s+j} b^{j+k} \binom{r}{s+j} \binom{s+j}{j} \\
&\qquad \times \binom{r-s-j}{k} \tilde{u}^1_{r-s,s} \tilde{u}^2_{s+j+k,r-s-j-k} \\
&= \sum_{s=0}^{r} \sum_{j=0}^{r-s} \sum_{k=j}^{r-s} (-1)^{s+j} b^{k} \binom{r}{s+j} \binom{s+j}{j} \\
&\qquad \times \binom{r-s-j}{k-j} \tilde{u}^1_{r-s,s} \tilde{u}^2_{s+k,r-s-k} \\
&= \sum_{s=0}^{r} \sum_{k=0}^{r-s} \frac{(-1)^s b^k r!}{k!\,(r-s-k)!s!} \tilde{u}^1_{r-s,s} \tilde{u}^2_{s+k,r-s-k} \\
&\qquad \times \sum_{j=0}^{k} (-1)^j \binom{k}{j} \\
&= \sum_{s=0}^{r} \sum_{k=0}^{r-s} \frac{(-1)^s b^k r!}{k!\,(r-s-k)!s!} \tilde{u}^1_{r-s,s} \tilde{u}^2_{s+k,r-s-k} 0^k \\
&= \sum_{s=0}^{r} (-1)^s \binom{r}{s} \tilde{u}^1_{r-s,s} \tilde{u}^2_{s,r-s}.
\end{aligned} \tag{48}$$

So, the theorem is true in this case. Similarly, we can prove that the theorem is true in the case of (18). From the principles of group action, we conclude that the theorem is true. □

Although the method presented in Theorem 9 is very general and can produce infinite number of affine differential invariants, there are affine differential invariants that cannot be constructed in this way. We need a method that can derive completely all affine differential invariants. Classical invariant theory of binary forms is a well-studied field. There are methods to derive a complete set of invariants of a system of binary forms under the action of the general linear group. The following theorem establishes a relation between classical invariant theory of binary forms and affine differential invariants of 2D functions.

Theorem 10. *Given a system of binary forms*

$$\llbracket_{i=1}^{q} \llbracket_{j=1}^{n} \sum_{k=0}^{j} \binom{j}{k} a_{k,j}^{i} X^{j-k} Y^{k} \tag{49}$$

and a set of smooth functions in variables x and y

$$\llbracket_{i=1}^{q} u^{i}(x, y), \tag{50}$$

the homogeneous and isobaric polynomial function

$$\mathcal{I}\left(\llbracket_{i=1}^{q} \llbracket_{j=1}^{n} \llbracket_{k=0}^{j} a_{k,j}^{i}\right) \tag{51}$$

is an invariant of the system of binary forms in (49) *with respect to the following linear transformation:*

$$\begin{aligned} X &= \lambda \widetilde{X} + \kappa \widetilde{Y}, \\ Y &= \eta \widetilde{X} + \mu \widetilde{Y} \end{aligned} \tag{52}$$

if and only if the homogeneous and isobaric polynomial function

$$\mathcal{I}\left(\llbracket_{i=1}^{q} \llbracket_{j=1}^{n} \llbracket_{k=0}^{j} u_{k,j-k}^{i}\right) \tag{53}$$

is an affine differential invariant with respect to affine transformation (6).

Proof. From Theorem 8, the necessary and sufficient conditions for the homogeneous and isobaric polynomial function

$$\mathcal{I}\left(\llbracket_{i=1}^{q} \llbracket_{j=1}^{n} \llbracket_{k=0}^{j} u_{k,j-k}^{i}\right) \tag{54}$$

to be an affine differential invariant are as follows:

$$\begin{aligned} &\mathcal{O}\mathcal{I}\left(\llbracket_{i=1}^{q} \llbracket_{j=1}^{n} \llbracket_{k=0}^{j} u_{k,j-k}^{i}\right) = 0, \\ &\mathcal{D}\mathcal{I}\left(\llbracket_{i=1}^{q} \llbracket_{j=1}^{n} \llbracket_{k=0}^{j} u_{k,j-k}^{i}\right) = 0, \\ &\sum_{i=1}^{q} \sum_{j=1}^{n} \sum_{k=0}^{j} k l_{Iijk} = \sum_{i=1}^{q} \sum_{j=1}^{n} \sum_{k=0}^{j} (j-k) l_{Iijk}. \end{aligned} \tag{55}$$

According to theorems of classical invariant theory (see book Sections I.4 and I.9 in [1], cf. chap.I and chap.II in [3]), the necessary and sufficient conditions for

$$\mathcal{I}\left(\llbracket_{i=1}^{q} \llbracket_{j=1}^{n} \llbracket_{k=0}^{j} a_{k,j}^{i}\right) \tag{56}$$

to be an invariant of the system of binary forms in (49) are as follows:

$$\begin{aligned} &\overline{\mathcal{O}}\mathcal{I}\left(\llbracket_{i=1}^{q} \llbracket_{j=1}^{n} \llbracket_{k=0}^{j} a_{k,j}^{i}\right) = 0, \\ &\overline{\mathcal{D}}\mathcal{I}\left(\llbracket_{i=1}^{q} \llbracket_{j=1}^{n} \llbracket_{k=0}^{j} a_{k,j}^{i}\right) = 0, \\ &\sum_{i=1}^{q} \sum_{j=1}^{n} \sum_{k=0}^{j} k l_{Iijk} = \sum_{i=1}^{q} \sum_{j=1}^{n} \sum_{k=0}^{j} (j-k) l_{Iijk}, \end{aligned} \tag{57}$$

where

$$\begin{aligned} \overline{\mathcal{O}} &= \sum_{i=1}^{q} \sum_{j=1}^{n} \sum_{k=0}^{j-1} (j-k)\, a_{k+1,j}^{i} \frac{\partial}{\partial a_{k,j}^{i}}, \\ \overline{\mathcal{D}} &= \sum_{i=1}^{q} \sum_{j=1}^{n} \sum_{k=1}^{j} k a_{k-1,j}^{i} \frac{\partial}{\partial a_{k,j}^{i}}. \end{aligned} \tag{58}$$

It is easy to see that, after a replacement of the corresponding variables, formula $\mathcal{O}\mathcal{I}(\llbracket_{i=1}^{q} \llbracket_{j=1}^{n} \llbracket_{k=0}^{j} u_{k,j-k}^{i})$ is the same as formula $\overline{\mathcal{O}}\mathcal{I}(\llbracket_{i=1}^{q} \llbracket_{j=1}^{n} \llbracket_{k=0}^{j} a_{k,j}^{i})$ and formula $\mathcal{D}\mathcal{I}(\llbracket_{i=1}^{q} \llbracket_{j=1}^{n} \llbracket_{k=0}^{j} u_{k,j-k}^{i})$ is the same as formula $\overline{\mathcal{D}}\mathcal{I}(\llbracket_{i=1}^{q} \llbracket_{j=1}^{n} \llbracket_{k=0}^{j} a_{k,j}^{i})$. This means that (55) and (57) have the same truth values. This ends the proof of the theorem. □

5. Affine Differential Invariants of Low Orders

By the methods provided in the previous section, it is easy to construct affine differential invariants of smooth functions up to any order. A general method for constructing affine differential invariants of smooth functions uses Theorem 8 directly. To construct all the differential invariants of degree d and order n, we can enumerate all homogeneous and isobaric polynomial functions of derivatives with unknown coefficients such that $\sum_{i=1}^{q} \sum_{j=1}^{n} \sum_{k=0}^{j} k l_{Iijk} = \sum_{i=1}^{q} \sum_{j=1}^{n} \sum_{k=0}^{j} (j-k) l_{Iijk}$. We then use the conditions $\mathcal{O}\mathcal{I}(\llbracket_{i=1}^{q} \llbracket_{j=1}^{n} \llbracket_{k=0}^{j} u_{k,j-k}^{i}) = 0$, $\mathcal{D}\mathcal{I}(\llbracket_{i=1}^{q} \llbracket_{j=1}^{n} \llbracket_{k=0}^{j} u_{k,j-k}^{i}) = 0$ to solve these unknown coefficients, cf. [1]. However, since there is a 1-1 correspondence between affine differential invariants and invariants of binary forms by Theorem 10, we do not need to derive affine differential invariants again. Low-degree invariants of binary forms are known completely [1–4]. There are general methods in the classical theory such as the transvection method which can derive complete sets of invariants up to any order.

We present a few affine differential invariants of low orders in this section. The invariants are in explicit form. This is important since application users may not be familiar with the mathematical background to derive the invariants.

There are two functional independent affine differential invariants of order two:

$$\begin{aligned} \mathcal{I}_{2,1}^{2} &= u_{20}u_{02} - u_{11}^{2}, \\ \mathcal{I}_{2,2}^{2} &= u_{20}u_{01}^{2} - 2u_{11}u_{01}u_{10} + u_{02}u_{10}^{2}. \end{aligned} \tag{59}$$

There are four functional independent affine differential invariants of order three. Two of them have weight three:

$$\begin{aligned} \mathcal{I}_{3,1}^{3} &= u_{30}u_{01}^{3} - 3u_{21}u_{10}u_{01}^{2} \\ &\quad + 3u_{12}u_{10}^{2}u_{01} - u_{03}u_{10}^{3}, \\ \mathcal{I}_{3,2}^{3} &= u_{30}u_{02}u_{01} - 2u_{21}u_{11}u_{01} - u_{21}u_{02}u_{10} \\ &\quad + u_{12}u_{20}u_{01} + 2u_{12}u_{11}u_{10} - u_{03}u_{20}u_{10}. \end{aligned} \tag{60}$$

One of them has weight four:

$$\mathcal{I}_{3,3}^{4} = u_{30}u_{12}u_{02} - u_{30}u_{03}u_{11} - u_{21}^{2}u_{02} + u_{21}u_{12}u_{11} + u_{21}u_{03}u_{20} - u_{12}^{2}u_{20}. \quad (61)$$

Another third-order affine differential invariant has weight six:

$$\mathcal{I}_{3,4}^{6} = u_{30}^{2}u_{03}^{2} - 6u_{30}u_{21}u_{12}u_{03} + 4u_{30}u_{12}^{3} - 3u_{21}^{2}u_{12}^{2} + 4u_{21}^{3}u_{03}. \quad (62)$$

There are five functional independent affine differential invariants of order four. One of them has weight six:

$$\mathcal{I}_{4,1}^{6} = u_{40}u_{22}u_{04} - u_{40}u_{13}^{2} + 2u_{31}u_{22}u_{13} - u_{31}^{2}u_{04} - u_{22}^{3}. \quad (63)$$

Four of them have weight four:

$$\begin{aligned}
\mathcal{I}_{4,2}^{4} &= u_{40}u_{04} - 4u_{31}u_{13} + 3u_{22}^{2},\\
\mathcal{I}_{4,3}^{4} &= u_{40}u_{01}^{4} - 4u_{31}u_{10}u_{01}^{3} + 6u_{22}u_{10}^{2}u_{01}^{2} - 4u_{13}u_{10}^{3}u_{01} + u_{04}u_{10}^{4},\\
\mathcal{I}_{4,4}^{4} &= u_{40}u_{02}^{2} - 4u_{31}u_{11}u_{02} + 2u_{22}u_{20}u_{02} + 4u_{22}u_{11}^{2} - 4u_{20}u_{13}u_{11} + u_{20}^{2}u_{04},\\
\mathcal{I}_{4,5}^{4} &= u_{40}u_{03}u_{01} - u_{31}\left(3u_{12}u_{01} + u_{03}u_{10}\right) + 3u_{22}\left(u_{21}u_{01} + u_{12}u_{10}\right) - u_{13}\left(u_{30}u_{01} + 3u_{21}u_{10}\right) + u_{04}u_{30}u_{10}.
\end{aligned} \quad (64)$$

Needless to say, the presented invariants all satisfy the properties (40). There are certainly other forms of invariants. The presented invariants are supposed to be the simplest ones.

6. Conclusion

We have presented an effective method to construct affine differential invariants of functions defined on the plane. The method is clear and simple. It is possible to generalize this method to construct differential invariants of functions defined on higher dimensional spaces. The proposed differential invariants may be applied in applications such as computer vision. These are the directions of future research.

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