Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 864865, 11 pages
http://dx.doi.org/10.1155/2013/864865

Research Article

A Fourth-Order Block-Grid Method for Solving Laplace's Equation on a Staircase Polygon with Boundary Functions in $C^{k,\lambda}$

A. A. Dosiyev and S. Cival Buranay

Department of Mathematics, Eastern Mediterranean University, Gazimagusa, North Cyprus, Mersin 10, Turkey

Correspondence should be addressed to A. A. Dosiyev; adiguzel.dosiyev@emu.edu.tr

Received 30 April 2013; Accepted 11 May 2013

Academic Editor: Allaberen Ashyralyev

The integral representations of the solution around the vertices of the interior reentered angles (on the "singular" parts) are approximated by the composite midpoint rule when the boundary functions are from $C^{4,\lambda}$, $0 < \lambda < 1$. These approximations are connected with the 9-point approximation of Laplace's equation on each rectangular grid on the "nonsingular" part of the polygon by the fourth-order gluing operator. It is proved that the uniform error is of order $O(h^4 + \varepsilon)$, where $\varepsilon > 0$ and h is the mesh step. For the p-order derivatives ($p = 0, 1, \ldots$) of the difference between the approximate and the exact solutions, in each " singular" part $O((h^4 + \varepsilon)r_j^{1/\alpha_j - p})$ order is obtained; here r_j is the distance from the current point to the vertex in question and $\alpha_j\pi$ is the value of the interior angle of the jth vertex. Numerical results are given in the last section to support the theoretical results.

1. Introduction

In the last two decades, among different approaches to solve the elliptic boundary value problems with singularities, a special emphasis has been placed on the construction of combined methods, in which differential properties of the solution in different parts of the domain are used (see [1, 2], and references therein).

In [2–7], a new combined difference-analytical method, called the block-grid method (BGM), is proposed for the solution of the Laplace equation on polygons, when the boundary functions on the sides causing the singular vertices are given as algebraic polynomials of the arc length. In the BGM, the given polygon is covered by a finite number of overlapping sectors around the singular vertices ("singular" parts) and rectangles for the part of the polygon which lies at a positive distance from these vertices ("nonsingular" part). The special integral representation in each "singular" part is approximated, and they are connected by the appropriate order gluing operator with the finite difference equations used in the "nonsingular" part of the polygon.

In [8, 9], the restriction on the boundary functions to be algebraic polynomials on the sides of the polygon causing the singular vertices in the BGM was removed. It was assumed that the boundary function on each side of the polygon is given from the Hölder classes $C^{k,\lambda}$, $0 < \lambda < 1$, and on the "nonsingular" part the 5-point scheme is used when $k = 2$ [8] and the 9-point scheme is used when $k = 6$ [9]. For the 5-point scheme a simple linear interpolation with 4 points is used. For the 9-point scheme an interpolation with 31 points is used to construct a gluing operator connecting the subsystems. Moreover, to connect the quadrature nodes which are at a distance of less than $4h$ from boundary of the polygon, a special representation of the harmonic function through the integrals of Poisson type for a half plane is used (see [9]).

In this paper the BGM is developed for the Dirichlet problem when the boundary function on each side of the polygon is from $C^{4,\lambda}$, by using the 9-point scheme on the "nonsingular" part with 16-point gluing operator for all quadrature nodes, including those near the boundary. The paper is organized as follows: in Section 2, the boundary value problem and the integral representations of the exact solution in each "singular" part are given. In Section 3, to support the aim of the paper, a Dirichlet problem on the rectangle for the known exact solution from $C^{k,\lambda}$, $k = 3, 4$, is solved using the 9-point scheme and the numerical results are illustrated. In Section 4, the system of block-grid equations and the convergence theorems are given. In Section 5 a highly accurate approximation of the coefficient of the leading singular term of the exact solution (stress intensity factor) is given. In Section 6 the method is illustrated for solving the problem in L-shaped polygon with the corner singularity. The conclusions are summarized in Section 7.

2. Dirichlet Problem on a Staircase Polygon

Let G be an open simply connected polygon, γ_j, $j = 1, 2, \ldots, N$, its sides, including the ends, enumerated counterclockwise, $\gamma = \gamma_1 \cup \cdots \cup \gamma_N$ the boundary of G, and $\alpha_j\pi$, $(\alpha_j = 1/2, 1, 3/2, 2)$, the interior angle formed by the sides γ_{j-1} and γ_j, $(\gamma_0 = \gamma_N)$. Denote by $A_j = \gamma_{j-1} \cap \gamma_j$ the vertex of the jth angle and by r_j, θ_j a polar system of coordinates with a pole in A_j, where the angle θ_j is taken counterclockwise from the side γ_j.

We consider the boundary value problem

$$\Delta u = 0 \quad \text{on } G, \qquad u = \varphi_j(s) \quad \text{on } \gamma_j, \quad 1 \le j \le N, \tag{1}$$

where $\Delta \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$, φ_j is a given function on γ_j of the arc length s taken along γ, and $\varphi_j \in C^{4,\lambda}(\gamma_j)$, $0 < \lambda < 1$; that is, φ_j has the fourth-order derivative on γ_j, which satisfies a Hölder condition with exponent λ.

At some vertices A_j, $(s = s_j)$ for $\alpha_j = 1/2$ the conjugation conditions

$$\varphi_{j-1}^{(2q)}(s_j) = (-1)^q \varphi_j^{(2q)}(s_j), \quad q = 0, 1 \tag{2}$$

are fulfilled. For the remaining vertices A_j, the values of φ_{j-1} and φ_j at A_j might be different. Let E be the set of all j, $(1 \le j \le N)$ for which $\alpha_j \ne 1/2$ or $\alpha_j = 1/2$, but (2) is not fulfilled. In the neighborhood of A_j, $j \in E$, we construct two fixed block sectors $T_j^i = T_j(r_{ji}) \subset G$, $i = 1, 2$, where $0 < r_{j2} < r_{j1} < \min\{s_{j+1} - s_j, s_j - s_{j-1}\}$, $T_j(r) = \{(r_j, \theta_j) : 0 < r_j < r,\ 0 < \theta_j < \alpha_j\pi\}$.

Let (see [10])

$$\begin{aligned} \varphi_{j0}(t) &= \varphi_j(s_j + t) - \varphi_j(s_j), \\ \varphi_{j1}(t) &= \varphi_{j-1}(s_j - t) - \varphi_{j-1}(s_j), \end{aligned} \tag{3}$$

$$\begin{aligned} Q_j(r_j, \theta_j) &= \varphi_j(s_j) + \frac{(\varphi_{j-1}(s_j) - \varphi_j(s_j))\theta_j}{\alpha_j\pi} \\ &\quad + \frac{1}{\pi}\sum_{k=0}^{1}\int_0^{\sigma_{jk}} \frac{\tilde{y}_j \varphi_{jk}(t^{\alpha_j})\,dt}{(t - (-1)^k \tilde{x}_j)^2 + \tilde{y}_j^2}, \end{aligned} \tag{4}$$

where

$$\begin{gathered} \tilde{x}_j = r_j^{1/\alpha_j} \cos\left(\frac{\theta_j}{\alpha_j}\right), \qquad \tilde{y}_j = r_j^{1/\alpha_j} \sin\left(\frac{\theta_j}{\alpha_j}\right), \\ \sigma_{jk} = \left|s_{j+1-k} - s_{j-k}\right|^{1/\alpha_j}. \end{gathered} \tag{5}$$

The function $Q_j(r_j, \theta_j)$ is harmonic on T_j^1 and satisfies the boundary conditions in (1) on $\gamma_{j-1} \cap \overline{T}_j^1$ and $\gamma_j \cap \overline{T}_j^1$, $j \in E$, except for the point A_j when $\varphi_{j-1}(s_j) \ne \varphi_j(s_j)$.

We formally set the value of $Q_j(r_j, \theta_j)$ and the solution u of the problem (1) at the vertex A_j equal to $(\varphi_{j-1}(s_j) + \varphi_j(s_j))/2$, $j \in E$.

Let

$$\begin{aligned} &R_j(r, \theta, \eta) \\ &\quad = \frac{1}{\alpha_j}\sum_{k=0}^{1}(-1)^k R\left(\left(\frac{r}{r_{j2}}\right)^{1/\alpha_j}, \frac{\theta}{\alpha_j}, (-1)^k\frac{\eta}{\alpha_j}\right), \\ &\qquad j \in E, \end{aligned} \tag{6}$$

where

$$R(r, \theta, \eta) = \frac{1 - r^2}{2\pi(1 - 2r\cos(\theta - \eta) + r^2)} \tag{7}$$

is the kernel of the Poisson integral for a unit circle.

Lemma 1 (see [10]). *The solution u of the boundary value problem* (1) *can be represented on $\overline{T}_j^2 \setminus V_j$, $j \in E$, in the form*

$$\begin{aligned} u(r_j, \theta_j) &= Q_j(r_j, \theta_j) \\ &\quad + \int_0^{\alpha_j\pi} R_j(r_j, \theta_j, \eta)(u(r_{j2}, \eta) - Q_j(r_{j2}, \eta))\,d\eta, \end{aligned} \tag{8}$$

where V_j is the curvilinear part of the boundary of T_j^2, and $Q_j(r_j, \theta_j)$ is the function defined by (4).

3. 9-Point Solution on Rectangles

Let $\Pi = \{(x, y) : 0 < x < a,\ 0 < y < b\}$ be a rectangle, with a/b being rational, γ_j, $j = 1, 2, 3, 4$ the sides, including the ends, enumerated counterclockwise, starting from the left side $(\gamma_0 \equiv \gamma_4, \gamma_5 \equiv \gamma_1)$, and $\gamma = \cup_{j=1}^4 \gamma_j$ the boundary of Π.

We consider the boundary value problem

$$\begin{gathered} \Delta u = 0 \quad \text{on } \Pi, \\ u = \varphi_j \quad \text{on } \gamma_j, \ j = 1, 2, 3, 4, \end{gathered} \tag{9}$$

where φ_j is the given function on γ_j.

Definition 2. One says that the solution u of the problem (9) belongs to $\widetilde{C}^{4,\lambda}(\overline{\Pi})$ if

$$\varphi_j \in C^{4,\lambda}(\gamma_j), \quad 0 < \lambda < 1, \ j = 1, 2, 3, 4, \tag{10}$$

and at the vertices $A_j = \gamma_{j-1} \cap \gamma_j$ the conjugation conditions

$$\varphi_j^{(2q)} = (-1)^q \varphi_{j-1}^{(2q)}, \quad q = 0, 1 \tag{11}$$

are satisfied.

Remark 3. From Theorem 3.1 in [11] it follows that the class of functions $\widetilde{C}^{4,\lambda}(\overline{\Pi})$ is wider than $C^{4,\lambda}(\overline{\Pi})$.

Let $h > 0$, with $a/h \geq 2$, $b/h \geq 2$ integers. We assign to Π^h a square net on Π, with step h, obtained with the lines $, y = 0, h, 2h, \ldots$. Let γ_j^h be a set of nodes on the interior of γ_j and let

$$\dot{\gamma}_j^h = \gamma_j \cap \gamma_{j+1}, \qquad \gamma^h = \cup_{j=1}^4 \left(\gamma_j^h \cup \dot{\gamma}_j^h\right), \\ \overline{\Pi}^h = \Pi^h \cup \gamma^h. \tag{12}$$

We consider the system of finite difference equations

$$u_h = B u_h \quad \text{on } \Pi^h, \\ u_h = \varphi_j \quad \text{on } \gamma_j^h, \ j = 1, 2, 3, 4, \tag{13}$$

where

$$Bu(x,y) \\ \equiv \frac{(u(x+h,y) + u(x,y+h) + u(x-h,y) + u(x,y-h))}{5} \\ + \Big(((u(x+h,y+h) + u(x-h,y+h) \\ + u(x-h,y-h) + u(x+h,y-h))) \\ \times 20^{-1} \Big). \tag{14}$$

On the basis of the maximum principle the unique solvability of the system of finite difference equations (13) follows (see [12, Chapter 4]).

Everywhere below we will denote constants which are independent of h and of the cofactors on their right by $c, c_0, c_1, \ldots$, generally using the same notation for different constants for simplicity.

Theorem 4. *Let u be the solution of problem* (9). *If $u \in \widetilde{C}^{4,\lambda}(\overline{\Pi})$, then*

$$\max_{\overline{\Pi}^h} |u_h - u| \leq c h^4, \tag{15}$$

where u_h is the solution of the system (13).

Proof. For the proof of this theorem see [13]. □

Let $\Pi' = \{(x,y) : -0.25 < x < 0.25, \ 0 < y < 1\}$ and let γ' be the boundary of Π'. We consider the Dirichlet problem

$$\Delta u = 0 \quad \text{on } \Pi', \\ u = v \quad \text{on } \gamma', \tag{16}$$

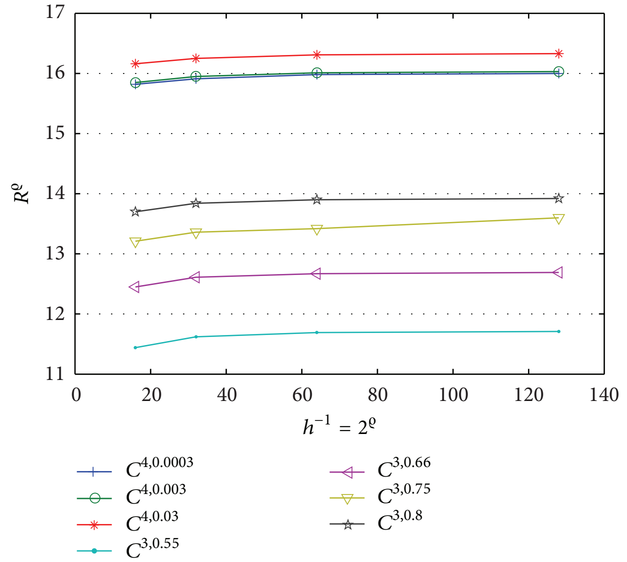

Figure 1: Dependence of the approximate solutions for the boundary functions from $C^{k,\lambda}$.

where $v = r^{k+\lambda}\cos(k+\lambda)\theta, r = \sqrt{x^2+y^2}$, $0 < \lambda < 1$, is the exact solution of this problem, which is from $C^{k,\lambda}(\overline{\Pi}')$.

We solve the problem (16) by approximating 9-point scheme when $k = 3, 4$ for the different values of λ.

In Figure 1, the order of numerical convergence

$$\mathfrak{R}_{\Pi^h}^{\varrho} = \frac{\max_{\Pi^h} |u_{2^{-\varrho}} - u|}{\max_{\Pi^h} |u_{2^{-(\varrho+1)}} - u|} \tag{17}$$

of the 9-point solution u_h, for different $h = 2^{-\varrho}$ and $\varrho = 4, 5, 6, 7$, is demonstrated. These results show that the order of numerical convergence, when the exact solution $u \in C^{k,\lambda}(\overline{\Pi})$, depends on k and λ and is $O(h^4)$ when $k = 4$, which supports estimation (15). Moreover, this dependence also requires the use of fourth-order gluing operator for all quadrature nodes in the construction of the system of block-grid equations, when the given boundary functions are from the Hölder classes $C^{4,\lambda}$.

4. System of Block-Grid Equations

In addition to the sectors T_j^1 and T_j^2 (see Section 2) in the neighborhood of each vertex $A_j, j \in E$ of the polygon G, we construct two more sectors T_j^3 and T_j^4, where $0 < r_{j4} < r_{j3} < r_{j2}, r_{j3} = (r_{j2} + r_{j4})/2$ and $T_k^3 \cap T_l^3 = \emptyset, k \neq l, \ k, l \in E$, and let $G_T = G \setminus (\cup_{j \in E} T_j^4)$.

We cover the given solution domain (a staircase polygon) by the finite number of sectors $T_j^1, j \in E$, and rectangles $\Pi_k \subset G_T, k = 1, 2, \ldots, M$, as is shown in Figure 2, for the case of L-shaped polygon, where $j = 1, M = 4$ (see also [2]). It is assumed that for the sides a_{1k} and a_{2k} of Π_k the quotient a_{1k}/a_{2k} is rational and $G = (\cup_{k=1}^M \Pi_k) \cup (\cup_{j \in E} T_j^3)$. Let η_k be the boundary of the rectangle Π_k, let V_j be

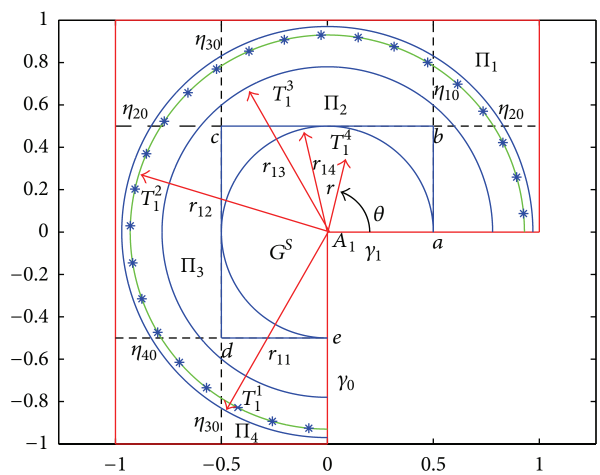

Figure 2: Description of BGM for the L-shaped domain.

the curvilinear part of the boundary of the sector T_j^2, and let $t_{kj} = \eta_k \cap \overline{T}_j^3$. We choose a natural number n and define the quantities $n(j) = \max\{4, [\alpha_j n]\}$, $\beta_j = \alpha_j \pi / n(j)$, and $\theta_j^m = (m - 1/2)\beta_j$, $j \in E$, $1 \le m \le n(j)$. On the arc V_j we take the points (r_{j2}, θ_j^m), $1 \le m \le n(j)$, and denote the set of these points by V_j^n. We introduce the parameter $h \in (0, \varkappa_0/4]$, where $\varkappa_0$ is a gluing depth of the rectangles Π_k, $k = 1, 2, \ldots, M$, and define a square grid on Π_k, $k = 1, 2, \ldots, M$, with maximal possible step $h_k \le \min\{h, \min\{a_{1k}, a_{2k}\}/6\}$ such that the boundary η_k lies entirely on the grid lines. Let Π_k^h be the set of grid nodes on Π_k, let η_k^h be the set of nodes on η_k, and let $\overline{\Pi}_k^h = \Pi_k^h \cup \eta_k^h$. We denote the set of nodes on the closure of $\eta_k \cap G_T$ by η_{k0}^h, the set of nodes on t_{kj} by t_{kj}^h, and the set of remaining nodes on η_k by η_{k1}^h.

Let

$$\omega^{h,n} = \left(\cup_{k=1}^{M} \eta_{k0}^h\right) \cup \left(\cup_{j\in E} V_j^n\right),$$
$$\overline{G}_T^{h,n} = \omega^{h,n} \cup \left(\cup_{k=1}^{M} \overline{\Pi}_k^h\right). \tag{18}$$

Let $\varphi = \{\varphi_j\}_{j=1}^N$, where $\varphi_j \in C^{4,\lambda}(\gamma_j)$, $0 < \lambda < 1$, is the given function in (1). We use the matching operator S^4 at the points of the set $\omega^{h,n}$ constructed in [14]. The value of $S^4(u_h, \varphi)$ at the point $P \in \omega^{h,n}$ is expressed linearly in terms of the values of u_h at the points P_k of the grid constructed on $\Pi_{k(P)}$, $(P \in \Pi_{k(P)})$ some part of whose boundary located in G is the maximum distance away from P, and in terms of the boundary values of $\varphi^{(m)}$, $m = 0, 1, 2, 3$ at a fixed number of points. Moreover $S^4(u_h, 0)$ has the representation

$$S^4(u_h, 0) = \sum_{0 \le l \le 15} \xi_l u_{h,l}, \tag{19}$$

where $u_{h,k} = u_h(P_k)$,

$$\xi_l \ge 0, \qquad \sum_{0 \le l \le 15} \xi_l = 1, \tag{20}$$

$$u - S^4(u, \varphi) = O\left(h^4\right). \tag{21}$$

Let $\omega_I^{h,n} \subset \omega^{h,n}$ be the set of such points $P \in \omega^{h,n}$, for which all points P_l in expression (19) are in $\cup_{k=1}^{M} \overline{\Pi}_k^h$. If some points P_l in (19) emerge through the side γ_m, then the set of such points P is denoted by $\omega_D^{h,n}$. According to the construction of S^4 in [14], the expression $S^4(u_h, \varphi)$ at each point $P \in \omega^{h,n} = \omega_I^{h,n} \cup \omega_D^{h,n}$ can be expressed as follows:

$$S^4(u_h, \varphi) = \begin{cases} S^4 u_h, & P \in \omega_I^{h,n}, \\ S^4\left(u_h - \displaystyle\sum_{k=0}^{3} a_k \operatorname{Re} z^k\right) + \left(\displaystyle\sum_{k=0}^{3} a_k \operatorname{Re} z^k\right)_P, & P \in \omega_D^{h,n}, \end{cases} \tag{22}$$

where

$$a_k = \frac{1}{k!} \left.\frac{d^k \varphi_m(s)}{ds^k}\right|_{s=s_P}, \quad k = 0, 1, 2, 3 \tag{23}$$

and s_P corresponds to such a point $Q \in \gamma_m$ for which the line PQ is perpendicular to γ_m.

Let

$$Q_j = Q_j\left(r_j, \theta_j\right), \qquad Q_{j2}^q = Q_j\left(r_{j2}, \theta_j^q\right). \tag{24}$$

The quantities in (24) are given by (4) and (5), which contain integrals that have not been computed exactly in the general case. Assume that approximate values Q_j^ε and $Q_{j2}^{q\varepsilon}$ of the quantities in (24) are known with accuracy $\varepsilon > 0$; that is,

$$\left|Q_j^\varepsilon - Q_j\right| \le c_1 \varepsilon, \qquad \left|Q_{j2}^{q\varepsilon} - Q_{j2}^q\right| \le c_2 \varepsilon, \tag{25}$$

where $j \in E$, $1 \le q \le n(j)$, and c_1, c_2 are constants independent of ε.

Consider the system of linear algebraic equations

$$\begin{aligned} &u_h^\varepsilon = B u_h^\varepsilon \quad \text{on } \Pi_k^h, \\ &u_h^\varepsilon = \varphi_m \quad \text{on } \eta_{k1}^h \cap \gamma_m, \\ &u_h^\varepsilon\left(r_j, \theta_j\right) \\ &\quad = Q_j^\varepsilon + \beta_j \sum_{q=1}^{n(j)} \left(u_h^\varepsilon\left(r_{j2}, \theta_j^q\right) - Q_{j2}^{q\varepsilon}\right) \\ &\qquad \times R_j\left(r_j, \theta_j, \theta_j^q\right) \quad \text{on } \left(r_j, \theta_j\right) \in t_{kj}^h, \\ &u_h^\varepsilon = S^4 u_h^\varepsilon \quad \text{on } \omega^{h,n}, \end{aligned} \tag{26}$$

where $1 \le k \le M$, $1 \le m \le N$, and $j \in E$.

Table 1: The order of convergence in the "nonsingular" part when $h = 2^{-\varrho}$ and $\varepsilon = 5 \times 10^{-13}$.

$(2^{-\varrho}, n)$	$\|\zeta_h^\varepsilon\|_{G^{NS}}$	$\mathfrak{R}^{\varrho}_{G^{NS}}$
$(2^{-4}, 60)$	1.609×10^{-8}	15.577
$(2^{-5}, 170)$	1.033×10^{-9}	
$(2^{-5}, 130)$	1.191×10^{-9}	16.690
$(2^{-6}, 150)$	7.136×10^{-11}	
$(2^{-5}, 140)$	1.136×10^{-9}	16.259
$(2^{-6}, 170)$	6.991×10^{-11}	
$(2^{-6}, 100)$	2.169×10^{-10}	17.096
$(2^{-7}, 130)$	1.269×10^{-11}	

Table 2: The order of convergence in the "singular" part when $h = 2^{-\varrho}$ and $\varepsilon = 5 \times 10^{-13}$.

$(2^{-\varrho}, n)$	$\|\zeta_h^\varepsilon\|_{G^{S}}$	$\mathfrak{R}^{\varrho}_{G^{S}}$
$(2^{-4}, 100)$	1.931×10^{-8}	16.078
$(2^{-5}, 150)$	1.182×10^{-9}	
$(2^{-5}, 130)$	1.294×10^{-9}	16.789
$(2^{-6}, 150)$	7.708×10^{-11}	
$(2^{-5}, 140)$	1.312×10^{-9}	17.967
$(2^{-6}, 170)$	7.304×10^{-11}	
$(2^{-6}, 100)$	2.389×10^{-10}	18.164
$(2^{-7}, 130)$	1.315×10^{-11}	

Table 3: The minimum errors of the solution over the pairs (h^{-1}, n) in maximum norm when $\varepsilon = 5 \times 10^{-13}$.

(h^{-1}, n)	$\|\zeta_h^\varepsilon\|_{G^{NS}}$	$\|\zeta_h^\varepsilon\|_{G^{S}}$	Iteration
(16, 70)	1.139×10^{-8}	1.572×10^{-8}	22
(32, 170)	1.033×10^{-9}	1.184×10^{-9}	23
(64, 170)	6.990×10^{-11}	7.304×10^{-11}	24
(128, 200)	8.628×10^{-12}	8.833×10^{-12}	25

Let $T_j^* = T_j(r_j^*)$ be the sector, where $r_j^* = (r_{j2} + r_{j3})/2$, $j \in E$, and let $u_h^\varepsilon(r_{j2}, \theta_j^q)$, $1 \le q \le n(j)$, $j \in E$, be the solution values of the system (26) on V_j^h (at the quadrature nodes). The function

$$U_h^\varepsilon\left(r_j, \theta_j\right) = Q_j\left(r_j, \theta_j\right) + \beta_j \sum_{q=1}^{n(j)} R_j\left(r_j, \theta_j, \theta_j^q\right)\left(u_h^\varepsilon\left(r_{j2}, \theta_j^q\right) - Q_{j2}^{q\varepsilon}\right), \tag{27}$$

defined on T_j^*, is called an approximate solution of the problem (1) on the closed block $\overline{T}_j^3$, $j \in E$.

Definition 5. The system (26) and (27) is called the system of block-grid equations.

Theorem 6. *There is a natural number n_0, such that for all $n \ge n_0$ and for any $\varepsilon > 0$ the system* (26) *has a unique solution.*

Proof. From the estimation (2.29) in [15] follows the existence of the positive constants n_0 and σ, such that for all $n \ge n_0$

$$\max_{(r_j, \theta_j) \in \overline{T}_j^3} \beta_j \sum_{q=1}^{n(j)} R_j\left(r_j, \theta_j, \theta_j^q\right) \le \sigma < 1. \tag{28}$$

The proof is obtained on the basis of principle of maximum by taking into account (14), (19), (20), and (28). □

Theorem 7. *There exists a natural number n_0, such that for all*

$$n \ge \max\left\{n_0, \left[\ln^{1+\varkappa} h^{-1}\right] + 1\right\}, \tag{29}$$

where $\varkappa > 0$ is a fixed number, and for any $\varepsilon > 0$ the following inequalities are valid:

$$\max_{\overline{G}_T^{h,n}} \left|u_h^\varepsilon - u\right| \le c\left(h^4 + \varepsilon\right), \tag{30}$$

$$\left|\frac{\partial^p}{\partial x^{p-q} \partial y^q}\left(U_h^\varepsilon\left(r_j, \theta_j\right) - u\left(r_j, \theta_j\right)\right)\right| \le c_p\left(h^4 + \varepsilon\right) \quad \text{on } \overline{T}_j^3, \tag{31}$$

for integer $1/\alpha_j$ when $p \ge 1/\alpha_j$,

$$\left|\frac{\partial^p}{\partial x^{p-q} \partial y^q}\left(U_h^\varepsilon\left(r_j, \theta_j\right) - u\left(r_j, \theta_j\right)\right)\right| \le \frac{c_p\left(h^4 + \varepsilon\right)}{r^{p-1/\alpha_j}} \quad \text{on } \overline{T}_j^3, \tag{32}$$

for any $1/\alpha_j$, if $0 \le p < 1/\alpha_j$,

$$\left|\frac{\partial^p}{\partial x^{p-q} \partial y^q}\left(U_h^\varepsilon\left(r_j, \theta_j\right) - u\left(r_j, \theta_j\right)\right)\right| \le \frac{c_p\left(h^4 + \varepsilon\right)}{r^{p-1/\alpha_j}} \quad \text{on } \overline{T}_j^3 \setminus A_j, \tag{33}$$

for noninteger $1/\alpha_j$, when $p > 1/\alpha_j$. Everywhere $0 \le q \le p$, u is the exact solution of the problem (1) *and $U_h^\varepsilon(r_j, \theta_j)$ is defined by formula* (27).

Proof. Let

$$\xi_h^\varepsilon = u_h^\varepsilon - u, \tag{34}$$

where u_h^ε is a solution of system (26) and u is the trace on $\overline{G}_T^{h,n}$ of the solution of (1). On the basis of (1), (26), and (34) the error ξ_h^ε satisfies the system of difference equations

$$\begin{aligned} &\xi_h^\varepsilon = B\xi_h^\varepsilon + r_h^1 \quad \text{on } \Pi_k^h, \\ &\xi_h^\varepsilon = 0 \quad \text{on } \eta_{k1}^h, \\ &\xi_h^\varepsilon\left(r_j, \theta_j\right) = \beta_j \sum_{q=1}^{n(j)} \xi_h^\varepsilon\left(r_{j2}, \theta_j^q\right) R_j\left(r_j, \theta_j, \theta_j^q\right) + r_{jh}^2, \quad \left(r_j, \theta_j\right) \in t_{kj}^h, \\ &\xi_h^\varepsilon = S^4 \xi_h^\varepsilon + r_h^3 \quad \text{on } \omega^{h,n}, \end{aligned} \tag{35}$$

Table 4: In $G^S \cap r \geq 0.2$, the minimum errors of the derivatives over the pairs (h^{-1}, n) in maximum norm when $\varepsilon = 5 \times 10^{-13}$.

(h^{-1}, n)	$\mathrm{Max}_{G^S \cap \{r\geq 0.2\}} r^{1/3} \left\| \frac{\partial U_h^\varepsilon}{\partial x} - \frac{\partial u}{\partial x} \right\|$	$\mathrm{Max}_{G^S \cap \{r\geq 0.2\}} r^{1/3} \left\| \frac{\partial U_h^\varepsilon}{\partial y} - \frac{\partial u}{\partial y} \right\|$
$(16, 70)$	3.895×10^{-7}	3.895×10^{-7}
$(32, 170)$	4.627×10^{-8}	4.627×10^{-8}
$(64, 170)$	1.124×10^{-9}	3.125×10^{-9}
$(128, 200)$	2.214×10^{-10}	2.233×10^{-10}

Table 5: In G^S, the minimum errors of the derivatives over the pairs (h^{-1}, n) in maximum norm when $\varepsilon = 5 \times 10^{-13}$.

(h^{-1}, n)	$\mathrm{Max}_{G^S} r^{1/3} \left\| \frac{\partial U_h^\varepsilon}{\partial x} - \frac{\partial u}{\partial x} \right\|$	$\mathrm{Max}_{G^S} r^{1/3} \left\| \frac{\partial U_h^\varepsilon}{\partial y} - \frac{\partial u}{\partial y} \right\|$
$(16, 70)$	9.663×10^{-6}	9.663×10^{-6}
$(32, 170)$	9.653×10^{-6}	9.653×10^{-6}
$(64, 170)$	9.649×10^{-6}	9.649×10^{-6}
$(128, 200)$	9.648×10^{-6}	9.648×10^{-6}

where $1 \leq k \leq M,\ j \in E$,

$$r_h^1 = Bu - u \quad \text{on } \cup_{k=1}^{M} \Pi_k^h, \tag{36}$$

$$\begin{aligned} r_{jh}^2 = \beta_j \sum_{q=1}^{n(j)} \left(u\left(r_{j2}, \theta_j^q\right) - Q_{j2}^{q\varepsilon} \right) R_j\left(r_j, \theta_j, \theta_j^q\right) \\ - \left(u - Q_j^\varepsilon\right) \quad \text{on } \cup_{k=1}^{M} \left(\cup_{j\in E} t_{kj}^h\right), \end{aligned} \tag{37}$$

$$\begin{aligned} &r_h^3 \\ &= \begin{cases} S^4 u - u & \text{on } \omega_I^{h,n}, \\ S^4\left(u - \sum_{k=0}^{3} a_k \operatorname{Re} z^k\right) - \left(u - \sum_{k=0}^{3} a_k \operatorname{Re} z^k\right)_P, & P \in \omega_D^{h,n}. \end{cases} \end{aligned} \tag{38}$$

On the basis of estimations (15), (21), (25), and Lemma 1 by analogy to the proof of Theorem 4.3 in [9] the proof of inequality (30) follows.

The function $U_h^\varepsilon(r_j, \theta_j)$ given by formula (27), defined on the closed sector $\overline{T}_j^*$, $j \in E$, where $r_j^* = (r_{j2} + r_{j3})/2$, and the integral representation (8) of the exact solution of the problem (1) is given on $\overline{T}_j^2 \setminus V_j$, $j \in E$, and then the difference function $\zeta_h^\varepsilon(r_j, \theta_j) = U_h^\varepsilon(r_j, \theta_j) - u(r_j, \theta_j)$ is defined on $\overline{T}_j^*$, $j \in E$ and

$$\zeta_h^\varepsilon\left(r_j^*, 0\right) = \zeta_h^\varepsilon\left(r_j^*, \alpha_j \pi\right) = 0, \quad j \in E. \tag{39}$$

On the basis of Lemma 6.11 from [16], (25), and (28), for $n \geq \max\{n_0, [\ln^{1+\varkappa} h^{-1}] + 1\}$, $\varkappa > 0$ is a fixed number, and we obtain

$$\left|\zeta_h^\varepsilon\left(r_j, \theta_j\right)\right| \leq c\left(h^4 + \varepsilon\right) \quad \text{on } \overline{T}_j^*,\ j \in E. \tag{40}$$

Furthermore, the function $\zeta_h^\varepsilon(r_j, \theta_j)$ continuous on $\overline{T}_j^*$ is a solution of the following Dirichlet problem:

$$\begin{aligned} \Delta \zeta_h^\varepsilon = 0 \quad \text{on } T_j^*, \\ \zeta_h^\varepsilon = 0 \quad \text{on } \gamma_m \cap \overline{T}_j^*,\ m = j-1, j, \\ \zeta_h^\varepsilon\left(r_j^*, \theta_j\right) = U_h^\varepsilon\left(r_j^*, \theta_j\right) - u\left(r_j^*, \theta_j\right), \quad 0 \leq \theta_j \leq \alpha_j \pi. \end{aligned} \tag{41}$$

Since $T_j^3 \subset \overline{T}_j^*$, on the basis of (39) and (40), from Lemma 6.12 in [16], inequalities (31)–(33) of Theorem 7 follow. □

5. Stress Intensity Factor

Let, in the condition $\varphi_j \in C^{4,\lambda}(\gamma_j)$, the exponent λ be such that

$$\left\{\alpha_j (4 + \lambda)\right\} \neq 0, \qquad \left\{2\alpha_j (4 + \lambda)\right\} \neq 0, \tag{42}$$

where $\{\cdot\}$ is the symbol of fractional part. These conditions for the given α_j can be fulfilled by decreasing λ.

On the basis of Section 2 of [11], a solution of the problem (1) can be represented in $\overline{T}_j^*$, $j \in E$, as follows:

$$\begin{aligned} u\left(x_j, y_j\right) = \tilde{u}\left(x_j, y_j\right) + \sum_{k=0}^{4} \mu_k^{(j)} \operatorname{Im}\left\{z^k \ln z\right\} \\ + \sum_{k=1}^{n_{\alpha_j}} \tau_k^{(j)} r_j^{k/\alpha_j} \sin \frac{k\theta_j}{\alpha_j}, \end{aligned} \tag{43}$$

where $n_{\alpha_j} = [\alpha_j(4+\lambda)]$, $[\cdot]$ is the integer part, $z = x_j + iy_j$, $\mu_k^{(j)}$ and $\tau_k^{(j)}$ are some numbers, and $\tilde{u}(x_j, y_j) \in C^{4,\lambda}(T_j^2)$ is the harmonic on T_j^2. By taking $\theta_j = \alpha_j \pi/2$, from the formula (43), it follows that the coefficient $\tau_1^{(j)}$ which is called the stress intensity factor can be represented as

$$\begin{aligned} \tau_1^{(j)} = \lim_{r_j \to 0} \frac{1}{r_j^{1/\alpha_j}} \Bigg(u\left(x_j, y_j\right) - \tilde{u}\left(x_j, y_j\right) \\ - \sum_{k=0}^{4} \mu_k^{(j)} \operatorname{Im}\left\{z^k \ln z\right\} \Bigg). \end{aligned} \tag{44}$$

TABLE 6: The stress intensity factor $\tau_{1,n}^{\varepsilon}$ for $n = 70, 170, 200$ when $\varepsilon = 5 \times 10^{-13}$.

h^{-1}	$\tau_{1,70}^{\varepsilon}$	$\tau_{1,170}^{\varepsilon}$	$\tau_{1,200}^{\varepsilon}$
16	1.000000014856688	1.000000017180415	1.000000017197438
32	1.000000005800844	1.000000001230267	1.000000001236709
64	1.000000004138169	1.000000000073107	1.000000000079938
128	1.000000004053153	1.000000000003531	1.000000000003267

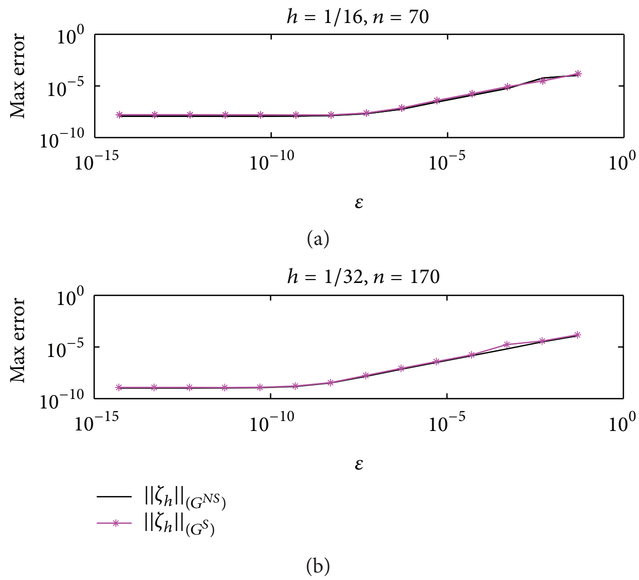

FIGURE 3: Dependence on ε for $h^{-1} = 16, 32$.

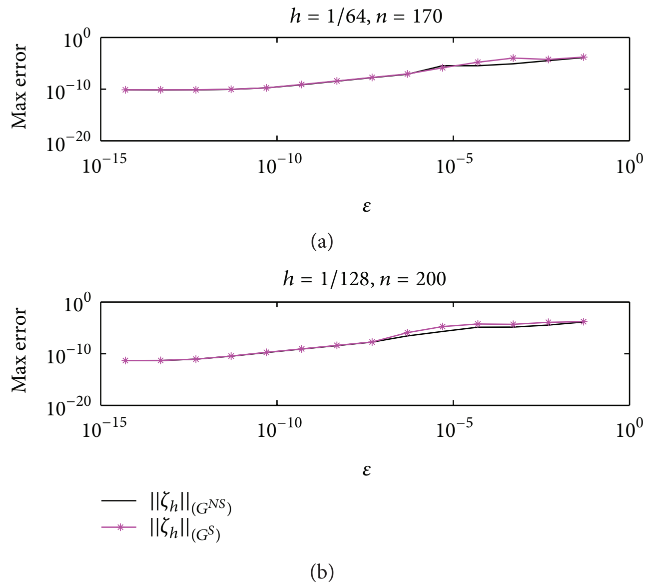

FIGURE 4: Dependence on ε for $h^{-1} = 64, 128$.

From formula (44) it follows that $\tau_1^{(j)}$ can be approximated by

$$\tau_{1,n}^{(j)\varepsilon} = \lim_{r_j \to 0} \frac{1}{r_j^{1/\alpha_j}} \left(U_h^{\varepsilon}(r_j, \theta_j) - \left(\varphi_j(s_j) + (\varphi_{j-1}(s_j) - \varphi_j(s_j)) \frac{\theta_j}{\alpha_j \pi} \right) \right). \tag{45}$$

Using formula (3), (4), and (27) from (45) for the stress intensity factor (see [17]), we obtain the next formula:

$$\tau_{1,n}^{(j)\varepsilon} = \frac{1}{\pi} \int_0^{\sigma_{j0}} \frac{\varphi_{j0}(t^{\alpha_j})\,dt}{t^2} + \frac{1}{\pi} \int_0^{\sigma_{j1}} \frac{\varphi_{j1}(t^{\alpha_j})\,dt}{t^2} + \frac{2}{n(j)\, r_{j2}^{1/\alpha_j}} \sum_{q=1}^{n(j)} \left(u_h^{\varepsilon}(r_{j2}, \theta_j^q) - Q_{j2}^{q\varepsilon} \right) \sin \frac{1}{\alpha_j} \theta_j^q. \tag{46}$$

This formula is obtained for the second-order BGM in [8].

6. Numerical Results

Let G be L-shaped and defined as follows:

$$G = \{(x, y) : -1 < x < 1,\ -1 < y < 1\} \setminus \Omega, \tag{47}$$

where $\Omega = \{(x, y) : 0 \le x \le 1,\ -1 \le y \le 0\}$ and γ is the boundary of G.

We consider the following problem:

$$\Delta u = 0 \quad \text{in } G, \qquad u = v(r, \theta) \quad \text{on } \gamma, \tag{48}$$

where

$$v(r, \theta) = r^{2/3} \sin\left(\frac{2}{3}\theta\right) + 0.0051 r^{16/3} \cos\left(\frac{16}{3}\theta\right) \tag{49}$$

is the exact solution of this problem.

We choose a "singular" part of G as

$$G^S = \{(x, y) : -0.5 < x < 0.5,\ -0.5 < y < 0.5\} \setminus \Omega_1, \tag{50}$$

where $\Omega_1 = \{(x, y) : 0 \le x \le 0.5,\ -0.5 \le y \le 0\}$. Then $G^{NS} = G \setminus G^S$ is a "nonsingular" part of G.

The given domain G is covered by four overlapping rectangles Π_k, $k = 1, \ldots, 4$, and by the block sector T_1^3

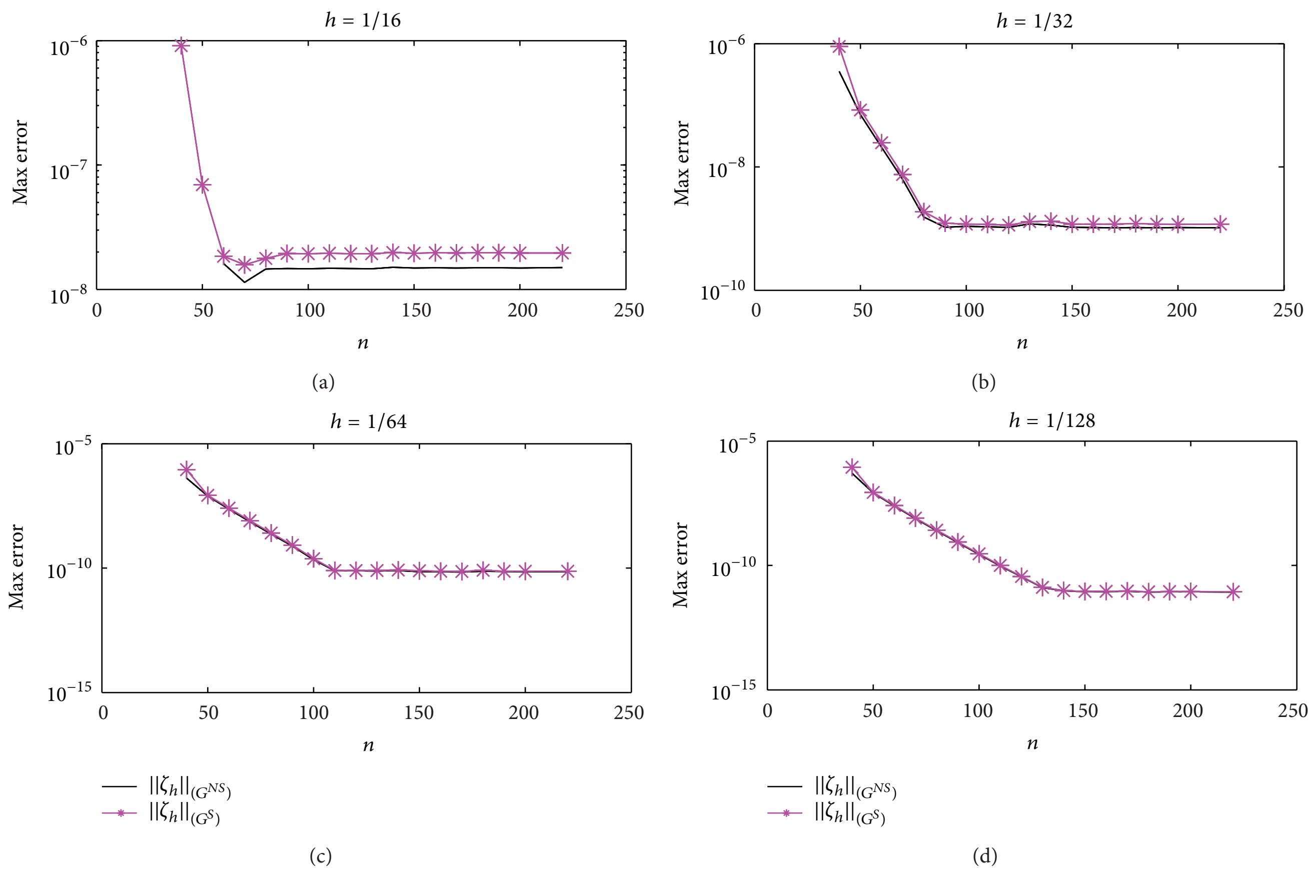

FIGURE 5: Maximum error depending on the number of quadrature nodes n.

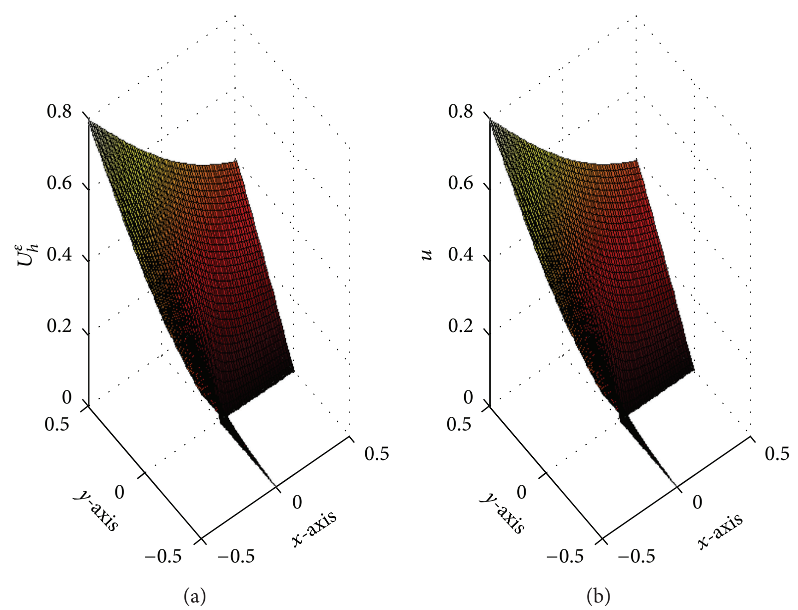

FIGURE 6: The approximate solution U_h^ε and the exact solution u in the "singular" part for $\varepsilon = 5 \times 10^{-13}$.

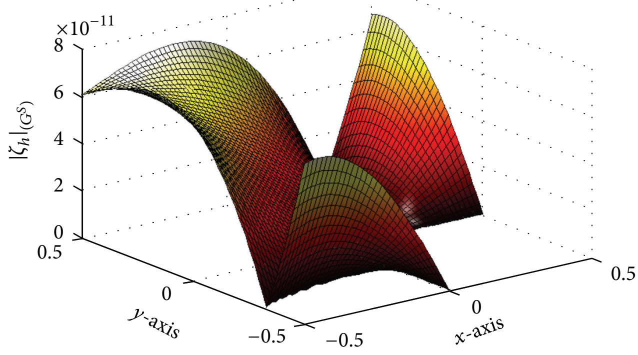

FIGURE 7: The error function in "singular" part when $\varepsilon = 5 \times 10^{-13}$.

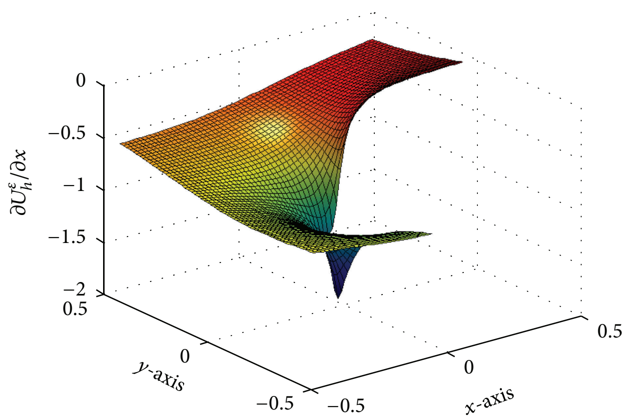

FIGURE 8: $\partial U_h^\varepsilon/\partial_x$ in the "singular" part.

(see Figure 2). For the boundary of G^S on G is the polygonal line $t_1 = abcde$. The radius r_{12} of sector T_1^2 is taken as 0.93. According to (49), the function $Q(r,\theta)$ in (4) is

$$Q(r,\theta) = \frac{0.0051}{\pi}\int_0^1 \frac{\tilde{y}t^8 dt}{(t-\tilde{x})^2+\tilde{y}^2} + \frac{0.0051}{\pi}\int_0^1 \frac{\tilde{y}t^8 dt}{(t-\tilde{x})^2+\tilde{y}^2}, \quad (51)$$

where $\tilde{x} = r^{2/3}\cos(2\theta/3)$ and $\tilde{y} = r^{2/3}\sin(2\theta/3)$. Since we have only one singular point, we omit subindices in (51). We calculate the values $Q^\varepsilon(r_{12},\theta^q)$ and $Q^\varepsilon(r,\theta)$ on the grids t_1^h, with an accuracy of ε using the quadrature formulae proposed in [10].

On the basis of (46) and (51), for the stress intensity factor, we have

$$\tau_{1,n}^\varepsilon = \frac{0.0102}{7\pi} + \frac{2}{n(0.93)^{2/3}}\sum_{q=1}^{n}\left(u_h^\varepsilon\left(0.93,\theta_j^q\right) - Q_{j2}^{q\varepsilon}\right)\sin\frac{2}{3}\theta_j^q. \quad (52)$$

Taking the zero approximation $u_h^{\varepsilon(0)} = 0$, the results of realization of the Schwarz iteration (see [2]) for the solution of the problem (48) are given in Tables 1, 2, 3, and 4. Tables

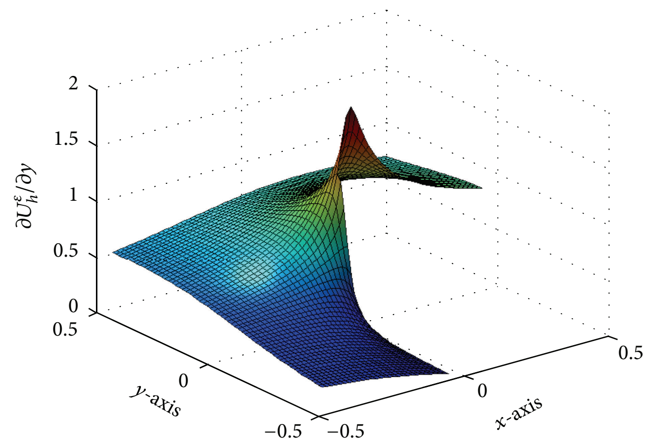

FIGURE 9: $\partial U_h^\varepsilon/\partial_y$ in the "singular" part.

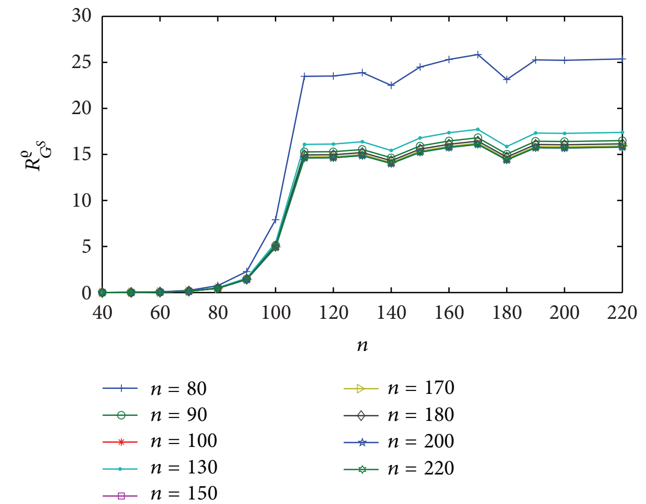

FIGURE 10: $\mathfrak{R}_{G^S}^\varrho$ when $\varrho = 5$ by fixing n for $h^{-1} = 32$ for different n values of $h^{-1} = 64$.

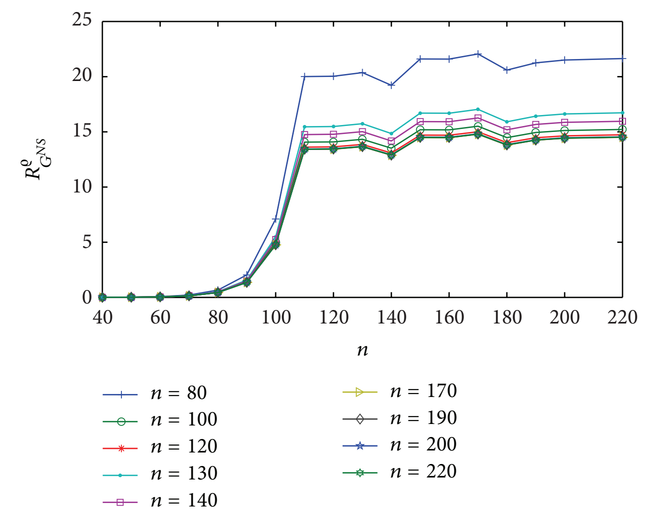

FIGURE 11: $\mathfrak{R}_{G^{NS}}^\varrho$ when $\varrho = 5$ by fixing n for $h^{-1} = 32$ for different n values of $h^{-1} = 64$.

1 and 2 represent the order of convergence. Table 6 shows a highly accurate approximation of the stress intensity factor by the proposed fourth order BGM

$$\mathfrak{R}^{\varrho}_{G^{NS}} = \frac{\max_{G^{NS}} \left|u^{\varepsilon}_{2^{-\varrho}} - u\right|}{\max_{G^{NS}} \left|u^{\varepsilon}_{2^{-(\varrho+1)}} - u\right|} \tag{53}$$

in the "nonsingular" and the order of convergence

$$\mathfrak{R}^{\varrho}_{G^{S}} = \frac{\max_{G^{S}} \left|U^{\varepsilon}_{2^{-\varrho}} - u\right|}{\max_{G^{S}} \left|U^{\varepsilon}_{2^{-(\varrho+1)}} - u\right|} \tag{54}$$

in the "singular" parts of G, respectively, for $\varepsilon = 5 \times 10^{-13}$, where ϱ is a positive integer. In Table 3, the minimal values over the pairs (h^{-1}, n) of the errors in maximum norm, of the approximate solution when $\varepsilon = 5 \times 10^{-13}$, are presented. The similar values of errors for the first-order derivatives are presented in Table 4, when $\partial Q/\partial x$ and $\partial Q/\partial y$ are approximated by fourth-order central difference formula on G^S for $r \geq 0.2$. For $r < 0.2$, the order of errors decreases down to 10^{-6}, which are presented in Table 5. This happens because the integrands in (51) are not sufficiently smooth for fourth-order differentiation formula. The order of accuracy of the derivatives for $r < 0.2$ can be increased if we use similar quadrature rules, which we used for the integrals in (51) for the derivatives of integrands also.

Figures 3 and 4 show the dependence on ε for different mesh steps h. Figure 5 demonstrates the convergence of the BGM with respect to the number of quadrature nodes for different mesh steps h. The approximate solution and the exact solution in the "singular" part are given in Figure 6, to illustrate the accuracy of the BGM. The error of the block-grid solution, when the function $Q(r, \theta)$ in (51) is calculated with an accuracy of $\varepsilon = 5 \times 10^{-13}$, is presented in Figure 7. Figures 8 and 9 show the singular behaviour of the first-order partial derivatives in the "singular" part. The ratios $\mathfrak{R}^{\varrho}_{G^S}$ and $\mathfrak{R}^{\varrho}_{G^{NS}}$, when $\varrho = 5$ with respect to different n values for $h^{-1} = 64$ and for a fixed value of n of $h^{-1} = 32$, are illustrated in Figures 10 and 11, respectively. These ratios show that the order of the convergence in both the "singular" and the "nonsingular" parts is asymptotically equal to 16 when n is kept fixed for $h^{-1} = 32$, and it is selected as large as possible ($n > 100$) for $h^{-1} = 64$.

7. Conclusions

In the block-grid method (BGM) for solving Laplace's equation, the restriction on the boundary functions to be algebraic polynomials on the sides of the polygon causing the singular vertices is removed. This condition is replaced by the functions from the Hölder classes $C^{4,\lambda}$, $0 < \lambda < 1$. In the integral representations around singular vertices (on the "singular" part), which are combined with the 9-point finite difference equations on the "nonsingular" part of the polygon, the boundary conditions are taken into account with the help of integrals of Poisson type for a half-plane. To connect the subsystems, a homogeneous fourth-order gluing operator is used. It is proved that the final uniform error is of order $O(h^4 + \varepsilon)$, where ε is the error of the approximation of the mentioned integrals and h is the mesh step. For the p-order derivatives ($p = 0, 1, \ldots$) of the difference between the approximate and the exact solutions, in each "singular" part $O((h^4 + \varepsilon) r_j^{1/\alpha_j - p})$ order is obtained. The method is illustrated in solving the problem in L-shaped polygon with the corner singularity. Dependence of the approximate solution and its errors on ε, h and the number of quadrature nodes n are demonstrated. Furthermore, by the constructed approximate solution on the "singular" part of the polygon, a highly accurate formula for the stress intensity factor is given.

From the error estimation formula (33) of Theorem 7 it follows that the error of the approximate solution on the block sectors decreases as $r_j^{1/\alpha_j}(h^4 + \varepsilon)$, which gives an additional accuracy of the BGM near the singular points.

The method and results of this paper are valid for multiply connected polygons.

References

[1] Z. C. Li, *Combined Methods for Elliptic Problems with Singularities, Interfaces and Infinities*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1998.

[2] A. A. Dosiyev, "The high accurate block-grid method for solving Laplace's boundary value problem with singularities," *SIAM Journal on Numerical Analysis*, vol. 42, no. 1, pp. 153–178, 2004.

[3] A. A. Dosiyev, "A block-grid method for increasing accuracy in the solution of the Laplace equa-tion on polygons," *Russian Academy of Sciences*, vol. 45, no. 2, pp. 396–399, 1992.

[4] A. A. Dosiyev, "A block-grid method of increased accuracy for solving Dirichlet's problem forLaplace's equation on polygons," *Computational Mathematics and Mathematical Physics*, vol. 34, no. 5, pp. 591–604, 1994.

[5] A. A. Dosiyev and S. Cival, "A difference-analytical method for solving Laplace's boundary valueproblem with singularities," in *Proceedings of Conference Dynamical Systems and Applications*, pp. 339–360, Antalya, Turkey, July 2004.

[6] A. A. Dosiyev and S. Cival, "A combined method for solving Laplace's boundary value problem with singularities," *International Journal of Pure and Applied Mathematics*, vol. 21, no. 3, pp. 353–367, 2005.

[7] A. A. Dosiyev and S. C. Buranay, "A fourth order accurate difference-analytical method for solving Laplace's boundary value problem with singularities," in *Mathematical Methods in Engineers*, K. Tas, J. A. T. Machado, and D. Baleanu, Eds., pp. 167–176, Springer, 2007.

[8] A. A. Dosiyev, S. Cival Buranay, and D. Subasi, "The block-grid method for solving Laplace's equation on polygons with nonanalytic boundary conditions," *Boundary Value Problems*, Article ID 468594, 22 pages, 2010.

[9] A. A. Dosiyev, S. C. Buranay, and D. Subasi, "The highly accurate block-grid method in solving Laplace's equation for nonanalytic boundary condition with corner singularity," *Computers & Mathematics with Applications*, vol. 64, no. 4, pp. 616–632, 2012.

[10] E. A. Volkov, "Approximate solution of Laplace's equation by the block method on polygons undernonanalytic boundary conditions," *Proceedings of the Steklov Institute of Mathematics*, no. 4, pp. 65–90, 1993.

[11] E. A. Volkov, "Differentiability properties of solutions of boundary value problems for the Laplaceand Poisson equations on a

rectangle," *Proceedings of the Steklov Institute of Mathematics*, vol. 77, pp. 101–126, 1965.

[12] A. A. Samarskii, *The Theory of Difference Schemes*, vol. 240 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 2001.

[13] A. A. Dosiyev, "On the maximum error in the solution of Laplace equation by finite difference method," *International Journal of Pure and Applied Mathematics*, vol. 7, no. 2, pp. 229–241, 2003.

[14] A. A. Dosiyev, "A fourth order accurate composite grids method for solving Laplace's boundary value problems with singularities," *Computational Mathematics and Mathematical Physics*, vol. 42, no. 6, pp. 867–884, 2002.

[15] E. A. Volkov, "An exponentially converging method for solving Laplace's equation on polygons," *Mathematics of the USSR-Sbornik*, vol. 37, no. 3, pp. 295–325, 1980.

[16] E. A. Volkov, *Block Method for Solving the Laplace Equation and for Constructing Conformal Mappings*, CRC Press, Boca Raton, Fla, USA, 1994.

[17] G. J. Fix, S. Gulati, and G. I. Wakoff, "On the use of singular functions with finite element approximations," *Journal of Computational Physics*, vol. 13, pp. 209–228, 1973.