Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 868035, 15 pages
http://dx.doi.org/10.1155/2013/868035

Research Article

A Finite Element Method for the Multiterm Time-Space Riesz Fractional Advection-Diffusion Equations in Finite Domain

Jingjun Zhao, Jingyu Xiao, and Yang Xu

Department of Mathematics, Harbin Institute of Technology, Harbin, Heilongjiang 150001, China

Correspondence should be addressed to Yang Xu; yangx@hit.edu.cn

Received 13 December 2012; Revised 28 February 2013; Accepted 28 February 2013

Academic Editor: Xinan Hao

We present an effective finite element method (FEM) for the multiterm time-space Riesz fractional advection-diffusion equations (MT-TS-RFADEs). We obtain the weak formulation of MT-TS-RFADEs and prove the existence and uniqueness of weak solution by the Lax-Milgram theorem. For multiterm time discretization, we use the Diethelm fractional backward finite difference method based on quadrature. For spatial discretization, we show the details of an FEM for such MT-TS-RFADEs. Then, stability and convergence of such numerical method are proved, and some numerical examples are given to match well with the main conclusions.

1. Introduction

Fractional differential equations are different from integer ones, in which the nature of the fractional derivative introduces the memory effect, thus increasing its modeling ability. Recently, many mathematical models with fractional derivatives have been successfully applied in biology, physics, chemistry, and biochemistry, hydrology, and finance [1–3]. The multiterm fractional differential equations have been widely studied in rheology, and, in many cases, the exact solutions are known [4, 5]. Summary of the fractional differential equations can be found in monographs [6–9]. As one of the main branch, fractional partial differential equations have attracted great attention. Therefore, the numerical treatment and supporting analysis of fractional order partial differential equations have become an important research topic that offers great potential.

The FEM is one of the effective numerical methods for solving traditional partial differential equations. For fractional partial differential equations, FEM also can be a useful and effective numerical method. In recent years, some valuable papers are concerned with the FEM for fractional differential equations. Adolfsson et al. [10, 11] considered an efficient numerical method to integrate the constitutive response of fractional order, viscoelasticity based on the FEM. Roop and Ervin [12–15] investigated the theoretical framework for the Galerkin finite element approximation to some kinds of fractional partial differential equations. Li et al. [16] considered numerical approximation of fractional differential equations with subdiffusion and superdiffusion by using difference method and finite element method. Li and Xu [17, 18] proposed a time-space spectral method for time and time-space fractional partial differential equation based on a weak formulation, and a detailed error analysis was carried out. Jiang and Ma [19] considered a high-order FEM for time fractional partial differential equations and proved the optimal order error estimates. Ford et al. [20] studied an FEM for time fractional partial differential equations.

Fractional advection-diffusion equations especially are important in describing and understanding the dispersion phenomena. Analytical solutions of such equations in finite domain have been obtained by Park in [21]. Also, the Riesz fractional advection-diffusion equations (RFADEs) with a symmetric fractional derivative (the Riesz fractional derivative) were derived from the kinetics of chaotic dynamics by [22] and summarized by [23]. Ciesielski and Leszczynski [24] presented a numerical solution for such equations based on the finite difference method.

One often sees RFADEs defined in terms of the fractional Laplacian as follows:

$$\frac{d}{dt}u(t,x) = -k_\alpha(-\Delta)^{\alpha/2}u(t,x) - k_\beta(-\Delta)^{\beta/2}u(t,x), \qquad (1)$$

for example, where u is a solute concentration, k_α and k_β represent the dispersion coefficient and the average fluid velocity. Here, the fractional Laplacian operator $-(-\Delta)^{\alpha/2}$ uses the Fourier transformation on an infinite domain, with a natural extension to include finite domains when the function $u(t,x)$ is subject to the zero Dirichlet boundary conditions (see [9]). Due to Lemma 1 in [25], the fractional Laplacian operator $-(-\Delta)^{\alpha/2}$ on an infinite domain $x \in (-\infty,\infty)$ is equivalent to the Riesz fractional derivative operator ${}^RD^{\alpha}_{|x|}$. In particular, the Riesz fractional derivatives include both the left and the right Riemann-Liouville derivatives that allow the modeling of flow regime impacts from either side of the domain. Yang et al. [25] investigated the numerical treatment for the RFADE with the Riesz space fractional derivative as

$$\frac{d}{dt}u(t,x) = k_\alpha\, {}^RD^{\alpha}_{|x|}u(t,x) + k_\beta\, {}^RD^{\beta}_{|x|}u(t,x), \quad (2)$$

where ${}^RD^{\alpha}_{|x|}$ is the Riesz space fractional operator defined in Section 2.

To increase the modeling ability, some authors considered the equations with the fractional order in both time and spatial variables in RFADEs, which include more information and hence are more interesting. For the time-space fractional advection-dispersion equations, Shen et al. [26] presented the fundamental solution and numerical solution of the Riesz fractional advection-dispersion equation with initial and boundary conditions on a bounded domain, and derived the stability and convergence of their proposed numerical methods. Then, for fractional advection-diffusion equations, Shen et al. [27] presented an explicit difference approximation and an implicit difference approximation for the time-space Riesz-Caputo fractional advection-diffusion equations with initial and boundary conditions on a finite domain.

All of the above papers only considered single-term fractional equations in time variable, where only one fractional differential operator appeared. In this paper, we consider the multiterm fractional differential equation, which includes more than one fractional derivative. For example, the so-called Bagley-Torvik equation [28] is

$$A\,{}^CD^2 y(t) + B\,{}^CD^{3/2} y(t) + Cy(t) = f(t), \quad (3)$$

where A, B, and C are certain constants and f is a given function. The Basset equation is, in [6],

$$D^1 y(t) + bD^\alpha y(t) + cy(t) = f(t), \quad y(0) = y_0, \quad (4)$$

where $0 < \alpha < 1$, b, and c are positive real numbers. This equation describes the forces that occur when a spherical object sinks in an incompressible viscous fluid.

Recently, some authors considered the applications of multiterm fractional differential equations [29] and the numerical methods for such equations [30–32]. At the same time, the multiterm fractional partial differential equations have been proposed in [33, 34]. The analytical solution and the numerical methods for multiterm time fractional wave-diffusion equations have been investigated in [35, 36]. This motivates us to consider the effective numerical solution for such multiterm fractional partial differential equations.

In this paper, we consider MT-TS-RFADEs in finite domain with the zero Dirichlet boundary conditions. The analytical solution of such MT-TS-RFADEs has been investigated by Jiang et al. in [37]. Here, we present an FEM for a simplified MT-TS-RFADEs and obtain the optimal order error estimates both in semidiscrete and fully discrete cases and derive the stability of such FEM. As far as we are aware, there are few research papers in the published literature written on this topic.

This paper is organized as follows. In Section 2, the preliminaries of the fractional calculation are shown. Then, we give the weak formulation of MT-TS-RFADEs and prove the existence and uniqueness of this problems by the well-known Lax-Milgram theorem. In Section 3, we present the convergence rate of Diethelm's fractional backward difference method (see [38, 39]) for time discretization. In Section 4, we propose a finite element method based on the weak formulations and carry out the error analysis. In Section 5, we prove the stability of such FEM for MT-TS-RFADEs. Finally, some numerical examples are considered in Section 6.

2. Existence and Uniqueness

We consider the MT-TS-RFADEs with multiterm time fractional derivatives and the Riesz space fractional derivatives in the following form:

$$P({}_0D_t)\,u(t,x) = k_\beta\, {}^RD^{\beta}_{|x|}u(t,x) + k_\gamma\, {}^RD^{\gamma}_{|x|}u(t,x) + f(t,x), \quad (5)$$

where $0 < \beta < 1$, $1 < \gamma < 2$, $x \in [0,X]$, and $t \in [0,T]$ are respectively the space and time variables and k_β, k_γ are positive constants. We consider this problem with the zero Dirichlet boundary value conditions and the initial value condition defined as follows:

$$\begin{aligned} u(t,0) &= u(t,X) = 0, \quad t \in [0,T], \\ u(0,x) &= \phi(x), \quad x \in (0,X). \end{aligned} \quad (6)$$

For nonzero boundary value conditions, we need to transform the problem into one with zero boundary value conditions before using the method in this paper.

Here, we consider the multiterm time fractional differential operator which has the subdiffusion process (see [16]). It is different from (2), which only has one integer order differential operator in time.

Note that the analytical solutions for MT-TS-RFADEs have been studied in [37], in which this problem is well defined. For this problem, some new techniques have been used, such as a spectral representation of the fractional Laplacian operator and the equivalent relationship between the fractional Laplacian operator and the Riesz fractional derivative.

For convenience, we introduce the following definitions and properties. The space derivatives ${}^RD^{\beta}_{|x|}u(t,x)$ and ${}^RD^{\gamma}_{|x|}u(t,x)$ are the Riesz space fractional derivatives of order β and γ, respectively. The definitions of them can be found in [40].

Let $\Gamma(\cdot)$ denote the gamma function. For any positive integer n and real number θ $(n-1<\theta<n)$, there are different definitions of fractional derivatives with order θ in [8]. During this paper, we consider the left, (right) Caputo derivative and left (right) Riemann-Liouville derivative defined as follows:

(i) the left Caputo derivative:

$$ {}_0^C D_t^\theta v(t) = \frac{1}{\Gamma(n-\theta)} \int_0^t \frac{1}{(t-\tau)^{\theta-n+1}} \left(\frac{d^n}{d\tau^n} v(\tau) \right) d\tau, \quad (7) $$

(ii) the right Caputo derivative:

$$ {}_t^C D_T^\theta v(t) = \frac{(-1)^n}{\Gamma(n-\theta)} \int_t^T \frac{1}{(\tau-t)^{\theta-n+1}} \left(\frac{d^n}{d\tau^n} v(\tau) \right) d\tau, \quad (8) $$

(iii) the left Riemann-Liouville derivative:

$$ {}_0^R D_t^\theta v(t) = \frac{1}{\Gamma(n-\theta)} \frac{d^n}{dt^n} \int_0^t \frac{v(\tau)}{(t-\tau)^{\theta-n+1}} d\tau, \quad (9) $$

(iv) the right Riemann-Liouville derivative:

$$ {}_t^R D_T^\theta v(t) = \frac{(-1)^n}{\Gamma(n-\theta)} \frac{d^n}{dt^n} \int_t^T \frac{v(\tau)}{(\tau-t)^{\theta-n+1}} d\tau. \quad (10) $$

The Riesz fractional operators ${}^R D_{|x|}^\beta$ and ${}^R D_{|x|}^\gamma$ in (5) can be defined by the left and the right Riemann-Liouville fractional derivatives.

Definition 1 (see [40]). The Riesz fractional derivatives of order θ for $n \in \mathbf{N}$, $n-1<\theta<n$, on a finite interval $0 \le x \le X$ is defined as

$$ \begin{aligned} &{}^R D_{|x|}^\theta v(t,x) \\ &= -C_\theta \left({}_0^R D_x^\theta v(t,x) + {}_x^R D_X^\theta v(t,x) \right), \quad x \in [0,X], \end{aligned} \quad (11) $$

where $C_\theta = 1/(2\cos(\pi\theta/2))$.

The multiterm fractional operator $P({}_0D_t)$ in (5) is

$$ P({}_0D_t) u(t,x) = \left({}_0^C D_t^\alpha + \sum_{i=1}^s a_i \, {}_0^C D_t^{\alpha_i} \right) u(t,x), \quad (12) $$

where $0<\alpha_s<\alpha_{s-1}<\cdots<\alpha_1<\alpha<1$ or $1<\alpha_s<\alpha_{s-1}<\cdots<\alpha_1<\alpha<2$ or $0<\alpha_s<\cdots<\alpha_{s_0}<1<\alpha_{s_0-1}<\cdots<\alpha_1<\alpha<2$, and $a_i \in \mathbf{R}$, $i=1,\ldots,s$, $s \in \mathbf{N}$. Here, ${}_0^C D_t^\alpha$ and ${}_0^C D_t^{\alpha_i}$ $(i=1,\ldots,s)$ denote the left Caputo fractional derivatives with respect to the time variable t of order $0<\alpha<1$ and $0<\alpha_s<\cdots<\alpha_1<1$. There are three cases for multiterm time-space fractional derivatives.

Case 1. If $0<\alpha_s<\cdots<\alpha_1<\alpha<1$, (5) is a generalized MT-TS-RFADE with multiterm time fractional diffusion terms with initial conditions given as (6). And especially, if $\alpha_i=0$, $i=1,\ldots,s$ and $\alpha=1$, then (5) becomes a space fractional advection-diffusion equation with the Riesz space fractional derivatives, which was discussed by Yang et al. [25].

Case 2. If $1<\alpha_s<\cdots<\alpha_1<\alpha<2$, (5) is a generalized MT-TS-RFADE with multiterm time fractional wave terms. In this case, the initial conditions are given as follows:

$$ u(0,x) = \phi(x), \quad u_t(0,x) = \psi(x), \quad x \in (0,X). \quad (13) $$

Case 3. If $0<\alpha_s<\cdots<\alpha_{s_0}<1<\alpha_{s_0-1}<\cdots<\alpha_1<\alpha<2$, (5) becomes a generalized MT-TS-RFADE, which we refer to as a multiterm time-space fractional mixed wave-diffusion equation. In this case, the initial conditions are also given by (13).

In this paper, we just consider Case 1 of (5) with $0<\alpha_s<\cdots<\alpha_1<\alpha<1$. Another two cases with $1<\alpha_s<\cdots<\alpha_1<\alpha<2$ and $0<\alpha_s<\cdots<\alpha_{s_0}<1<\alpha_{s_0-1}<\cdots<\alpha_1<\alpha<2$ will be studied in our following work.

In order to establish the weak formulation of the problem (5), we need some preparatory work. We use definitions of functional spaces and derive some properties related to these spaces. Let $C^\infty(0,T)$ denote the space of infinitely differentiable functions on $(0,T)$, and let $C_0^\infty(0,T)$ denote the space of infinitely differentiable functions with compact support in $(0,T)$. Let $L_2(\mathbb{Q})$ be the space of measurable functions whose square is the Lebesgue integrable in $\mathbb{Q}$, which may denote a domain $\mathbb{Q} = I$ or Ω or $I \times \Omega$, where $I=[0,T]$ denotes the time domain and $\Omega=[0,X]$ denotes the space domain. The inner product and norm of $L_2(\mathbb{Q})$ are defined by

$$ \begin{gathered} (u,v)_{L_2(\mathbb{Q})} = \int_{\mathbb{Q}} uv \, d\mathbb{Q}, \\ \|u\|_{L_2(\mathbb{Q})} = (u,u)_{L_2(\mathbb{Q})}^{1/2}, \\ \forall u, v \in L_2(\mathbb{Q}). \end{gathered} \quad (14) $$

For any real $\sigma>0$, we define the spaces ${}^l H_0^\sigma(\mathbb{Q})$ and ${}^r H_0^\sigma(\mathbb{Q})$ to be the closure of $C_0^\infty(\mathbb{Q})$ with respect to the norms $\|v\|_{{}^l H_0^\sigma(\mathbb{Q})}$ and $\|v\|_{{}^r H_0^\sigma(\mathbb{Q})}$, respectively, where

$$ \begin{gathered} \|v\|_{{}^l H_0^\sigma(\mathbb{Q})} := \left(\|v\|_{L_2(\mathbb{Q})}^2 + |v|_{{}^l H_0^\sigma(\mathbb{Q})}^2 \right)^{1/2}, \\ |v|_{{}^l H_0^\sigma(\mathbb{Q})}^2 := \left\| {}_0^R D_t^\sigma v \right\|_{L_2(\mathbb{Q})}^2, \\ \|v\|_{{}^r H_0^\sigma(\mathbb{Q})} := \left(\|v\|_{L_2(\mathbb{Q})}^2 + |v|_{{}^r H_0^\sigma(\mathbb{Q})}^2 \right)^{1/2}, \\ |v|_{{}^r H_0^\sigma(\mathbb{Q})}^2 := \left\| {}_t^R D_T^\sigma v \right\|_{L_2(\mathbb{Q})}^2. \end{gathered} \quad (15) $$

In the usual Sobolev space $H_0^\sigma(\mathbb{Q})$, we also have the definition

$$ \begin{gathered} \|v\|_{H_0^\sigma(\mathbb{Q})} := \left(\|v\|_{L_2(\mathbb{Q})}^2 + |v|_{H_0^\sigma(\mathbb{Q})}^2 \right)^{1/2}, \\ |v|_{H_0^\sigma(\mathbb{Q})}^2 := \left(\frac{\left({}_0^R D_t^\sigma v, {}_t^R D_T^\sigma v \right)_{L_2(\mathbb{Q})}}{\cos(\pi\sigma)} \right). \end{gathered} \quad (16) $$

From [18], for $\sigma>0$, $\sigma \ne n-1/2$, the spaces ${}^l H_0^\sigma(\mathbb{Q})$, ${}^r H_0^\sigma(\mathbb{Q})$, and $H_0^\sigma(\mathbb{Q})$ are equal, and their seminorms are all equivalent to $|\cdot|_{H_0^\sigma(\mathbb{Q})}$.

We now give some results for fractional operators on these spaces.

Lemma 2 (see [8, 18]). (1) *For real* $0 < \theta < 1,\ 0 < \delta < 1$, *if* $v(0) = 0,\ x \in (0, X)$, *then*

$$
\begin{aligned}
{}_0^R D_x^{\theta+\delta} v(x) &= \left({}_0^R D_x^{\theta}\right)\left({}_0^R D_x^{\delta}\right) v(x) \\
&= \left({}_0^R D_x^{\delta}\right)\left({}_0^R D_x^{\theta}\right) v(x), \quad \forall v \in H^{\theta+\delta}(0, X).
\end{aligned}
\tag{17}
$$

(2) *Let* $0 < \theta < 1$. *Then, one has*

$$
\begin{aligned}
&\left({}_0^R D_x^{\theta} w, v\right)_{L_2(0,X)} = \left(w, {}_x^R D_X^{\theta} v\right)_{L_2(0,X)}, \\
&\qquad \forall w \in H^{\theta}(0, X),\ v \in C_0^{\infty}(0, X).
\end{aligned}
\tag{18}
$$

Lemma 3 (see [18]). *Let* $0 < \theta < 2,\ \theta \neq 1$. *Then, for any* $w, v \in H_0^{\theta/2}(0, X)$, *then*

$$
\left({}_0^R D_x^{\theta} w, v\right)_{L_2(0,X)} = \left({}_0^R D_x^{\theta/2} w, {}_t^R D_X^{\theta/2} v\right)_{L_2(0,X)}. \tag{19}
$$

Following from the definitions and lemmas above, we define the space

$$
\begin{aligned}
B^{\alpha/2,\gamma/2}(I \times \Omega) := \;& H^{\alpha/2}\left(I, L_2(\Omega)\right) \cap H^{\alpha_1/2}\left(I, L_2(\Omega)\right) \\
& \cap \cdots \cap H^{\alpha_s/2}\left(I, L_2(\Omega)\right) \cap L_2\left(I, H_0^{\beta/2}(\Omega)\right) \\
& \cap L_2\left(I, H_0^{\gamma/2}(\Omega)\right),
\end{aligned}
\tag{20}
$$

when $0 < \alpha_s < \cdots < \alpha_1 < \alpha < 1,\ 0 < \beta < 1$ and $1 < \gamma < 2$.

Then, one obtains

$$
B^{\alpha/2,\gamma/2}(I \times \Omega) = H^{\alpha/2}\left(I, L_2(\Omega)\right) \cap L_2\left(I, H_0^{\gamma/2}(\Omega)\right). \tag{21}
$$

Here, $B^{\alpha/2,\gamma/2}(I \times \Omega)$ is a Banach space with respect to the following norm:

$$
\begin{aligned}
\|v\|_{B^{\alpha/2,\gamma/2}(I\times\Omega)} :&= \left(\|v\|^2_{H^{\alpha/2}(I,L_2(\Omega))} + \|v\|^2_{L_2\left(I,H_0^{\gamma/2}(\Omega)\right)}\right)^{1/2}, \\
&\qquad \forall v \in B^{\alpha/2,\gamma/2}(I \times \Omega),
\end{aligned}
\tag{22}
$$

where

$$
H^{\alpha/2}\left(I, L_2(\Omega)\right) := \left\{v; \|v(t,\cdot)\|_{L_2(\Omega)} \in H^{\alpha/2}(I)\right\}, \tag{23}
$$

endowed with the norm

$$
\|v\|_{H^{\alpha/2}(I,L_2(\Omega))} := \left\| \|v(t,\cdot)\|_{L_2(\Omega)} \right\|_{H^{\alpha/2}(I)}. \tag{24}
$$

For obtaining a suitable weak solution for problem (5) with the Caputo time fractional derivation, we consider the connection between the Caputo and the Riemann-Liouville fractional definition. Based on the definitions of the Caputo and the Riemann-Liouville fractional differential operators, we have an immediate consequence, for any real order $\theta > 0$, in [38],

$$
\begin{aligned}
{}_0^C D_t^{\theta} f(t) &= \frac{1}{\Gamma(-\theta)} \int_0^t (t-s)^{-\theta-1} \left(f(s) - T_{n-1}[f;0](s)\right) ds \\
&= {}_0^R D_t^{\theta} f(t) - {}_0^R D_t^{\theta} T_{n-1}[f;0](t),
\end{aligned}
\tag{25}
$$

where $T_{n-1}[f;0](t)$ denotes the Taylor polynomial for f of order $n-1$, centered at 0,

$$
T_{n-1}[f;0](t) = \sum_{k=0}^{n} \frac{t^k}{k!} \frac{d^k}{dt^k} f(0), \tag{26}
$$

where $f(t) \in C^{n-1}[0, T]$.

Based on (25), we make

$$
\overline{f}(t, x) = f(t, x) + \frac{u_0(x)\, t^{-\alpha}}{\Gamma(1-\alpha)} + \sum_{i=1}^{s} a_i \frac{u_0(x)\, t^{-\alpha_i}}{\Gamma(1-\alpha_i)}. \tag{27}
$$

Therefore, that led to the following weak formulation of (5). Let $B^{\alpha/2,\gamma/2}(I \times \Omega)'$ be the dual space of $B^{\alpha/2,\gamma/2}(I \times \Omega)$. For $\overline{f} \in B^{\alpha/2,\gamma/2}(I \times \Omega)'$, we find $u(t, x) \in B^{\alpha/2,\gamma/2}(I \times \Omega)$ such that

$$
\mathscr{A}(u, v) = \mathscr{F}(v), \quad v \in B^{\alpha/2,\gamma/2}(I \times \Omega), \tag{28}
$$

where the bilinear form $\mathscr{A}(\cdot,\cdot)$ is, based on Lemmas 2 and 3,

$$
\begin{aligned}
\mathscr{A}(u, v) := \;& \left({}_0^R D_t^{\alpha/2} u, {}_t^R D_T^{\alpha/2} v\right)_{L_2(\mathscr{Q})} \\
& + \sum_{i=1}^{s} a_i \left({}_0^R D_t^{\alpha_i/2} u, {}_t^R D_T^{\alpha_i/2} v\right)_{L_2(\mathscr{Q})} \\
& + k_\beta C_\beta \Big[\left({}_0^R D_x^{\beta/2} u, {}_x^R D_X^{\beta/2} v\right)_{L_2(\mathscr{Q})} \\
& \qquad + \left({}_x^R D_X^{\beta/2} u, {}_0^R D_x^{\beta/2} v\right)_{L_2(\mathscr{Q})}\Big] \\
& + k_\gamma C_\gamma \Big[\left({}_0^R D_x^{\gamma/2} u, {}_x^R D_X^{\gamma/2} v\right)_{L_2(\mathscr{Q})} \\
& \qquad + \left({}_x^R D_X^{\gamma/2} u, {}_0^R D_x^{\gamma/2} v\right)_{L_2(\mathscr{Q})}\Big],
\end{aligned}
\tag{29}
$$

and the functional $\mathscr{F}(\cdot)$ is given by

$$
\mathscr{F}(v) := \left\langle \overline{f}, v \right\rangle_{L_2(\mathscr{Q})'}. \tag{30}
$$

Lemma 4 (see [17]). *For real* $\theta > 0$, $v \in C_0^{\infty}(\mathbf{R})$, *then*

$$
\begin{aligned}
\left({}_{-\infty}^R D_t^{\theta} v(t), {}_t^R D_{\infty}^{\theta} v(t)\right)_{L_2(\mathbf{R})} &= \cos(\pi\theta) \left\|{}_{-\infty}^R D_t^{\theta} v(t)\right\|^2_{L_2(\mathbf{R})}, \\
\left({}_{-\infty}^R D_t^{\theta} v(t), {}_t^R D_{\infty}^{\theta} v(t)\right)_{L_2(\mathbf{R})} &= \cos(\pi\theta) \left\|{}_t^R D_{\infty}^{\theta} v(t)\right\|^2_{L_2(\mathbf{R})}.
\end{aligned}
\tag{31}
$$

Based on Lemma 4, we can prove the following existence and uniqueness theorem. During this paper, we use the expression $A \lesssim B$ ($A \gtrsim B$) to mean that there exists a positive real number c such that $A \leq cB$ ($A \geq cB$). At the same time, we denote $A \cong B$ to mean that $A \lesssim B \lesssim A$, which means there exist positive real numbers c_1, c_2 such that $A \leq c_1 B$ and $B \leq c_2 A$ (i.e., $(1/c_1)A \leq B \leq c_2 A$).

Theorem 5. *Assume that* $0 < \alpha_1 < \cdots < \alpha_s < \alpha < 1,\ 0 < \beta < 1,\ 1 < \gamma < 2$ *and* $f \in B^{\alpha/2,\gamma/2}(I \times \Omega)'$. *Then, system* (28) *has a unique solution in* $B^{\alpha/2,\gamma/2}(I \times \Omega)$. *Furthermore,*

$$\|u\|_{B^{\alpha/2,\gamma/2}(I\times\Omega)} \lesssim \left\|\overline{f}\right\|_{B^{\alpha/2,\gamma/2}(I\times\Omega)'}. \quad (32)$$

Proof. The existence and uniqueness of the solution is guaranteed by the well-known Lax-Milgram theorem. First, from the equivalence of ${}^{l}H_0^{\alpha}(I\times\Omega)$, ${}^{r}H_0^{\alpha}(I\times\Omega)$, and $H_0^{\alpha}(I\times\Omega)$, for all $u, v \in B^{\alpha/2,\gamma/2}(I\times\Omega)$, it follows that

$$\begin{aligned}
|\mathscr{A}(u,v)| = \Big| & \left({}_0^R D_t^{\alpha/2} u, {}_t^R D_T^{\alpha/2} v\right)_{L_2(\mathscr{Q})} \\
& + \sum_{i=1}^{s} a_i \left({}_0^R D_t^{\alpha_i/2} u, {}_t^R D_T^{\alpha_i/2} v\right)_{L_2(\mathscr{Q})} \\
& + k_\beta C_\beta \Big[\left({}_0^R D_x^{\beta/2} u, {}_x^R D_X^{\beta/2} v\right)_{L_2(\mathscr{Q})} \\
& \qquad + \left({}_x^R D_X^{\beta/2} u, {}_0^R D_x^{\beta/2} v\right)_{L_2(\mathscr{Q})} \Big] \\
& + k_\gamma C_\gamma \Big[\left({}_0^R D_x^{\gamma/2} u, {}_x^R D_X^{\gamma/2} v\right)_{L_2(\mathscr{Q})} \\
& \qquad + \left({}_x^R D_X^{\gamma/2} u, {}_0^R D_x^{\gamma/2} v\right)_{L_2(\mathscr{Q})} \Big] \Big| \\
\lesssim & \left\|{}_0^R D_t^{\alpha/2} u\right\|_{L_2(\mathscr{Q})} \left\|{}_0^R D_t^{\alpha/2} v\right\|_{L_2(\mathscr{Q})} \\
& + \sum_{i=1}^{s} a_i \left\|{}_0^R D_t^{\alpha_i/2} u\right\|_{L_2(\mathscr{Q})} \left\|{}_0^R D_t^{\alpha_i/2} v\right\|_{L_2(\mathscr{Q})} \\
& + k_\beta C_\beta \Big[\left\|{}_0^R D_x^{\beta/2} u\right\|_{L^2(\mathscr{Q})} \left\|{}_x^R D_X^{\beta/2} v\right\|_{L^2(\mathscr{Q})} \\
& \qquad + \left\|{}_x^R D_X^{\beta/2} u\right\|_{L^2(\mathscr{Q})} \left\|{}_0^R D_x^{\beta/2} v\right\|_{L^2(\mathscr{Q})} \Big] \\
& + k_\gamma C_\gamma \Big[\left\|{}_0^R D_x^{\gamma/2} u\right\|_{L^2(\mathscr{Q})} \left\|{}_x^R D_X^{\gamma/2} v\right\|_{L^2(\mathscr{Q})} \\
& \qquad + \left\|{}_x^R D_X^{\gamma/2} u\right\|_{L^2(\mathscr{Q})} \left\|{}_0^R D_x^{\gamma/2} v\right\|_{L^2(\mathscr{Q})} \Big] \\
\lesssim & \|u\|_{H^{\alpha/2}(I,L_2(\Omega))} \|v\|_{H^{\alpha/2}(I,L_2(\Omega))} \\
& + \sum_{i=1}^{s} a_i \|u\|_{H^{\alpha_i/2}(I,L_2(\Omega))} \|v\|_{H^{\alpha_i/2}(I,L_2(\Omega))} \\
& + k_\beta \|u\|_{L^2(I,H^{\beta/2}(\Omega))} \|v\|_{L^2(I,H^{\beta/2}(\Omega))} \\
& + k_\gamma \|u\|_{L^2(I,H^{\gamma/2}(\Omega))} \|v\|_{L^2(I,H^{\gamma/2}(\Omega))} \\
\lesssim & \|u\|_{B^{\alpha/2,\gamma/2}(I\times\Omega)} \|v\|_{B^{\alpha/2,\gamma/2}(I\times\Omega)},
\end{aligned} \quad (33)$$

$$|\mathscr{F}(v)| \lesssim \left\|\overline{f}\right\|_{B^{\alpha/2,\gamma/2}(I\times\Omega)'} \|v\|_{B^{\alpha/2,\gamma/2}(I\times\Omega)}. \quad (34)$$

This implies the continuity of the bilinear form $\mathscr{A}(\cdot,\cdot)$ and the right-hand side function $\mathscr{F}(v)$.

We next prove the coercivity of the bilinear operator $\mathscr{A}(\cdot,\cdot)$. Note that

$$\begin{aligned}
\left({}_0^R D_t^{\alpha/2}\varphi, {}_t^R D_T^{\alpha/2}\varphi\right)_{L_2(\mathscr{Q})} &= \left({}_{-\infty}^R D_t^{\alpha/2}\widetilde{\varphi}, {}_t^R D_\infty^{\alpha/2}\widetilde{\varphi}\right)_{L_2(\mathbf{R})} \\
&= \cos\left(\frac{\pi\alpha}{2}\right) \cdot \left\|{}_{-\infty}^R D_t^{\alpha/2}\widetilde{\varphi}\right\|_{L_2(\mathbf{R})} \\
&= \cos\left(\frac{\pi\alpha}{2}\right) \cdot \left\|{}_0^R D_t^{\alpha/2}\varphi\right\|_{L_2(\mathscr{Q})},
\end{aligned} \quad (35)$$

for all $\varphi \in C_0^{\infty}(\mathscr{Q})$, where $\widetilde{\varphi}$ is the extension of φ by zero outside of $(0,T)$. Thus, we find that $({}_0^R D_t^{\alpha/2} v, {}_t^R D_T^{\alpha/2} v)_{L_2(\mathscr{Q})}$ is nonnegative for $v \in H^{\alpha/2}(0,T)$, $0 < \alpha < 1$ since $\cos(\pi\alpha/2)$ is nonnegative for $0 < \alpha < 1$. That is the same for fractional operators with α_i, $i = 1, \ldots, s$.

From the above analysis, we have

$$\begin{aligned}
\mathscr{A}(v,v) = & \left({}_0^R D_t^{\alpha/2} v, {}_t^R D_T^{\alpha/2} v\right)_{L_2(\mathscr{Q})} \\
& + \sum_{i=1}^{s} a_i \left({}_0^R D_t^{\alpha_i/2} v, {}_t^R D_T^{\alpha_i/2} v\right)_{L_2(\mathscr{Q})} \\
& + 2k_\beta C_\beta \left({}_0^R D_x^{\beta/2} v, {}_x^R D_X^{\beta/2} v\right)_{L_2(\mathscr{Q})} \\
& + 2k_\gamma C_\gamma \left({}_0^R D_x^{\gamma/2} v, {}_x^R D_X^{\gamma/2} v\right)_{L_2(\mathscr{Q})} \\
\cong & \cos\left(\frac{\pi\alpha}{2}\right) \left({}_0^R D_t^{\alpha/2} v, {}_0^R D_t^{\alpha/2} v\right)_{L_2(\mathscr{Q})} \\
& + \sum_{i=1}^{s} a_i \cos\left(\frac{\pi\alpha_i}{2}\right) \left({}_0^R D_t^{\alpha_i/2} v, {}_0^R D_t^{\alpha_i/2} v\right)_{L_2(\mathscr{Q})} \\
& + k_\beta \left({}_0^R D_x^{\beta/2} v, {}_0^R D_x^{\beta/2} v\right)_{L_2(\mathscr{Q})} \\
& + k_\gamma \left({}_0^R D_x^{\gamma/2} v, {}_0^R D_x^{\gamma/2} v\right)_{L_2(\mathscr{Q})} \\
\gtrsim & \|v\|^2_{B^{\alpha/2,\gamma/2}(I\times\Omega)}.
\end{aligned} \quad (36)$$

By using the well-known Lax-Milgram theorem, there exists a unique solution $u \in B^{\alpha/2,\gamma/2}(I\times\Omega)$ such that (28) holds.

To prove the stability estimate (32), by using (34) and (36), we take $v = u$ in (28) and obtain

$$\begin{aligned}
\|u\|^2_{B^{\alpha/2,\gamma/2}(I\times\Omega)} &\lesssim \mathscr{A}(u,u) \\
&= \mathscr{F}(u) \lesssim \left\|\overline{f}\right\|_{B^{\alpha/2,\gamma/2}(I\times\Omega)'} \|u\|_{B^{\alpha/2,\gamma/2}(I\times\Omega)},
\end{aligned} \quad (37)$$

which implies that $\| u\|_{B^{\alpha/2,\gamma/2}(I\times\Omega)} \lesssim \| \overline{f}\|_{B^{\alpha/2,\gamma/2}(I\times\Omega)'}$. □

3. Time Discretization

In this section, we consider the Diethelm fractional backward difference method based on quadrature, which was independently introduced by [39], for ordinary fractional differential equations. Here, we consider this method for the time discretization of (5) and derive the convergence rate for the time-discretization of MT-TS-RFADEs.

Lemma 6 (see [25]). *For $n-1<\theta<n$ $(n=1,2,\ldots)$ and a function $u(x)$ defined on an unbounded domain $(-\infty<x<\infty)$, the following equality holds:*

$$\begin{aligned} {}^{R}D^{\theta}_{|x|}u(x) &= -\frac{1}{2\cos(\pi\theta/2)}\left({}^{R}_{-\infty}D^{\theta}_{x}u + {}^{R}_{x}D^{\theta}_{\infty}u\right) \\ &= -(-\Delta)^{\theta/2}u(x), \end{aligned} \tag{38}$$

where Δ is the Laplacian.

Definition 7 (see [26]). Suppose the Laplacian has a complete set of orthonormal eigenfunctions ϕ_n corresponding to eigenvalues λ_n^2 on a bounded region $\mathcal{Q}$, that is, $-\Delta\phi_n=\lambda_n^2\phi_n$ on a bounded region $\mathcal{Q}$; $B(\phi)=0$ on $\partial\mathcal{Q}$ is one of the standard three homogeneous boundary conditions. Let

$$\mathbf{F}=\left\{f=\sum_{n=1}^{\infty}c_n\phi_n,\ c_n=\langle f,\phi_n\rangle,\ \sum_{n=1}^{\infty}|c_n|^2|\lambda_n|^{\theta}\leq\infty\right\}, \tag{39}$$

then for any $f\in\mathbf{F}$, $(-\Delta)^{\theta/2}$ is defined by

$$(-\Delta)^{\theta/2}f=\sum_{n=1}^{\infty}c_n\left(\lambda_n^2\right)^{\theta/2}\phi_n. \tag{40}$$

Here, for any $t\in I$, we make a continuous prolongation in space for function $u(t,x)$ as follows:

$$\tilde{u}(t,x)=\begin{cases}u(t,x), & x\in[0,X],\\ 0, & x\in(-\infty,0)\cup(X,\infty).\end{cases} \tag{41}$$

Let $u(t)$ denote the one-variable function of $\tilde{u}(t,x)$, and let $f(t)$ and $u(0)$ denote the one-variable functions as $f(t,\cdot)$ and $\tilde{u}(0,\cdot)$, respectively. By the definition (12) of $P({}_0D_t)$ and Lemma 6, (5) is equivalent to

$$\begin{aligned} &{}^{C}_{0}D^{\alpha}_{t}u(t)+\sum_{i=1}^{s}a_i\,{}^{C}_{0}D^{\alpha_i}_{t}u(t)+k_\beta(-\Delta)^{\beta/2}u(t) \\ &\quad+k_\gamma(-\Delta)^{\gamma/2}u(t)=f(t). \end{aligned} \tag{42}$$

Let $A=-\Delta$, $D(A)=H_0^1(\Omega)\cap H^2(\Omega)$. Then, the system (5) and the initial condition (6) can be rewritten in the abstract form, for $0\leq t\leq T$,

$$\begin{aligned} &{}^{C}_{0}D^{\alpha}_{t}u(t)+\sum_{i=1}^{s}a_i\,{}^{C}_{0}D^{\alpha_i}_{t}u(t)+k_\beta A^{\beta/2}u(t) \\ &\quad+k_\gamma A^{\gamma/2}u(t)=f(t), \end{aligned} \tag{43}$$

$$u(0)=\phi_0. \tag{44}$$

Based on (25), (43) can be rewritten as

$$\begin{aligned} &{}^{R}_{0}D^{\alpha}_{t}[u-\phi_0](t)+\sum_{i=1}^{s}a_i\,{}^{R}_{0}D^{\alpha_i}_{t}[u-\phi_0](t) \\ &\quad+k_\beta A^{\beta/2}u(t)+k_\gamma A^{\gamma/2}u(t)=f(t), \end{aligned} \tag{45}$$

where $0<t<T$, $0<\alpha_s<\cdots<\alpha_1<\alpha<1$.

Let $0=t_0<t_1<\cdots<t_N=T$ be a partition of $[0,T]$. Without loss of generality, for $0<\alpha<1$ and fix $t_n, n=1,2,\ldots,N$, we have

$${}^{R}_{0}D^{\alpha}_{t}[u-\phi_0](t_j)=\frac{t_j^{-\alpha}}{\Gamma(-\alpha)}\int_0^1 g(\theta)\,\theta^{-1-\alpha}\,d\theta, \tag{46}$$

where $g(\theta)=u(t_j-t_j\theta)-\phi_0$. Here, the integral is a Hadamard finite-part integral, in [38, 39].

Now, for every n, we replace the integral by a first-degree compound quadrature formula with equispaced nodes $0, 1/n, 2/n, \ldots, 1$ and obtain

$$\int_0^1 g(\theta)\,\theta^{-1-\alpha}d\theta=\sum_{k=0}^{n}\alpha_{kn}^{(\alpha)}g\left(\frac{k}{n}\right)+R_n^{(\alpha)}(g), \quad 0<\alpha<1, \tag{47}$$

where the weights $\alpha_{kn}^{(\alpha)}$ are

$$\alpha(1-\alpha)n^{-\alpha}\alpha_{kn}^{(\alpha)}=\begin{cases}-1, & k=0,\\ 2k^{1-\alpha}-(k-1)^{1-\alpha}-(k+1)^{1-\alpha}, & \\ \qquad\text{for } k=1,2,\ldots,n-1, & \\ (\alpha-1)k^{-\alpha}-(k-1)^{1-\alpha}+k^{1-\alpha}, & \\ \qquad\text{for } k=n, & \end{cases} \tag{48}$$

and the remainder term $R_n^{(\alpha)}(g)$ satisfies

$$\left\|R_n^{(\alpha)}(g)\right\|\leq\gamma_\alpha n^{\alpha-2}\sup_{0\leq t\leq T}\left\|g''(t)\right\|, \tag{49}$$

where $\gamma_\alpha>0$ is a constant.

Thus, we have

$$\begin{aligned} &{}^{R}_{0}D^{\alpha}_{t}[u-\phi_0](t_n) \\ &=\frac{t_n^{-\alpha}}{\Gamma(-\alpha)}\left(\sum_{k=0}^{n}\alpha_{kn}^{(\alpha)}\left(u(t_n-t_k)-\phi_0\right)+R_n^{(\alpha)}(g)\right) \\ &=\Delta t^{-\alpha}\sum_{k=0}^{n}\omega_{kn}^{(\alpha)}\left(u(t_n-t_k)-\phi_0\right)+\frac{t_n^{-\alpha}}{\Gamma(-\alpha)}R_n^{(\alpha)}(g), \end{aligned} \tag{50}$$

where $\omega_{kn}^{(\alpha)}=n^{-\alpha}\alpha_{kn}^{(\alpha)}/\Gamma(-\alpha)$.

Let $t=t_n$, we can write (45) as

$$\begin{aligned} &\Delta t^{-\alpha}\sum_{k=0}^{n}\omega_{kn}^{(\alpha)}\left(u(t_n-t_k)-\phi_0\right) \\ &\quad+\sum_{i=1}^{s}a_i\Delta t^{-\alpha_i}\sum_{k=0}^{n}\omega_{kn}^{(\alpha_i)}\left(u(t_n-t_k)-\phi_0\right) \\ &\quad+k_\beta A^{\beta/2}u(t_n)+k_\gamma A^{\gamma/2}u(t_n) \\ &=f(t_n)-\frac{t_n^{-\alpha}}{\Gamma(-\alpha)}R_n^{(\alpha)}(g) \\ &\quad-\sum_{i=1}^{s}a_i\frac{t_n^{-\alpha_i}}{\Gamma(-\alpha_i)}R_n^{(\alpha_i)}(g), \end{aligned} \tag{51}$$

for $n=1,2,3,\ldots$, where $\omega_{kn}^{(\alpha_i)}=n^{-\alpha_i}\alpha_{kn}^{(\alpha_i)}/\Gamma(-\alpha_i)$.

Denote U^n as the approximation of $u(t_n)$. We can define the following time stepping method:

$$\Delta t^{-\alpha}\sum_{k=0}^{n}\omega_{kn}^{(\alpha)}\left(U^{n-k}-U^{0}\right)+\sum_{i=1}^{s}a_i\Delta t^{-\alpha_i}\sum_{k=0}^{n}\omega_{kn}^{(\alpha_i)}\left(U^{n-k}-U^{0}\right)+k_\beta A^{\beta/2}U^n+k_\gamma A^{\gamma/2}U^n=f_n, \tag{52}$$

where $f_n = f(t_n)$.

Lemma 8 (see [38]). *For $0<\alpha<1$, let the sequence $\{d_j\}$ $j=1,2,\ldots$ be given by $d_1=1$ and $d_j=1+\alpha(1-\alpha)j^{-\alpha}\sum_{k=1}^{j-1}\alpha_{kj}d_{j-k}$, $j=2,3,\ldots$. Then, $1\le d_j\le(\sin(\pi\alpha))/(\pi\alpha(1-\alpha))j^{\alpha}$, $j=1,2,3,\ldots$.*

Let $e^n = U^n - u(t_n)$ denote the error in t_n. Let $\|\cdot\|$ be any given norm. Then, we have the following error estimate.

Theorem 9. *Let $u(t_n)$ and U^n be the solutions of* (43) *and* (52)*, respectively, and Δt is the time step size. Then, one has*

$$\max_{n=0,1,\ldots,N}\left\|U^n-u(t_n)\right\|=O\left(\Delta t^{2-\alpha}+\sum_{i=0}^{s}a_i\Delta t^{2-\alpha_i}\right). \tag{53}$$

Proof. Subtracting (52) from (51), we obtain the error equation

$$\Delta t^{-\alpha}\sum_{k=0}^{n}\omega_{kn}^{(\alpha)}\left(e^{n-k}-e^{0}\right)+\sum_{i=1}^{s}a_i\Delta t^{-\alpha_i}\sum_{k=0}^{n}\omega_{kn}^{(\alpha_i)}\left(e^{n-k}-e^{0}\right)+k_\beta A^{\beta/2}e^n+k_\gamma A^{\gamma/2}e^n=-\frac{t_n^{-\alpha}}{\Gamma(-\alpha)}R_n^{(\alpha)}(g)-\sum_{i=1}^{s}a_i\frac{t_n^{-\alpha_i}}{\Gamma(-\alpha_i)}R_n^{(\alpha_i)}(g). \tag{54}$$

Note that $e^0=\phi_0-U^0=0$ and denote

$$\begin{aligned}e^n={}&\left(\Delta t^{-\alpha}\omega_{0n}^{(\alpha)}+\sum_{i=1}^{s}a_i\Delta t^{-\alpha_i}\omega_{0n}^{(\alpha_i)}+k_\beta A^{\beta/2}+k_\gamma A^{\gamma/2}\right)^{-1}\\&\times\left(\Delta t^{-\alpha}\sum_{k=1}^{n}\omega_{kn}^{(\alpha)}e^{n-k}+\sum_{i=1}^{s}a_i\Delta t^{-\alpha_i}\sum_{k=1}^{n}\omega_{kn}^{(\alpha_i)}e^{n-k}-\frac{t_n^{-\alpha}}{\Gamma(-\alpha)}R_n^{(\alpha)}(g)-\sum_{i=1}^{s}a_i\frac{t_n^{-\alpha_i}}{\Gamma(-\alpha_i)}R_n^{(\alpha_i)}(g)\right)\\={}&\left(\alpha_{0n}^{(\alpha)}+\sum_{i=1}^{s}a_i\Delta t^{\alpha-\alpha_i}\frac{\Gamma(-\alpha)}{\Gamma(-\alpha_i)}\alpha_{0n}^{(\alpha_i)}+k_\beta A^{\beta/2}t_n^{\alpha}\Gamma(-\alpha)+k_\gamma A^{\gamma/2}t_n^{\alpha}\Gamma(-\alpha)\right)^{-1}\\&\times\left(\sum_{k=1}^{n}\alpha_{kn}^{(\alpha)}e^{n-k}+\sum_{i=1}^{s}a_i\Delta t^{\alpha-\alpha_i}\frac{\Gamma(-\alpha)}{\Gamma(-\alpha_i)}\times\sum_{k=1}^{n}\alpha_{kn}^{(\alpha_i)}e^{n-k}-R_n^{(\alpha)}(g)-\sum_{i=1}^{s}a_i\frac{\Gamma(-\alpha)}{\Gamma(-\alpha_i)}t_n^{\alpha-\alpha_i}R_n^{(\alpha_i)}(g)\right).\end{aligned} \tag{55}$$

Thus, we have

$$\begin{aligned}\left\|e^n\right\|\le{}&\left\|\left(\alpha_{0n}^{(\alpha)}+\sum_{i=1}^{s}a_i\Delta t^{\alpha-\alpha_i}\frac{\Gamma(-\alpha)}{\Gamma(-\alpha_i)}\alpha_{0n}^{(\alpha_i)}+k_\beta A^{\beta/2}t_n^{\alpha}\Gamma(-\alpha)+k_\gamma A^{\gamma/2}t_n^{\alpha}\Gamma(-\alpha)\right)^{-1}\right\|\\&\times\left(\sum_{k=1}^{n}\alpha_{kn}^{(\alpha)}\left\|e^{n-k}\right\|+\sum_{i=1}^{s}a_i\Delta t^{\alpha-\alpha_i}\frac{\Gamma(-\alpha)}{\Gamma(-\alpha_i)}\sum_{k=1}^{j}\alpha_{kn}^{(\alpha_i)}\left\|e^{n-k}\right\|+\left\|R_n^{(\alpha)}(g)\right\|+\sum_{i=1}^{s}a_i\frac{\Gamma(-\alpha)}{\Gamma(-\alpha_i)}t_n^{\alpha-\alpha_i}\left\|R_n^{(\alpha_i)}(g)\right\|\right).\end{aligned} \tag{56}$$

From Definition 7, note that A is a positive definite elliptic operator with all of eigenvalues $\lambda>0$. Since $\alpha_{0n}^{(\alpha_i)}<0$, $i=1,\ldots,s$, $\alpha_{0n}^{(\alpha)}<0$ and $\Gamma(-\alpha)<0$ when $0<\alpha<1$, we have

$$\left\|\left(\alpha_{0n}^{(\alpha)}+\sum_{i=1}^{s}a_i\Delta t^{\alpha-\alpha_i}\frac{\Gamma(-\alpha)}{\Gamma(-\alpha_i)}\alpha_{0n}^{(\alpha_i)}+k_\beta A^{\beta/2}t_n^{\alpha}\Gamma(-\alpha)+k_\gamma A^{\gamma/2}t_n^{\alpha}\Gamma(-\alpha)\right)^{-1}\right\|$$

$$= \sup_{\lambda>0} \left\| \left(\alpha_{0n}^{(\alpha)} + \sum_{i=1}^{s} a_i \Delta t^{\alpha-\alpha_i} \frac{\Gamma(-\alpha)}{\Gamma(-\alpha_i)} \alpha_{0n}^{(\alpha_i)} + \lambda^{\beta/2} t_n^{\alpha} \Gamma(-\alpha) + \lambda^{\gamma/2} t_n^{\alpha} \Gamma(-\alpha) \right)^{-1} \right\|$$
$$< \left(-\alpha_{0n}^{(\alpha)} - \sum_{i=1}^{s} a_i \Delta t^{\alpha-\alpha_i} \frac{\Gamma(-\alpha)}{\Gamma(-\alpha_i)} \alpha_{0n}^{(\alpha_i)} \right)^{-1}. \tag{57}$$

Hence,

$$\begin{aligned} \|e^n\| &\le \left(-\alpha_{0n}^{(\alpha)}\right)^{-1} \sum_{k=1}^{n} \alpha_{kn}^{(\alpha)} \|e^{n-k}\| + \sum_{i=1}^{s} \left(-\alpha_{0n}^{(\alpha_i)}\right)^{-1} \sum_{k=1}^{n} \alpha_{kn}^{(\alpha_i)} \|e^{n-k}\| \\ &\quad + \left(-\alpha_{0n}^{(\alpha)}\right)^{-1} \|R_n^{(\alpha)}(g)\| + \sum_{i=1}^{s} \left(-\alpha_{0n}^{(\alpha_i)}\right)^{-1} \|R_n^{(\alpha_i)}(g)\| \\ &\le \alpha(1-\alpha) n^{-\alpha} \sum_{k=1}^{n} \alpha_{kn}^{(\alpha)} \|e^{n-k}\| + \sum_{i=1}^{s} \alpha_i (1-\alpha_i) n^{-\alpha_i} \sum_{k=1}^{n} \alpha_{kn}^{(\alpha_i)} \|e^{n-k}\| \\ &\quad + \alpha(1-\alpha) \gamma_\alpha N^{-2} \sup_{0\le t\le T} \|u''\| + \sum_{i=1}^{s} \alpha_i (1-\alpha_i) \gamma_{\alpha_i} N^{-2} \sup_{0\le t\le T} \|u''\|. \end{aligned} \tag{58}$$

Note that $e^0 = \phi_0 - U^0 = 0$. Denote $d_1 = 1$ and

$$\begin{aligned} d_n &= 1 + \alpha(1-\alpha) n^{-\alpha} \sum_{k=1}^{n-1} \alpha_{kn}^{(\alpha)} d_{n-k}, \quad n = 2, 3, \ldots, N, \\ d_n^i &= 1 + \alpha_i (1-\alpha_i) n^{-\alpha_i} \sum_{k=1}^{j-1} \alpha_{kn}^{(\alpha_i)} d_{n-k}, \quad n = 2, 3, \ldots, N, \ i = 1, 2, \ldots, s. \end{aligned} \tag{59}$$

Then, by induction and Lemma 8, we have

$$\begin{aligned} \|e^n\| &\lesssim \alpha(1-\alpha) N^{-2} \sup_{0\le t\le T} \|u''(t)\| \cdot d_n + \sum_{i=1}^{s} a_i \alpha_i (1-\alpha_i) N^{-2} \sup_{0\le t\le T} \|u''(t)\| \cdot d_n^i \\ &\le N^{-2} \sup_{0\le t\le T} \|u''(t)\| \frac{\sin(\pi\alpha)}{\pi} n^{\alpha} + \sum_{i=1}^{s} a_i N^{-2} \sup_{0\le t\le T} \|u''(t)\| \frac{\sin(\pi\alpha_i)}{\pi} n^{\alpha_i} \\ &= O\left(\Delta t^{2-\alpha} + \sum_{i=1}^{s} a_i \Delta t^{2-\alpha_i} \right). \end{aligned} \tag{60}$$

□

4. Space Discretization

In this section, we consider the space discretization of (5) with homogeneous boundary condition. Using the FEM, we obtain the numerical approximation solution in a finite domain. Then, we prove the convergence rate of this method. Let $\Omega = [0, X]$ be an interval in one-dimensional space. All of the results in this section can be generalized into the cases of high dimension.

Based on the time discretization in Section 3 and (25), we need to find $u(t, \cdot) \in H_0^{\gamma/2}(\Omega)$ such that

$$\begin{aligned} &\left({}_0^R D_t^{\alpha} u(t,x), v\right)_{L_2(\Omega)} + \sum_{i=1}^{s} a_i \left({}_0^R D_t^{\alpha_i} u(t,x), v\right)_{L_2(\Omega)} \\ &\quad + k_\beta C_\beta \left[\left({}_0^R D_x^{\beta/2} u(t,x), {}_x^R D_X^{\beta/2} v\right)_{L_2(\Omega)} + \left({}_x^R D_X^{\beta/2} u(t,x), {}_0^R D_x^{\beta/2} v\right)_{L_2(\Omega)} \right] \\ &\quad + k_\gamma C_\gamma \left[\left({}_0^R D_x^{\gamma/2} u(t,x), {}_x^R D_X^{\gamma/2} v\right)_{L_2(\Omega)} + \left({}_x^R D_X^{\gamma/2} u(t,x), {}_0^R D_x^{\gamma/2} v\right)_{L_2(\Omega)} \right] \\ &= \left(\overline{f}(t,x), v\right)_{L_2(\Omega)}, \quad \forall v \in H_0^{\gamma/2}(\Omega). \end{aligned} \tag{61}$$

Let h denote the maximal length of intervals in Ω, and let r be any nonnegative integer. We denote the norm in $H^r(\Omega)$ by $\|\cdot\|_{H^r(\Omega)}$. Let $S_h \subset H_0^{\gamma/2}$ be a family of finite element space with the accuracy of order $r \ge 2$; that is, S_h consists of continuous functions on the closure $\overline{\Omega}$ of Ω which are polynomials of degree at most $r-1$ in each interval and which vanish outside Ω_h such that for small h, $v \in H^{\eta}(\Omega) \cap H_0^{\gamma/2}(\Omega)$,

$$\begin{aligned} &\inf_{\chi\in S_h} \left(\|(v-\chi)\|_{L_2(\Omega)} + h\|\nabla(v-\chi)\|_{L_2(\Omega)} \right) \\ &\quad \lesssim h^{\eta} \|v\|_{H^{\eta}(\Omega)}, \quad 1 \le \eta \le r. \end{aligned} \tag{62}$$

The semidiscrete problem of (5) is to find the approximate solution $u_h(t) = u_h(t, \cdot) \in S_h$ for each t such that

$$\begin{aligned} &\left({}_0^R D_t^{\alpha} u_h(t), \chi\right)_{L_2(\Omega)} + \sum_{i=1}^{s} a_i \left({}_0^R D_t^{\alpha_i} u_h(t), \chi\right)_{L_2(\Omega)} \\ &\quad + k_\beta C_\beta \left[\left({}_0^R D_x^{\beta/2} u_h(t), {}_x^R D_X^{\beta/2} \chi\right)_{L_2(\Omega)} + \left({}_x^R D_X^{\beta/2} u_h(t), {}_0^R D_x^{\beta/2} \chi\right)_{L_2(\Omega)} \right] \end{aligned}$$

$$+ k_\gamma C_\gamma \left[\left({}_0^R D_x^{\gamma/2} u_h(t), {}_x^R D_X^{\gamma/2} \chi \right)_{L_2(\Omega)} + \left({}_x^R D_X^{\gamma/2} u_h(t), {}_0^R D_x^{\gamma/2} \chi \right)_{L_2(\Omega)} \right] = \left(\overline{f}(t), \chi \right)_{L_2(\Omega)}, \quad \forall \chi \in S_h. \tag{63}$$

Let $R_h : H^{\gamma/2}(\Omega) \rightarrow S_h$ be the elliptic projection defined by

$$\begin{aligned} & k_\beta C_\beta \left({}_0^R D_x^{\beta/2} R_h u, {}_x^R D_X^{\beta/2} \chi \right)_{L_2(\Omega)} \\ &\quad + k_\beta C_\beta \left({}_x^R D_X^{\beta/2} R_h u, {}_0^R D_x^{\beta/2} \chi \right)_{L_2(\Omega)} \\ &\quad + k_\gamma C_\gamma \left({}_0^R D_x^{\gamma/2} R_h u, {}_x^R D_X^{\gamma/2} \chi \right)_{L_2(\Omega)} \\ &\quad + k_\gamma C_\gamma \left({}_x^R D_X^{\gamma/2} R_h u, {}_0^R D_x^{\gamma/2} \chi \right)_{L_2(\Omega)} \\ &= k_\beta C_\beta \left({}_0^R D_x^{\beta/2} u, {}_x^R D_X^{\beta/2} \chi \right)_{L_2(\Omega)} \\ &\quad + k_\beta C_\beta \left({}_x^R D_X^{\beta/2} u, {}_0^R D_x^{\beta/2} \chi \right)_{L_2(\Omega)} \\ &\quad + k_\gamma C_\gamma \left({}_0^R D_x^{\gamma/2} u, {}_x^R D_X^{\gamma/2} \chi \right)_{L_2(\Omega)} \\ &\quad + k_\gamma C_\gamma \left({}_x^R D_X^{\gamma/2} u, {}_0^R D_x^{\gamma/2} \chi \right)_{L_2(\Omega)}, \quad \forall \chi \in S_h. \end{aligned} \tag{64}$$

Lemma 10. *Assume that* (62) *holds. Then, for R_h defined by* (64) *and any $v \in H^\eta(\Omega) \cap H_0^{\theta/2}(\Omega)$, one has*

$$\left\| {}_0^R D_x^{\theta/2} (R_h v - v) \right\|_{L_2(\Omega)} \lesssim h^{\eta-\theta/2} \|v\|_{H^\eta(\Omega)}. \tag{65}$$

Proof. Let I_h be the projection operator from $H^\eta(\Omega) \cap H_0^{\theta/2}(\Omega)$ to S_h. From the definition of L_2-norm, we obtain

$$\begin{aligned} & \left({}_0^R D_x^{\theta/2} (R_h v - v), {}_x^R D_X^{\theta/2} (R_h v - v) \right)_{L_2(\Omega)} \\ &\quad + \left({}_x^R D_X^{\theta/2} (R_h v - v), {}_0^R D_x^{\theta/2} (R_h v - v) \right)_{L_2(\Omega)} \\ &= \left({}_0^R D_x^{\theta/2} (R_h v - v), {}_x^R D_X^{\theta/2} (R_h v - I_h v) \right)_{L_2(\Omega)} \\ &\quad + \left({}_x^R D_X^{\theta/2} (R_h v - v), {}_0^R D_x^{\theta/2} (R_h v - I_h v) \right)_{L_2(\Omega)} \\ &\quad + \left({}_0^R D_x^{\theta/2} (R_h v - v), {}_x^R D_X^{\theta/2} (I_h v - v) \right)_{L_2(\Omega)} \\ &\quad + \left({}_x^R D_X^{\theta/2} (R_h v - v), {}_0^R D_x^{\theta/2} (I_h v - v) \right)_{L_2(\Omega)}. \end{aligned} \tag{66}$$

Let $\chi = R_h v - I_h v \in S_h$. From (64), we obtain

$$\begin{aligned} & \left({}_0^R D_x^{\theta/2} (R_h v - v), {}_x^R D_X^{\theta/2} (R_h v - v) \right)_{L_2(\Omega)} \\ &\quad + \left({}_x^R D_X^{\theta/2} (R_h v - v), {}_0^R D_x^{\theta/2} (R_h v - v) \right)_{L_2(\Omega)} \\ &\leq \left\| {}_0^R D_x^{\theta/2} (R_h v - v) \right\|_{L_2(\Omega)} \left\| {}_x^R D_X^{\theta/2} (I_h v - v) \right\|_{L_2(\Omega)} \\ &\quad + \left\| {}_x^R D_X^{\theta/2} (R_h v - v) \right\|_{L_2(\Omega)} \left\| {}_0^R D_x^{\theta/2} (I_h v - v) \right\|_{L_2(\Omega)} \\ &\leq \left(\left\| {}_0^R D_x^{\theta/2} (R_h v - v) \right\|_{L_2(\Omega)} + \left\| {}_x^R D_X^{\theta/2} (R_h v - v) \right\|_{L_2(\Omega)} \right) \\ &\quad \times \left(\left\| {}_x^R D_X^{\theta/2} (I_h v - v) \right\|_{L_2(\Omega)} + \left\| {}_0^R D_x^{\theta/2} (I_h v - v) \right\|_{L_2(\Omega)} \right). \end{aligned} \tag{67}$$

Note that

$$\begin{aligned} & \left({}_0^R D_x^{\theta/2} (R_h v - v), {}_x^R D_X^{\theta/2} (R_h v - v) \right)_{L_2(\Omega)} \\ &\quad + \left({}_x^R D_X^{\theta/2} (R_h v - v), {}_0^R D_x^{\theta/2} (R_h v - v) \right)_{L_2(\Omega)} \\ &\gtrsim \left(\left\| {}_0^R D_x^{\theta/2} (R_h v - v) \right\|_{L_2(\Omega)} + \left\| {}_x^R D_X^{\theta/2} (R_h v - v) \right\|_{L_2(\Omega)} \right)^2. \end{aligned} \tag{68}$$

Thus, combining (67) and (68), we obtain

$$\begin{aligned} & \left\| {}_0^R D_x^{\theta/2} (R_h v - v) \right\|_{L_2(\Omega)} + \left\| {}_x^R D_X^{\theta/2} (R_h v - v) \right\|_{L_2(\Omega)} \\ &\quad \lesssim \left\| {}_0^R D_x^{\theta/2} (I_h v - v) \right\|_{L_2(\Omega)} + \left\| {}_x^R D_X^{\theta/2} (I_h v - v) \right\|_{L_2(\Omega)}. \end{aligned} \tag{69}$$

By (62), we see that

$$\left\| v - I_h v \right\|_{L_2(\Omega)} + h \left\| \nabla (v - I_h v) \right\|_{L_2(\Omega)} \lesssim h^\eta |v|_{H^\eta(\Omega)}. \tag{70}$$

By interpolation properties, we obtain

$$\begin{aligned} \left\| {}_0^R D_x^{\theta/2} (v - I_h v) \right\|_{L_2(\Omega)} &\leq \left| (v - I_h v) \right|_{H^{\theta/2}(\Omega)} \\ &\lesssim h^{\eta-\theta/2} |v|_{H^{\eta/2}(\Omega)}. \end{aligned} \tag{71}$$

Similarly, we have

$$\begin{aligned} \left\| {}_x^R D_X^{\theta/2} (v - I_h v) \right\|_{L_2(\Omega)} &\leq \left| (v - I_h v) \right|_{H^{\theta/2}(\Omega)} \\ &\lesssim h^{\eta-\theta/2} |v|_{H^{\eta/2}(\Omega)}. \end{aligned} \tag{72}$$

Combining (71) and (72), we can obtain (65). □

Theorem 11. *For $0 < \alpha_s < \cdots < \alpha_1 < \alpha < 1$, let u_h and u be the solutions of* (63) *and* (61), *respectively. Then, it holds*

$$\left\| u(t,x) - u_h(t,x) \right\|_{H^{\gamma/2}(\Omega)} = O\left(\Delta x^{\eta-\gamma/2} \right). \tag{73}$$

Proof. We write

$$u_h - u = \vartheta + \rho, \tag{74}$$

where $\vartheta = u_h - R_h u$, $\rho = R_h u - u$. The second term is easily bounded by Lemma 10 and has the obvious estimate

$$\|\rho(x)\|^2_{H^{\gamma/2}(\Omega)} \lesssim h^{2\eta-\gamma}\|u\|^2_{H^{\eta}(\Omega)}. \quad (75)$$

In order to estimate ϑ, for all $\chi \in S_h$, we get

$$\begin{aligned}
&\left({}_0^R D_t^{\alpha}\vartheta(t), \chi\right)_{L_2(\Omega)} + \sum_{i=1}^{s} a_i \left({}_0^R D_t^{\alpha_i}\vartheta(t), \chi\right)_{L_2(\Omega)} \\
&\quad + k_\beta C_\beta \left[\left({}_0^R D_x^{\beta/2}\vartheta(t), {}_x^R D_X^{\beta/2}\chi\right)_{L_2(\Omega)} + \left({}_x^R D_X^{\beta/2}\vartheta(t), {}_0^R D_x^{\beta/2}\chi\right)_{L_2(\Omega)}\right] \\
&\quad + k_\gamma C_\gamma \left[\left({}_0^R D_x^{\gamma/2}\vartheta(t), {}_x^R D_X^{\gamma/2}\chi\right)_{L_2(\Omega)} + \left({}_x^R D_X^{\gamma/2}\vartheta(t), {}_0^R D_x^{\gamma/2}\chi\right)_{L_2(\Omega)}\right] \\
&= \left({}_0^R D_t^{\alpha}\left(u_h(t) - R_h u(t)\right), \chi\right)_{L_2(\Omega)} + \sum_{i=1}^{s} a_i \left({}_0^R D_t^{\alpha_i}\left(u_h(t) - R_h u(t)\right), \chi\right)_{L_2(\Omega)} \\
&\quad + k_\beta C_\beta \left[\left({}_0^R D_x^{\beta/2}\left(u_h(t) - R_h u(t)\right), {}_x^R D_X^{\beta/2}\chi\right)_{L_2(\Omega)} + \left({}_x^R D_X^{\beta/2}\left(u_h(t) - R_h u(t)\right), {}_0^R D_x^{\beta/2}\chi\right)_{L_2(\Omega)}\right] \\
&\quad + k_\gamma C_\gamma \left[\left({}_0^R D_x^{\gamma/2}\left(u_h(t) - R_h u(t)\right), {}_x^R D_X^{\gamma/2}\chi\right)_{L_2(\Omega)} + \left({}_x^R D_X^{\gamma/2}\left(u_h(t) - R_h u(t)\right), {}_0^R D_x^{\gamma/2}\chi\right)_{L_2(\Omega)}\right] \\
&= \left({}_0^R D_t^{\alpha} u_h(t), \chi\right)_{L_2(\Omega)} + \sum_{i=1}^{s} a_i \left({}_0^R D_t^{\alpha_i} u_h(t), \chi\right)_{L_2(\Omega)} \\
&\quad + k_\beta C_\beta \left({}_0^R D_x^{\beta/2} u_h(t), {}_x^R D_X^{\beta/2}\chi\right)_{L_2(\Omega)} + k_\beta C_\beta \left({}_x^R D_X^{\beta/2} u_h(t), {}_0^R D_x^{\beta/2}\chi\right)_{L_2(\Omega)} \\
&\quad + k_\gamma C_\gamma \left({}_0^R D_x^{\gamma/2} u_h(t), {}_x^R D_X^{\gamma/2}\chi\right)_{L_2(\Omega)} + k_\gamma C_\gamma \left({}_x^R D_X^{\gamma/2} u_h(t), {}_0^R D_x^{\gamma/2}\chi\right)_{L_2(\Omega)} \\
&\quad - \Bigg(\left({}_0^R D_t^{\alpha} R_h u(t), \chi\right)_{L_2(\Omega)} + \sum_{i=1}^{s} a_i \left({}_0^R D_t^{\alpha_i} R_h u(t), \chi\right)_{L_2(\Omega)} \\
&\quad + k_\beta C_\beta \left({}_0^R D_x^{\beta/2} R_h u(t), {}_x^R D_X^{\beta/2}\chi\right)_{L_2(\Omega)} + k_\beta C_\beta \left({}_x^R D_X^{\beta/2} R_h u(t), {}_0^R D_x^{\beta/2}\chi\right)_{L_2(\Omega)} \\
&\quad + k_\gamma C_\gamma \left({}_0^R D_x^{\gamma/2} R_h u(t), {}_x^R D_X^{\gamma/2}\chi\right)_{L_2(\Omega)} + k_\gamma C_\gamma \left({}_x^R D_X^{\gamma/2} R_h u(t), {}_0^R D_x^{\gamma/2}\chi\right)_{L_2(\Omega)}\Bigg) \\
&= \left(\overline{f}(t), \chi\right)_{L_2(\Omega)} - \Bigg(\left({}_0^R D_t^{\alpha} R_h u(t), \chi\right)_{L_2(\Omega)} + \sum_{i=1}^{s} a_i \left({}_0^R D_t^{\alpha_i} R_h u(t), \chi\right)_{L_2(\Omega)} \\
&\quad + k_\beta C_\beta \left({}_0^R D_x^{\beta/2} R_h u(t), {}_x^R D_X^{\beta/2}\chi\right)_{L_2(\Omega)} + k_\beta C_\beta \left({}_x^R D_X^{\beta/2} R_h u(t), {}_0^R D_x^{\beta/2}\chi\right)_{L_2(\Omega)} \\
&\quad + k_\gamma C_\gamma \left({}_0^R D_x^{\gamma/2} R_h u(t), {}_x^R D_X^{\gamma/2}\chi\right)_{L_2(\Omega)} + k_\gamma C_\gamma \left({}_x^R D_X^{\gamma/2} R_h u(t), {}_0^R D_x^{\gamma/2}\chi\right)_{L_2(\Omega)}\Bigg) \\
&= \left(P\left({}_0D_t\right) u(t), \chi\right)_{L_2(\Omega)} - \left(P\left({}_0D_t\right) R_h u(t), \chi\right)_{L_2(\Omega)} \\
&= \left(\left(I - R_h\right) P\left({}_0D_t\right) u(t), \chi\right)_{L_2(\Omega)} \\
&= -\left(P\left({}_0D_t\right) \rho(t), \chi\right)_{L_2(\Omega)}.
\end{aligned} \quad (76)$$

Choosing $\chi = \vartheta(t)$ and integrating on both sides with respect to t on $[0, T]$, we obtain

$$\begin{aligned}
&\left(P\left({}_0D_t\right)\vartheta(t), \vartheta(t)\right)_{L_2(\Omega)} \\
&\quad + k_\beta C_\beta \left({}_0^R D_x^{\beta/2}\vartheta(t), {}_x^R D_X^{\beta/2}\vartheta(t)\right)_{L_2(\Omega)} \\
&\quad + k_\beta C_\beta \left({}_x^R D_X^{\beta/2}\vartheta(t), {}_0^R D_x^{\beta/2}\vartheta(t)\right)_{L_2(\Omega)} \\
&\quad + k_\gamma C_\gamma \left({}_0^R D_x^{\gamma/2}\vartheta(t), {}_x^R D_X^{\gamma/2}\vartheta(t)\right)_{L_2(\Omega)} \\
&\quad + k_\gamma C_\gamma \left({}_x^R D_X^{\gamma/2}\vartheta(t), {}_0^R D_x^{\gamma/2}\vartheta(t)\right)_{L_2(\Omega)} \\
&= -\left(P\left({}_0D_t\right)\rho(t), \vartheta(t)\right)_{L_2(\Omega)}.
\end{aligned} \quad (77)$$

By Lemmas 2–4, for any small $\epsilon > 0$, we have

$$\begin{aligned}
&A_\alpha \left({}_0^R D_t^{\alpha/2}\vartheta(t), {}_0^R D_t^{\alpha/2}\vartheta(t)\right)_{L_2(\Omega)} \\
&\quad + A_{\alpha_i} \sum_{i=1}^{s} a_i \left({}_0^R D_t^{\alpha_i/2}\vartheta(t), {}_0^R D_t^{\alpha_i/2}\vartheta(t)\right)_{L_2(\Omega)} \\
&\quad + k_\beta C_\beta \left[\left({}_0^R D_x^{\beta/2}\vartheta(t), {}_x^R D_X^{\beta/2}\vartheta(t)\right)_{L_2(\Omega)} + \left({}_x^R D_X^{\beta/2}\vartheta(t), {}_0^R D_x^{\beta/2}\vartheta(t)\right)_{L_2(\Omega)}\right]
\end{aligned}$$

$$
\begin{aligned}
&+ k_\gamma C_\gamma \Big[\left({}_0^R D_x^{\gamma/2} \vartheta(t), {}_x^R D_X^{\gamma/2} \vartheta(t)\right)_{L_2(\Omega)} \\
&\quad + \left({}_x^R D_X^{\gamma/2} \vartheta(t), {}_0^R D_x^{\gamma/2} \vartheta(t)\right)_{L_2(\Omega)} \Big] \\
&= A_\alpha \left({}_0^R D_t^{\alpha/2} \rho(t), {}_0^R D_t^{\alpha/2} \vartheta(t)\right)_{L_2(\Omega)} \\
&\quad + \sum_{i=1}^{s} a_i A_{\alpha_i} \left({}_0^R D_t^{\alpha_i/2} \rho(t), {}_0^R D_t^{\alpha_i/2} \vartheta(t)\right)_{L_2(\Omega)} \\
&\le C_\epsilon \left(A_\alpha \left\| {}_0^R D_t^{\alpha/2} \rho(t) \right\|_{L_2(\Omega)}^2 + \sum_{i=1}^{s} a_i A_{\alpha_i} \left\| {}_0^R D_t^{\alpha_i/2} \rho(t) \right\|_{L_2(\Omega)}^2 \right) \\
&\quad + \epsilon \left(A_\alpha \left\| {}_0^R D_t^{\alpha/2} \vartheta(t) \right\|_{L_2(\Omega)}^2 + \sum_{i=1}^{s} a_i A_{\alpha_i} \left\| {}_0^R D_t^{\alpha_i/2} \vartheta(t) \right\|_{L_2(\Omega)}^2 \right),
\end{aligned} \tag{78}
$$

where C_ϵ is a constant respect to ϵ, $A_\alpha = \cos(\pi\alpha/2)$ and $A_{\alpha_i} = \cos((\pi\alpha_i)/2)$, $i = 1, \dots, s$.

For sufficiently small $\epsilon > 0$, by (75), we obtain

$$
\begin{aligned}
&\left(P\left({}_0D_t\right) \vartheta(t), \vartheta(t)\right)_{L_2(\Omega)} \\
&\le C_\epsilon \left(\left\| {}_0^R D_t^{\alpha/2} \rho(t) \right\|_{L_2(\Omega)}^2 + \sum_{i=1}^{s} a_i \left\| {}_0^R D_t^{\alpha_i/2} \rho(t) \right\|_{L_2(\Omega)}^2 \right) \\
&\le C_\epsilon \left(h^{\eta-\gamma/2}\right)^2 \left(\left\| {}_0^R D_t^{\alpha/2} u(t) \right\|_{H^\eta(\Omega)}^2 + \sum_{i=1}^{s} a_i \left\| {}_0^R D_t^{\alpha_i/2} u(t) \right\|_{H^\eta(\Omega)}^2 \right).
\end{aligned} \tag{79}
$$

Combining (75) with (79), we obtain that

$$
\left\| u - u_h \right\|_{H^{\gamma/2}(\Omega)} = \left\| \rho + \vartheta \right\|_{H^{\gamma/2}(\Omega)} = O\left(h^{\eta-\gamma/2}\right). \tag{80}
$$

□

For the reason that the time and space fractional derivatives, we introduce the complete form of this FEM. In view of space discretization, we first pose the finite-dimensional problem to find $u_h(t, \cdot) \in S_h$ such that (63) holds.

In terms of the basis $\{\psi_i\}_{i=1}^{M-1} \subseteq S_h$, write $u_h(t,x) = \sum_{j=1}^{M-1} u_j(t)\psi_j(x)$, and insert it into (63). After the time discretization on t_n, in Section 3, we get

$$
\begin{aligned}
&\sum_{j=1}^{M-1} \Delta t^{-\alpha} \sum_{k=1}^{n} \omega_{kn}^{(\alpha)} U_j^{n-k} \left(\psi_j, \psi_i\right)_{L_2(\Omega)} \\
&\quad + \sum_{i=1}^{s} a_i \sum_{j=1}^{M-1} \Delta t^{-\alpha_i} \sum_{k=1}^{n} \omega_{kn}^{(\alpha_i)} U_j^{n-k} \left(\psi_j, \psi_i\right)_{L_2(\Omega)} \\
&\quad + k_\beta C_\beta \sum_{j=1}^{M-1} U_j^n \Big[\left({}_0^R D_x^{\beta/2} \psi_j, {}_x^R D_X^{\beta/2} \psi_i\right)_{L_2(\Omega)} + \left({}_x^R D_X^{\beta/2} \psi_j, {}_0^R D_x^{\beta/2} \psi_i\right)_{L_2(\Omega)} \Big] \\
&\quad + k_\gamma C_\gamma \sum_{j=1}^{M-1} U_j^n \Big[\left({}_0^R D_x^{\gamma/2} \psi_j, {}_x^R D_X^{\gamma/2} \psi_i\right)_{L_2(\Omega)} + \left({}_x^R D_X^{\gamma/2} \psi_j, {}_0^R D_x^{\gamma/2} \psi_i\right)_{L_2(\Omega)} \Big] \\
&= \left(\overline{f}, \psi_i\right)_{L_2(\Omega)}, \quad \forall \psi_i \in S_h, \; i = 1, \dots, M-1.
\end{aligned} \tag{81}
$$

To obtain the value of $\{U_j^n\}_{j=1}^{M-1}$, Let $U_j = (U_j^1, U_j^2, \dots, U_j^n)^T$. From (81), we obtain a vector equation

$$
\begin{aligned}
&\Delta t^{-\alpha} \omega_{0n}^{(\alpha)} M U_n + \sum_{i=1}^{s} a_i \Delta t^{-\alpha_i} \omega_{0n}^{(\alpha_i)} M U_n \\
&\quad + k_\beta C_\beta \left(S_1^{(\beta/2)} + S_2^{(\beta/2)}\right) U_n \\
&\quad + k_\gamma C_\gamma \left(S_1^{(\gamma/2)} + S_2^{(\gamma/2)}\right) U_n \\
&= M F_n - \Delta t^{-\alpha} \sum_{k=1}^{n} \omega_{kn}^{(\alpha)} M U_{n-k} \\
&\quad - \sum_{i=1}^{s} \Delta t^{-\alpha_i} \sum_{k=1}^{n} \omega_{kn}^{(\alpha_i)} M U_{n-k},
\end{aligned} \tag{82}
$$

where $M = \{(\psi_j, \psi_i)_{L_2(\Omega)}\}_{j,i=1}^{M-1}$ is the mass matrix; $S_1^{(\theta)}$ and $S_2^{(\theta)}$ are stiffness matrices with $\theta = \beta/2, \gamma/2$, as follows:

$$
\begin{aligned}
S_1^{(\theta)} &= \left\{ \left({}_0^R D_x^{\theta} \psi_j, {}_x^R D_X^{\theta} \psi_i\right)_{L_2(\Omega)} \right\}_{j,i=1}^{M-1}, \\
S_2^{(\theta)} &= \left\{ \left({}_x^R D_X^{\theta} \psi_j, {}_0^R D_x^{\theta} \psi_i\right)_{L_2(\Omega)} \right\}_{j,i=1}^{M-1},
\end{aligned} \tag{83}
$$

and $F_N = (\overline{f}_1, \dots, \overline{f}_{M-1})^T$ is a vector valued function. Then, from (82), we can obtain the solution U_n.

5. Stability of the Numerical Method

In this section, we analyze the stability of the FEM for MT-TS-RFADEs (5). Now, we do some preparation before proof. Based on the definition of coefficients $\omega_{kn}^{(\alpha)}$ in Section 3, we can obtain the following lemma easily.

Lemma 12. *For $0 < \alpha < 1$, the coefficients $\omega_{kn}^{(\alpha)}$, $(k = 1, \ldots, n)$ satisfy the following properties:*

(i) $\omega_{0n}^{(\alpha)} > 0$ *and* $\omega_{kn}^{(\alpha)} < 0$ *for* $k = 1, 2, \ldots, n$,

(ii) $\Gamma(2-\alpha)\sum_{k=1}^{n}\omega_{kn}^{(\alpha)} = (1-\alpha)n^{-\alpha} + 1$.

Now, we report the stability theorem of this FEM for MT-TS-RFADEs as follows.

Theorem 13. *The FEM defined in* (81) *is unconditionally stable.*

Proof. Let U^n denote the approximation to u_h at $t = t_n$ and $\chi(\cdot) = U^n(\cdot)$ and the right-hand side $\overline{f} = 0$. From (63), we have

$$\begin{aligned}
&\Delta t^{-\alpha}\left\{\frac{1}{\Gamma(2-\alpha)}\left(U^n, U^n\right)_{L_2(\Omega)} + \sum_{k=1}^{n-1}\omega_{kn}^{(\alpha)}\left(U^{n-k}, U^n\right)_{L_2(\Omega)} + \omega_{nn}^{(\alpha)}\left(U^0, U^n\right)_{L_2(\Omega)}\right\} \\
&+ \sum_{i=1}^{s} a_i \Delta t^{-\alpha_i}\left\{\frac{1}{\Gamma(2-\alpha_i)}\left(U^n, U^n\right)_{L_2(\Omega)} + \sum_{k=1}^{n-1}\omega_{kn}^{(\alpha_i)}\left(U^{n-k}, U^n\right)_{L_2(\Omega)} + \omega_{nn}^{(\alpha_i)}\left(U^0, U^n\right)_{L_2(\Omega)}\right\} \\
&+ k_\beta C_\beta\left[\left({}_0^R D_x^{\beta/2} U^n, {}_x^R D_X^{\beta/2} U^n\right)_{L_2(\Omega)} + \left({}_x^R D_X^{\beta/2} U^n, {}_0^R D_x^{\beta/2} U^n\right)_{L_2(\Omega)}\right] \\
&+ k_\gamma C_\gamma\left[\left({}_0^R D_x^{\gamma/2} U^n, {}_x^R D_X^{\gamma/2} U^n\right)_{L_2(\Omega)} + \left({}_x^R D_X^{\gamma/2} U^n, {}_0^R D_x^{\gamma/2} U^n\right)_{L_2(\Omega)}\right] = 0.
\end{aligned} \tag{84}$$

Using the Cauchy-Schwarz inequality, $\pm(U^{n-k}, U^n) \le (1/2)(\| U^{n-k}\|_{L_2(\Omega)}^2 + \| U^n\|_{L_2(\Omega)}^2)$ for $k = 0, 1, 2, \ldots, n$, by Definition 1 and Lemma 12, we obtain

$$\begin{aligned}
&\left(\frac{\Delta t^{-\alpha}}{2\Gamma(2-\alpha)}\left(1 + (1-\alpha)n^{-\alpha}\right) + \sum_{i=1}^{s} a_i \frac{\Delta t^{-\alpha_i}}{2\Gamma(2-\alpha_i)}\left(1 + (1-\alpha_i)n^{-\alpha_i}\right)\right)\left\|U^n\right\|_{L_2(\Omega)}^2 \\
&+ k_\beta\left\|{}_0^R D_x^{\beta/2} U^n\right\|_{L_2(\Omega)}^2 + k_\gamma\left\|{}_0^R D_x^{\gamma/2} U^n\right\|_{L_2(\Omega)}^2 \\
&\le \frac{\Delta t^{-\alpha}}{2}\left[-\sum_{k=1}^{n-1}\omega_{kn}^{(\alpha)}\left\|U^{n-k}\right\|_{L_2(\Omega)}^2 - \omega_{nn}^{(\alpha)}\left\|U^0\right\|_{L_2(\Omega)}^2\right] \\
&+ \sum_{i=1}^{s} a_i \frac{\Delta t^{-\alpha_i}}{2}\left[-\sum_{k=1}^{n-1}\omega_{kn}^{(\alpha_i)}\left\|U^{n-k}\right\|_{L_2(\Omega)}^2 - \omega_{nn}^{(\alpha_i)}\left\|U^0\right\|_{L_2(\Omega)}^2\right].
\end{aligned} \tag{85}$$

We prove the stability of (63) by induction. From the beginning, we have

$$\begin{aligned}
&\left(\frac{\Delta t^{-\alpha}}{2\Gamma(2-\alpha)}\left(1 + (1-\alpha)\right) + \sum_{i=1}^{s} a_i \frac{\Delta t^{-\alpha_i}}{2\Gamma(2-\alpha_i)}\left(1 + (1-\alpha_i)\right)\right)\left\|U^1\right\|_{L_2(\Omega)}^2 \\
&\le \left(\frac{\Delta t^{-\alpha}}{2\Gamma(2-\alpha)}\left(1 - (1-\alpha)\right) + \sum_{i=1}^{s} a_i \frac{\Delta t^{-\alpha_i}}{2\Gamma(2-\alpha_i)}\left(1 - (1-\alpha_i)\right)\right)\left\|U^0\right\|_{L_2(\Omega)}^2.
\end{aligned} \tag{86}$$

The induction basis $\| U^1\|_{L_2(\Omega)} \le \| U^0\|_{L_2(\Omega)}$ is presupposed. For the induction step, we have $\| U^n\|_{L_2(\Omega)} \le \| U^{n-1}\|_{L_2(\Omega)} \le \cdots \le \| U^0\|_{L_2(\Omega)}$. Then, using this result, by Lemma 12, we obtain

$$\begin{aligned}
&\left(\frac{\Delta t^{-\alpha}\left(1 + (1-\alpha)(n+1)^{-\alpha}\right)}{2\Gamma(2-\alpha)} + \sum_{i=1}^{s} a_i \frac{\Delta t^{-\alpha_i}\left(1 + (1-\alpha_i)(n+1)^{-\alpha_i}\right)}{2\Gamma(2-\alpha_i)}\right)\left\|U^{n+1}\right\|_{L_2(\Omega)}^2 \\
&\le \left(\frac{\Delta t^{-\alpha}\left(1 - (1-\alpha)(n+1)^{-\alpha}\right)}{2\Gamma(2-\alpha)} + \sum_{i=1}^{s} a_i \frac{\Delta t^{-\alpha_i}\left(1 - (1-\alpha_i)(n+1)^{-\alpha_i}\right)}{2\Gamma(2-\alpha_i)}\right)\left\|U^0\right\|_{L_2(\Omega)}^2.
\end{aligned} \tag{87}$$

Here, $0 < 1-\alpha < 1$ and $0 < 1-\alpha_i < 1$ for $i = 1, \ldots, s$. After squaring at both sides of the above inequality, we obtain $\| U^{n+1}\|_{L_2(\Omega)} \le \| U^0\|_{L_2(\Omega)}$. □

6. Numerical Tests

Based on the above analysis, we present three numerical examples for MT-TS-RFADEs to demonstrate the efficiency of our theoretical analysis. The main purpose is to check the convergence behavior of numerical solutions with respect to time step size Δt and space step size Δx, which have been shown in Theorems 9 and 11.

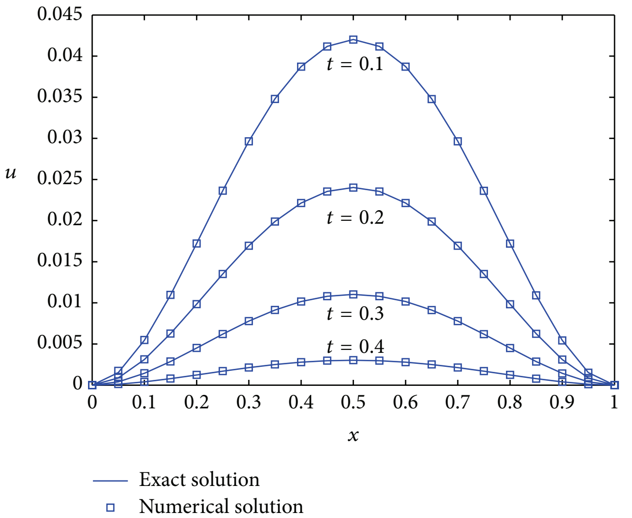

FIGURE 1: Numerical solution and exact solution for Example 1.

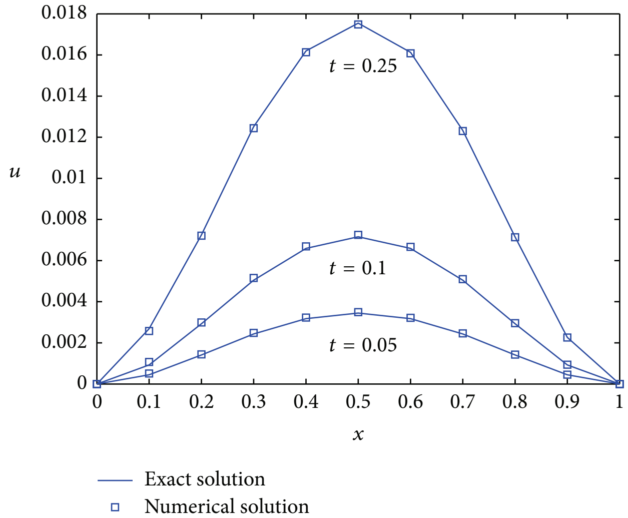

FIGURE 2: Numerical solution and exact solution for Example 2.

Example 1. Consider MT-TS-RFADE, in $t \in [0, 0.5], x \in [0, 1]$,

$$\begin{gathered}{}_0^C D_t^{\alpha} u(t,x) + {}_0^C D_t^{\alpha_1} u(t,x) - {}^R D_{|x|}^{\beta} u(t,x) \\ - {}^R D_{|x|}^{\gamma} u(t,x) = f(t,x), \\ u(0,x) = x^2(1-x)^2, \quad x \in (0,1), \\ u(t,0) = u(t,1) = 0, \quad t \in [0, 0.5].\end{gathered} \tag{88}$$

The exact solution of (88) is $u(t,x) = (2t-1)^2 x^2 (1-x)^2$. From the definition of the Riemann-Liouville differential operator, it holds

$${}_a^R D_t^{\alpha} (t-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} (t-a)^{\beta-\alpha}, \tag{89}$$

where a is a positive constant. From (89), we can choose right-hand side function $f(t,x)$ to satisfy (88).

Choosing $\alpha = 0.9$ and $\alpha_1 = 0.2$ in time fractional operators and $\beta = 0.4\gamma = 1.6$ in the space Riesz fractional operators, we can obtain the numerical approximation to the exact solution of (88) on finite domain $[0, 0.5] \times [0, 1]$, with space step size $\Delta x = 0.05$ and time step size $\Delta t = 0.01$. In Figure 1, one can see that the numerical solution matches well with the exact solution.

Example 2. Consider MT-TS-RFADE (88) with conditions as follows:

$$\begin{gathered} u(0,x) = 0, \quad x \in [0,1], \\ u(t,0) = u(t,1) = 0, \quad t \in [0, 0.5].\end{gathered} \tag{90}$$

Let the exact solution is $u(t,x) = t^{0.9} x^2 (1-x)^2$. We choose $\alpha_1 = 0.3$, $\alpha = 0.4$, $\beta = 0.4$, $\gamma = 1.7$ and obtain the numerical solution and exact solution when $t = 0.05, 0.1, 0.25$. The results have been shown in Figure 2, where the exact solution is noted by lines and numerical solution is noted by squares. Here, space step size is $\Delta x = 0.1$; time step size is $\Delta t = 0.01$.

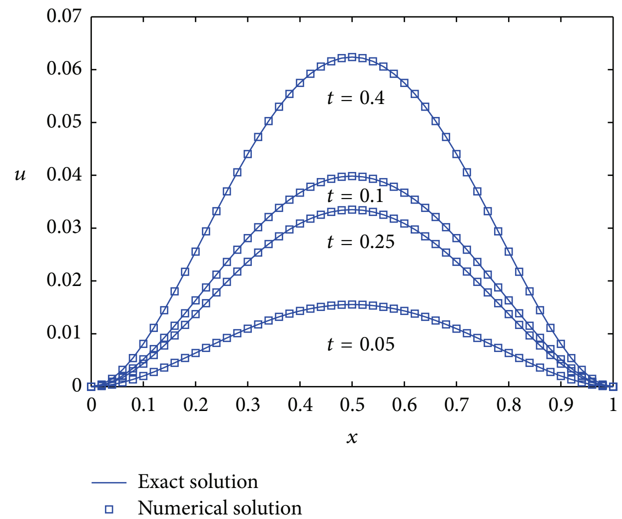

FIGURE 3: Numerical solution and exact solution for Example 3.

Example 3. Consider MT-TS-RFADE (88) with the zero Dirichlet boundary conditions, for $t \in [0, 1]$, $x \in [0, 1]$. We require that the exact solution is $u(t,x) = \sin(2\pi t) x^2 (1-x)^2$.

For this example, in the first test, we obtain the numerical solution and exact solution when $t = 0.05, 0.1, 0.25, 0.4$ in Figure 3, where we choose time step size $\Delta t = 0.01$ and space step size $\Delta x = 0.01$ with $\alpha_1 = 0.3$, $\alpha = 0.5$, and $\beta = 0.9$, $\gamma = 1.9$.

In the second test, we check the convergence rates of numerical solutions with respect to the fractional orders α_1, α, β, and γ. We fix $\alpha_1 = 0.2$, $\alpha = 0.8$, $\beta = 0.8$, and $\gamma = 1.8$ and choose $\Delta x = 0.001$ which is small enough such

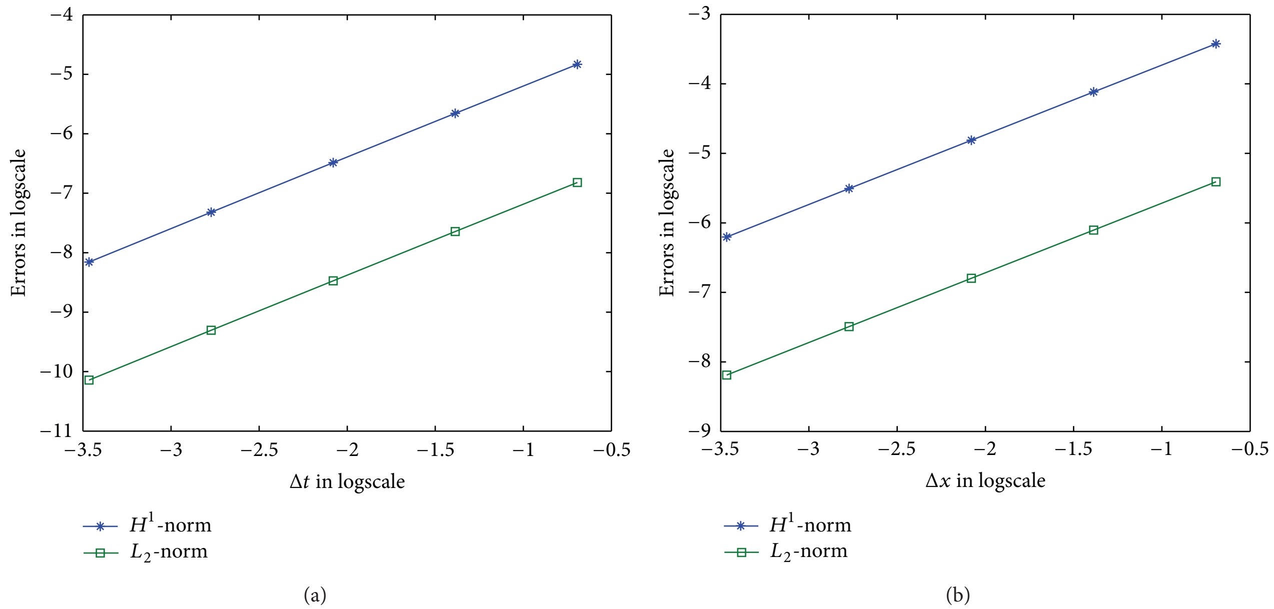

FIGURE 4: H^1-norm and L_2-norm of errors for Example 3; here, $\Delta x = 0.001$ (a) and $\Delta t = 0.001$ (b).

TABLE 1: Convergence rate in time for Example 3.

Δx	Δt	H^1-norm	L_2-norm	cvge. rate
0.001	1/2	7.9601×10^{-3}	1.0925×10^{-3}	
0.001	1/4	3.4906×10^{-3}	4.7906×10^{-4}	1.1893
0.001	1/8	1.5238×10^{-3}	2.0914×10^{-4}	1.1957
0.001	1/16	6.6273×10^{-4}	9.0956×10^{-5}	1.2012
0.001	1/32	2.8667×10^{-4}	3.9343×10^{-5}	1.2090

TABLE 2: Convergence rate in space for Example 3.

Δt	Δx	H^1-norm	L_2-norm	cvge. rate
0.001	1/2	3.2591×10^{-2}	4.4731×10^{-3}	
0.001	1/4	1.6288×10^{-2}	2.2356×10^{-3}	1.0006
0.001	1/8	8.1364×10^{-3}	1.1167×10^{-3}	1.0014
0.001	1/16	4.0602×10^{-3}	5.5723×10^{-4}	1.0029
0.001	1/32	2.0220×10^{-3}	2.7751×10^{-4}	1.0057

that the space discretization errors are negligible as compared with the time errors. Choosing step size $\Delta t = 1/2^i$ $(i = 1, \ldots, 5)$, we present Table 1 with the convergence rate which is equal to 1.2, as Theorem 9 predicted. Table 2 shows the spatial approximate convergence rate, by fixing $\Delta t = 0.001$ and choosing $\Delta x = 1/2^i$ $(i = 1, \ldots, 5)$. From Theorem 11, the convergence rate should be equal to or less than 1.1 (i.e., $\eta - \gamma/2$ for $\eta = 2$ and $\gamma = 1.8$). In Table 2, the numerical results match well with such conclusion. Here, we also report both the L_2-norm and H^1-norm of errors in Figure 4.

Acknowledgments

The authors are grateful to the referees for their valuable comments. This work is supported by the National Natural Science Foundation of China (11101109 and 11271102), the Natural Science Foundation of Hei-long-jiang Province of China (A201107), and SRF for ROCS, SEM.

References

[1] R. Metzler and J. Klafter, “The random walk’s guide to anomalous diffusion: a fractional dynamics approach,” *Physics Reports*, vol. 339, no. 1, pp. 1–77, 2000.

[2] G. M. Zaslavsky, “Chaos, fractional kinetics, and anomalous transport,” *Physics Reports*, vol. 371, no. 6, pp. 461–580, 2002.

[3] R. Metzler and J. Klafter, “The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics,” *Journal of Physics A*, vol. 37, no. 31, pp. R161–R208, 2004.

[4] W. G. Gloeckle and T. F. Nonnenmacher, “Fractional integral operators and Fox functions in the theory of viscoelasticity,” *Macromolecules*, vol. 24, no. 24, pp. 6426–6434, 1991.

[5] H. Schiessel, R. Metzler, A. Blumen, and T. F. Nonnenmacher, “Generalized vis- coelastic models: their fractional equations with solutions,” *Journal of Physics A*, vol. 28, no. 23, pp. 6567–6584, 1995.

[6] K. Diethelm, *The Analysis of Fractional Differential Equations*, Springer, Berlin, Germany, 2004.

[7] E. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific Publishing, New York, NY, USA, 2000.

[8] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, Calif, USA, 1999.

[9] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach Science Publishers, Philadelphia, Pa, USA, 1993.

[10] K. Adolfsson, M. Enelund, and S. Larsson, “Adaptive discretization of an integro-differential equation with a weakly singular convolution kernel,” *Computer Methods in Applied Mechanics and Engineering*, vol. 192, no. 51-52, pp. 5285–5304, 2003.

[11] K. Adolfsson, M. Enelund, and S. Larsson, "Adaptive discretization of fractional order viscoelasticity using sparse time history," *Computer Methods in Applied Mechanics and Engineering*, vol. 193, no. 42–44, pp. 4567–4590, 2004.

[12] J. P. Roop, "Computational aspects of FEM approximation of fractional advection dispersion equations on bounded domains in R^2," *Journal of Computational and Applied Mathematics*, vol. 193, no. 1, pp. 243–268, 2006.

[13] V. J. Ervin and J. P. Roop, "Variational formulation for the stationary fractional advection dispersion equation," *Numerical Methods for Partial Differential Equations*, vol. 22, no. 3, pp. 558–576, 2006.

[14] V. J. Ervin and J. P. Roop, "Variational solution of fractional advection dispersion equations on bounded domains in R^d," *Numerical Methods for Partial Differential Equations*, vol. 23, no. 2, pp. 256–281, 2007.

[15] V. J. Ervin, N. Heuer, and J. P. Roop, "Numerical approximation of a time dependent, nonlinear, space-fractional diffusion equation," *SIAM Journal on Numerical Analysis*, vol. 45, no. 2, pp. 572–591, 2007.

[16] C. Li, Z. Zhao, and Y. Chen, "Numerical approximation of nonlinear fractional differential equations with subdiffusion and superdiffusion," *Computers & Mathematics with Applications*, vol. 62, no. 3, pp. 855–875, 2011.

[17] X. J. Li and C. J. Xu, "A space-time spectral method for the time fractional diffusion equation," *SIAM Journal on Numerical Analysis*, vol. 47, no. 3, pp. 2108–2131, 2009.

[18] X. J. Li and C. J. Xu, "Existence and uniqueness of the weak solution of the space-time fractional diffusion equation and a spectral method approximation," *Communications in Computational Physics*, vol. 8, no. 5, pp. 1016–1051, 2010.

[19] Y. Jiang and J. Ma, "High-order finite element methods for time-fractional partial differential equations," *Journal of Computational and Applied Mathematics*, vol. 235, no. 11, pp. 3285–3290, 2011.

[20] N. J. Ford, J. Xiao, and Y. Yan, "A finite element method for time fractional partial differential equations," *Fractional Calculus and Applied Analysis*, vol. 14, no. 3, pp. 454–474, 2011.

[21] Y. S. Park and J. J. Baik, "Analytical solution of the advection-diffusion equation for a ground-level finite area source," *Atmospheric Environment*, vol. 42, no. 40, pp. 9063–9069, 2008.

[22] A. I. Saichev and G. M. Zaslavsky, "Fractional kinetic equations: solutions and applications," *Chaos*, vol. 7, no. 4, pp. 753–764, 1997.

[23] G. M. Zaslavsky, "Chaos, fractional kinetics, and anomalous transport," *Physics Reports*, vol. 371, no. 6, pp. 461–580, 2002.

[24] M. Ciesielski and J. Leszczynski, "Numerical solutions of a boundary value problem for the anomalous diffusion equation with the Riesz fractional derivative," in *Proceedings of the 16th International Conference on Computer Methods in Mechanics*, pp. 21–24, 2005.

[25] Q. Yang, F. Liu, and I. Turner, "Numerical methods for fractional partial differential equations with Riesz space fractional derivatives," *Applied Mathematical Modelling*, vol. 34, no. 1, pp. 200–218, 2010.

[26] S. Shen, F. Liu, V. Anh, and I. Turner, "The fundamental solution and numerical solution of the Riesz fractional advection-dispersion equation," *IMA Journal of Applied Mathematics*, vol. 73, no. 6, pp. 850–872, 2008.

[27] S. Shen, F. Liu, and V. Anh, "Numerical approximations and solution techniques for the space-time Riesz-Caputo fractional advection-diffusion equation," *Numerical Algorithms*, vol. 56, no. 3, pp. 383–403, 2011.

[28] K. Diethelm and N. J. Ford, "Numerical solution of the Bagley-Torvik equation," *BIT Numerical Mathematics*, vol. 42, no. 3, pp. 490–507, 2002.

[29] V. S. Erturk, S. Momani, and Z. Odibat, "Application of generalized differential transform method to multi-order fractional differential equations," *Communications in Nonlinear Science and Numerical Simulation*, vol. 13, no. 8, pp. 1642–1654, 2008.

[30] K. Diethelm and N. J. Ford, "Multi-order fractional differential equations and their numerical solution," *Applied Mathematics and Computation*, vol. 154, no. 3, pp. 621–640, 2004.

[31] E. H. Doha, A. H. Bhrawy, and S. S. Ezz-Eldien, "Efficient Chebyshev spectral methods for solving multi-term fractional orders differential equations," *Applied Mathematical Modelling*, vol. 35, no. 12, pp. 5662–5672, 2011.

[32] S. Momani, "A numerical scheme for the solution of multi-order fractional differential equations," *Applied Mathematics and Computation*, vol. 182, no. 1, pp. 761–770, 2006.

[33] V. Daftardar-Gejji and S. Bhalekar, "Boundary value problems for multi-term fractional differential equations," *Journal of Mathematical Analysis and Applications*, vol. 345, no. 2, pp. 754–765, 2008.

[34] A. Golbabai and K. Sayevand, "Analytical modelling of fractional advection-dispersion equation defined in a bounded space domain," *Mathematical and Computer Modelling*, vol. 53, no. 9-10, pp. 1708–1718, 2011.

[35] H. Jiang, F. Liu, I. Turner, and K. Burrage, "Analytical solutions for the multi-term time-fractional diffusion-wave/diffusion equations in a finite domain," *Computers & Mathematics with Applications*, vol. 64, no. 10, pp. 3377–3388, 2012.

[36] F. Liu, M. M. Meerschaert, R. J. McGough, P. Zhuang, and Q. Liu, "Numerical methods for solving the multi-term time-fractional wave-diffusion equation," *Fractional Calculus and Applied Analysis*, vol. 16, no. 1, pp. 9–25, 2013.

[37] H. Jiang, F. Liu, I. Turner, and K. Burrage, "Analytical solutions for the multi-term time-space Caputo-Riesz fractional advection-diffusion equations on a finite domain," *Journal of Mathematical Analysis and Applications*, vol. 389, no. 2, pp. 1117–1127, 2012.

[38] K. Diethelm, "Generalized compound quadrature formulae for finite-part integrals," *IMA Journal of Numerical Analysis*, vol. 17, no. 3, pp. 479–493, 1997.

[39] K. Diethelm, "An algorithm for the numerical solution of differential equations of fractional order," *Electronic Transactions on Numerical Analysis*, vol. 5, no. 1, pp. 1–6, 1997.

[40] R. Gorenflo and F. Mainardi, "Random walk models for space-fractional diffusion processes," *Fractional Calculus & Applied Analysis*, vol. 1, no. 2, pp. 167–191, 1998.