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Research Article

Inequalities Similar to Hilbert's Inequality

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In the present paper, we establish some new inequalities similar to Hilbert's type inequalities. Our results provide some new estimates to these types of inequalities.

1. Introduction

The well-known classical Hilbert's double-series inequality can be stated as follows [1, page 253].

Theorem A. *If $p_1, p_2 > 1$ such that $1/p_1 + 1/p_2 \geq 1$ and $0 < \lambda = 2 - 1/p_1 - 1/p_2 = 1/q_1 + 1/q_2 \leq 1$, where, as usual, q_1 and q_2 are the conjugate exponents of p_1 and p_2, respectively, then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} \leq K\left(\sum_{m=1}^{\infty}a_m^{p_1}\right)^{1/p_1}\left(\sum_{n=1}^{\infty}b_n^{p_2}\right)^{1/p_2}, \quad (1)$$

where $K = K(p_1, p_2)$ depends on p_1 and p_2 only.

In recent years, several authors [1–18] have given considerable attention to Hilbert's double-series inequality together with its integral version, inverse version, and various generalizations. In particular, Pachpatte [11] established an inequality similar to inequality (1) as follows.

Theorem 1. *Let $p > 1$ be constant and $1/p + 1/q = 1$. If $a(s)$ and $b(t)$ are real-valued functions defined for $\{0, 1, \ldots, m\}$ and $\{0, 1, \ldots, n\}$, respectively, and $a(0) = b(0) = 0$. Moreover, define the operators ∇ by $\nabla u(t) = u(t) - u(t-1)$. Then,*

$$\sum_{s=1}^{m}\sum_{t=1}^{n}\frac{|a(s)||b(t)|}{qs^{p-1}+pt^{q-1}} \leq \frac{1}{pq}m^{(p-1)/p}n^{(q-1)/q} \times\left(\sum_{s=1}^{m}(m-s+1)|\nabla a(s)|^{p}\right)^{1/p} \times\left(\sum_{t=1}^{n}(n-t+1)|\nabla b(t)|^{q}\right)^{1/q}. \quad (2)$$

The first aim of this paper is to establish a new inequality similar to Hilbert's type inequality. Our result provides new estimates to this type of inequality.

Theorem 2. *Let $p > 1$ be constants, and $1/p + 1/q = 1$. For $i = 1, 2$, let $a_i(s_i, t_i)$ be real-valued functions defined for (s_i, t_i), where $s_i = 1, 2, \ldots, m_i$; $t_i = 1, 2, \ldots, n_i$, and let m_i, n_i be natural numbers. Let $a_i(0, t_i) = a_i(s_i, 0) = 0$, and define the operators ∇_1, ∇_2 by*

$$\nabla_1 v_i(s_i, t_i) = v_i(s_i, t_i) - v_i(s_i - 1, t_i), \quad \nabla_2 v_i(s_i, t_i) = v_i(s_i, t_i) - v_i(s_i, t_i - 1). \quad (3)$$

Then,

$$\sum_{s_1=1}^{m_1}\sum_{t_1=1}^{n_1}\left(\sum_{s_2=1}^{m_2}\sum_{t_2=1}^{n_2}\left(|a_1(s_1,t_1)|^{p}+|a_2(s_2,t_2)|^{q}\right) \times\left(\Gamma_{p,q}(s_1,t_1,s_2,t_2) \cdot\max\left\{p(s_1t_1)^{p/q}, q(s_2t_2)^{q/p}\right\}\right)^{-1}\right)$$

$$\leq \frac{1}{pq}(m_1n_1)^{1/q}(m_2n_2)^{1/p} \times \left(\sum_{s_1=1}^{m_1}\sum_{t_1=1}^{n_1}(m_1-s_1+1)(n_1-t_1+1) \times |\nabla_2\nabla_1 a_1(s_1,t_1)|^p\right)^{1/p} \times \left(\sum_{s_2=1}^{m_2}\sum_{t_2=1}^{n_2}(m_2-s_2+1)(n_2-t_2+1) \times |\nabla_2\nabla_1 a_2(s_2,t_2)|^p\right)^{1/p}, \quad (4)$$

where

$$\Gamma_{p,q}(s_1,t_1,s_2,t_2) = pqS\left(\sum_{\xi_1=1}^{s_1}\sum_{\eta_1=1}^{t_1}|\nabla_2\nabla_1 a_1(\xi_1,\eta_1)|^p \times \left(\sum_{\xi_2=1}^{s_2}\sum_{\eta_2=1}^{t_2}|\nabla_2\nabla_1 a_2(\xi_2,\eta_2)|^q\right)^{-1}\right), \quad (5)$$

$$S(h) = \frac{h^{1/(h-1)}}{e\log h^{1/(h-1)}}, \quad h \neq 1. \quad (6)$$

Remark 3. Inequality (4) is just a similar version of the following inequality established by Pachpatte [11]:

$$\sum_{s=1}^{x}\sum_{t=1}^{y}\left(\sum_{k=1}^{z}\sum_{r=1}^{w}\frac{|a(s,t)||b(k,r)|}{q(st)^{p-1}+p(kr)^{q-1}}\right) \leq \frac{1}{pq}(xy)^{1/q}(zw)^{1/p}\left(\sum_{s=1}^{x}\sum_{t=1}^{y}(x-s+1) \times (y-t+1) \times |\nabla_2\nabla_1 a(s,t)|^p\right)^{1/p} \times \left(\sum_{k=1}^{z}\sum_{r=1}^{w}(z-k+1)(w-r+1) \times |\nabla_2\nabla_1 b(k,r)|^q\right)^{1/q}. \quad (7)$$

On the other hand, let $a_1(s_1,t_1)$ and $a_2(s_2,t_2)$ change to $a_1(s_1)$ and $a_2(s_2)$, respectively, and, with appropriate transformation, we have

$$\sum_{s_1=1}^{m_1}\sum_{s_2=1}^{m_2}\frac{|a_1(s_1)|^p+|a_2(s_2)|^q}{\Gamma_{p,q}(s_1,s_2)\cdot\max\{ps_1^{p/q},qs_2^{q/p}\}} \leq \frac{1}{pq}m_1^{1/q}m_2^{1/p} \times \left(\sum_{\tau_1=1}^{m_1}(m_1-\tau_1+1)|\nabla a_1(\tau_1)|^p\right)^{1/p} \times \left(\sum_{\tau_2=1}^{m_2}(m_2-\tau_2+1)|\nabla a_2(\tau_2)|^q\right)^{1/q}, \quad (8)$$

where

$$\Gamma_{p,q}(s_1,s_2) = pqS\left(\frac{\sum_{\tau_1=1}^{s_1}|\nabla a_1(\tau_1)|^p}{\sum_{\tau_2=1}^{s_2}|\nabla a_2(\tau_2)|^q}\right), \quad (9)$$

and $S(h)$ is as in (6). This is just a similar version of inequality (2) in Theorem 1.

The integral analogue of inequality (1) in Theorem A is as follows [1, page 254].

Theorem B. *Let p_1, p_2, q_1, q_2, and λ be as in Theorem A. If $f \in L^p(0,\infty)$ and $g \in L^q(0,\infty)$, then*

$$\iint_0^\infty \frac{f(x)g(x)}{(x+y)^\lambda}dx\,dy \leq K\left(\int_0^\infty f^p(x)\,dx\right)^{1/p}\left(\int_0^\infty g^q(y)\,dy\right)^{1/q}, \quad (10)$$

where $K = K(p,q)$ depends on p and q only.

In [11], Pachpatte also established a similar version of inequality (10) as follows.

Theorem 4. *Let $p > 1$ be constants, and $1/p + 1/q = 1$. If $f(s)$ and $g(t)$ are real-valued continuous functions defined on $[0,x)$ and $[0,y)$, respectively, and let $f(0) = g(0) = 0$. Then,*

$$\int_0^x\int_0^y \frac{|f(s)||g(t)|}{qs^{p-1}+pt^{q-1}}dt\,ds \leq \frac{1}{pq}x^{(p-1)/p}y^{(q-1)/q} \times \left(\int_0^x (x-s)|f'(s)|^p ds\right)^{1/p} \times \left(\int_0^y (y-t)|g'(t)|^q dt\right)^{1/q}. \quad (11)$$

Another aim of this paper is to establish a new integral inequality similar to Hilbert's type inequality.

Theorem 5. *Let $p > 1$, and $1/p + 1/q = 1$. For $i = 1, 2$, let $h_i \geq 1$, $f_i(s_i, t_i)$ be real-valued differentiable functions defined on $[0, x_i) \times [0, y_i)$, where $x_i \in (0, \infty)$, $y_i \in (0, \infty)$, and $f_i(0, t_i) = f_i(s_i, 0) = 0$. As usual, partial derivatives of f_i are denoted by $D_1 f_i, D_2 f_i, D_{12} f_i = D_{21} f_i$, and so forth. Let*

$$D_{12}^{*} f_i(s_i, t_i) = D_2\left(h_i f_i^{h_i - 1}(s_i, t_i) \cdot D_1 f_i(s_i, t_i)\right). \quad (12)$$

Then,

$$\int_0^{x_1}\int_0^{y_1}\left(\int_0^{x_2}\int_0^{y_2}\frac{\left|f_1^{h_1}(s_1,t_1)\right|^p + \left|f_2^{h_2}(s_2,t_2)\right|^q}{L_{p,q}(s_1,t_1,s_2,t_2)\max\left\{p(s_1t_1)^{p/q}, q(s_2t_2)^{q/p}\right\}}ds_1dt_1\right)ds_2dt_2$$
$$\leq \frac{1}{pq}(x_1y_1)^{1/q}(x_2y_2)^{1/p}\left(\int_0^{x_1}\int_0^{y_1}(x_1-s_1)(y_1-t_1)\left|D_{12}^{*}f_1(s_1,t_1)\right|^p dt_1ds_1\right)^{1/p} \quad (13)$$
$$\times\left(\int_0^{x_2}\int_0^{y_2}(x_2-s_2)(y_2-t_2)\left|D_{12}^{*}f_2(s_2,t_2)\right|^q dt_2ds_2\right)^{1/q},$$

where

$$L_{p,q}(s_1,t_1,s_2,t_2) = pqS\left(\frac{\int_0^{s_1}\int_0^{t_1}\left|D_{12}^{*}f_1(\xi_1,\eta_1)\right|^p d\eta_1 d\xi_1}{\int_0^{s_2}\int_0^{t_2}\left|D_{12}^{*}f_2(\xi_2,\eta_2)\right|^q d\eta_2 d\xi_2}\right), \quad (14)$$

and $S(h)$ is as in (6).

Remark 6. Inequality (13) is just a similar version of the following inequality established by Pachpatte [11]:

$$\int_0^{x_1}\int_0^{y_1}\left(\int_0^{x_2}\int_0^{y_2}\frac{\left|f_1(s_1,t_1)\right|\left|f_2(s_2,t_2)\right|}{q(s_1t_1)^{p-1} + p(s_2t_2)^{q-1}}ds_1dt_1\right)ds_2dt_2$$
$$\leq \frac{1}{pq}(x_1y_1)^{1/q}(x_2y_2)^{1/p}$$
$$\times\left(\int_0^{x_1}\int_0^{y_1}(x_1-s_1)(y_1-t_1)\times\left|D_2D_1f_1(s_1,t_1)\right|^p ds_1dt_1\right)^{1/p}$$
$$\times\left(\int_0^{x_2}\int_0^{y_2}(x_2-s_2)(y_2-t_2)\times\left|D_2D_1f_2(s_2,t_2)\right|^q ds_2dt_2\right)^{1/q}. \quad (15)$$

On the other hand, let $f_1(s_1, t_1)$ and $f_2(s_2, t_2)$ change to $f_1(s_1)$ and $f_2(s_2)$, respectively, and, with appropriate transformation, we have

$$\int_0^{x_1}\int_0^{x_2}\frac{\left|f_1^{h_1}(s_1)\right|^p + \left|f_2^{h_2}(s_2)\right|^q}{L_{p,q}(s_1,s_2)\ \max\left\{ps_1^{p/q}, qs_2^{q/p}\right\}}ds_1dt_1$$
$$\leq \frac{1}{pq}x_1^{1/q}x_2^{1/p}\left(\int_0^{x_1}(x_1-s_1)\left|f_1'(s_1)\right|^p ds_1\right)^{1/p}$$
$$\times\left(\int_0^{x_2}(x_2-s_2)\left|f_2'(s_2)\right|^q ds_2\right)^{1/q}, \quad (16)$$

where

$$L_{p,q}(s_1,s_2) = pqS\left(\frac{\int_0^{s_1}(s_1-\sigma_1)\left|f_1'(\sigma_1)\right|^p d\sigma_1}{\int_0^{s_2}(s_2-\sigma_2)\left|f_2'(\sigma_2)\right|^q d\sigma_2}\right). \quad (17)$$

This is just a similar version of inequality (11) in Theorem 4.

2. Proof of Theorems

Proof of Theorem 2. From the hypotheses of Theorem 2, we have

$$\left|a_1(s_1,t_1)\right| \leq \sum_{\xi_1=1}^{s_1}\sum_{\eta_1=1}^{t_1}\left|\nabla_2\nabla_1 a_1(\xi_1,\eta_1)\right|,$$
$$\left|a_2(s_2,t_2)\right| \leq \sum_{\xi_2=1}^{s_2}\sum_{\eta_2=1}^{t_2}\left|\nabla_2\nabla_1 a_2(\xi_2,\eta_2)\right|. \quad (18)$$

By using Hölder's inequality and noticing the reverse Young's inequality [19],

$$s_1^{1/\alpha}s_2^{1/\beta}S\left(\frac{s_1}{s_2}\right) \geq \frac{s_1}{\alpha} + \frac{s_2}{\beta}, \quad (19)$$

for positive real numbers s_1, s_2 and $1/\alpha + 1/\beta = 1$, $\alpha > 1$, where $S(h)$ is as in (6). Hence,

$$\begin{aligned}
&\frac{|a_1(s_1,t_1)|^p + |a_2(s_2,t_2)|^q}{\max\left\{p(s_1t_1)^{p/q}, q(s_2t_2)^{q/p}\right\}} \\
&\quad \le \frac{1}{p}\sum_{\xi_1=1}^{s_1}\sum_{\eta_1=1}^{t_1}|\nabla_2\nabla_1 a_1(\xi_1,\eta_1)|^p \\
&\qquad + \frac{1}{q}\sum_{\xi_2=1}^{s_2}\sum_{\eta_2=1}^{t_2}|\nabla_2\nabla_1 a_2(\xi_2,\eta_2)|^q \\
&\quad \le S\left(\frac{\sum_{\xi_1=1}^{s_1}\sum_{\eta_1=1}^{t_1}|\nabla_2\nabla_1 a_1(\xi_1,\eta_1)|^p}{\sum_{\xi_2=1}^{s_2}\sum_{\eta_2=1}^{t_2}|\nabla_2\nabla_1 a_2(\xi_2,\eta_2)|^q}\right) \\
&\qquad \times\left(\sum_{\xi_1=1}^{s_1}\sum_{\eta_1=1}^{t_1}|\nabla_2\nabla_1 a_1(\xi_1,\eta_1)|^p\right)^{1/p} \\
&\qquad \times\left(\sum_{\xi_2=1}^{s_2}\sum_{\eta_2=1}^{t_2}|\nabla_2\nabla_1 a_2(\xi_2,\eta_2)|^q\right)^{1/q}.
\end{aligned} \tag{20}$$

Dividing both sides of (20) by

$$\Gamma_{p,q}(s_1,t_1,s_2,t_2) = pqS\left(\frac{\sum_{\xi_1=1}^{s_1}\sum_{\eta_1=1}^{t_1}|\nabla_2\nabla_1 a_1(\xi_1,\eta_1)|^p}{\sum_{\xi_2=1}^{s_2}\sum_{\eta_2=1}^{t_2}|\nabla_2\nabla_1 a_2(\xi_2,\eta_2)|^q}\right), \tag{21}$$

taking the sum of both sides of (20) over t_i and s_i from 1 to m_i and n_i $(i = 1,2)$, respectively, and making use of Hölder's inequality, we have

$$\begin{aligned}
&\sum_{s_1=1}^{m_1}\sum_{t_1=1}^{n_1}\left(\sum_{s_2=1}^{m_2}\sum_{t_2=1}^{n_2}\left(|a_1(s_1,t_1)|^p + |a_2(s_2,t_2)|^q\right)\right. \\
&\qquad \times\left(\Gamma_{p,q}(s_1,t_1,s_2,t_2)\right. \\
&\qquad\qquad \left.\left.\cdot\max\left\{p(s_1t_1)^{p/q}, q(s_2t_2)^{q/p}\right\}\right)^{-1}\right) \\
&\quad \le \frac{1}{pq}\sum_{s_1=1}^{m_1}\sum_{t_1=1}^{n_1}\left(\sum_{\xi_1=1}^{s_1}\sum_{\eta_1=1}^{t_1}|\nabla_2\nabla_1 a_1(\xi_1,\eta_1)|^p\right)^{1/p} \\
&\qquad \times\sum_{s_2=1}^{m_2}\sum_{t_2=1}^{n_2}\left(\sum_{\xi_2=1}^{s_2}\sum_{\eta_2=1}^{t_2}|\nabla_2\nabla_1 a_2(\xi_2,\eta_2)|^q\right)^{1/q}. \\
&\quad \le \frac{1}{pq}(m_1n_1)^{1/q} \\
&\qquad \times\left(\sum_{s_1=1}^{m_1}\sum_{t_1=1}^{n_1}\sum_{\xi_1=1}^{s_1}\sum_{\eta_1=1}^{t_1}|\nabla_2\nabla_1 a_1(\xi_1,\eta_1)|^p\right)^{1/p} \\
&\qquad \times(m_2n_2)^{1/p} \\
&\qquad \times\left(\sum_{s_2=1}^{m_2}\sum_{t_2=1}^{n_2}\sum_{\xi_2=1}^{s_2}\sum_{\eta_2=1}^{t_2}|\nabla_2\nabla_1 a_2(\xi_2,\eta_2)|^q\right)^{1/q}. \\
&\quad = \frac{1}{pq}(m_1n_1)^{1/q}(m_2n_2)^{1/p} \\
&\qquad \times\left(\sum_{\xi_1=1}^{m_1}\sum_{\eta_1=1}^{n_1}(m_1-\xi_1+1)\right. \\
&\qquad\qquad \left.\times(n_1-\eta_1+1)|\nabla_2\nabla_1 a_1(\xi_1,\eta_1)|^p\right)^{1/p} \\
&\qquad \times\left(\sum_{\xi_2=1}^{m_2}\sum_{\eta_2=1}^{n_2}(m_2-\xi_2+1)(n_2-\eta_2+1)\right. \\
&\qquad\qquad \left.\times|\nabla_2\nabla_1 a_2(\xi_2,\eta_2)|^q\right)^{1/q} \\
&\quad = \frac{1}{pq}(m_1n_1)^{1/q}(m_2n_2)^{1/p} \\
&\qquad \times\left(\sum_{s_1=1}^{m_1}\sum_{t_1=1}^{n_1}(m_1-s_1+1)(n_1-t_1+1)\right. \\
&\qquad\qquad \left.\times|\nabla_2\nabla_1 a_1(s_1,t_1)|^p\right)^{1/p} \\
&\qquad \times\left(\sum_{s_2=1}^{m_2}\sum_{t_2=1}^{n_2}(m_2-s_2+1)(n_2-t_2+1)\right. \\
&\qquad\qquad \left.\times|\nabla_2\nabla_1 a_2(s_2,t_2)|^p\right)^{1/p}.
\end{aligned} \tag{22}$$

This completes the proof. □

Proof of Theorem 5. From the hypotheses of Theorem 5, we obtain for $i = 1,2$:

$$\begin{aligned}
f_i^{h_i}(s_i,t_i) &= f_i^{h_i}(s_i,t_i) - f_i^{h_i}(0,t_i) - f_i^{h_i}(s_i,0) + f_i^{h_i}(0,0) \\
&= \int_0^{s_i} D_1 f_i^{h_i}(\xi_i,t_i)\,d\xi_i - \int_0^{s_i} D_1 f_i^{h_i}(\xi_i,0)\,d\xi_i \\
&= \int_0^{s_i}\left(D_1 f_i^{h_i}(\xi_i,t_i) - D_1 f_i^{h_i}(\xi_i,0)\right)d\xi_i
\end{aligned}$$

$$= \int_0^{s_i}\int_0^{t_i} D_2\left(h_i f_i^{h_i-1}(\xi_i,\eta_i)\cdot D_1 f_i(\xi_i,\eta_i)\right) d\eta_i d\xi_i$$
$$= \int_0^{s_i}\int_0^{t_i} D_{12}^* f_i(\xi_i,\eta_i)\, d\eta_i d\xi_i. \tag{23}$$

From (23), Hölder's integral inequality and in view of the reverse Young's inequality (19), we have

$$\frac{\left|f_1^{h_1}(s_1,t_1)\right|^p + \left|f_2^{h_2}(s_2,t_2)\right|^q}{\max\left\{p(s_1t_1)^{p/q}, q(s_2t_2)^{q/p}\right\}} \le \frac{1}{p}\int_0^{s_1}\int_0^{t_1}\left|D_{12}^* f_1(\xi_1,\eta_1)\right|^p d\eta_1 d\xi_1 + \frac{1}{q}\int_0^{s_2}\int_0^{t_2}\left|D_{12}^* f_2(\xi_2,\eta_2)\right|^q d\eta_2 d\xi_2.$$
$$\le S\left(\frac{\int_0^{s_1}\int_0^{t_1}\left|D_{12}^* f_1(\xi_1,\eta_1)\right|^p d\eta_1 d\xi_1}{\int_0^{s_2}\int_0^{t_2}\left|D_{12}^* f_2(\xi_2,\eta_2)\right|^q d\eta_2 d\xi_2}\right) \times\left(\int_0^{s_1}\int_0^{t_1}\left|D_{12}^* f_1(\xi_1,\eta_1)\right|^p d\eta_1 d\xi_1\right)^{1/p} \times\left(\int_0^{s_2}\int_0^{t_2}\left|D_{12}^* f_2(\xi_2,\eta_2)\right|^q d\eta_2 d\xi_2\right)^{1/q}. \tag{24}$$

Integrating both sides of (24) over s_i and t_i from 1 to x_i and y_i $(i = 1, 2)$, respectively, and by using Hölder's integral inequality, we arrive at

$$\int_0^{x_1}\int_0^{y_1}\left(\int_0^{x_2}\int_0^{y_2}\frac{\left|f_1^{h_1}(s_1,t_1)\right|^p + \left|f_2^{h_2}(s_2,t_2)\right|^q}{L_{p,q}(s_1,t_1,s_2,t_2)\max\left\{p(s_1t_1)^{p/q}, q(s_2t_2)^{q/p}\right\}} ds_1 dt_1\right) ds_2 dt_2$$
$$\le \frac{1}{pq}\int_0^{x_1}\int_0^{y_1}\left(\int_0^{s_1}\int_0^{t_1}\left|D_{12}^* f_1(\xi_1,\eta_1)\right|^p d\eta_1 d\xi_1\right)^{1/p} ds_1 dt_1 \times\int_0^{x_2}\int_0^{y_2}\left(\int_0^{s_2}\int_0^{t_2}\left|D_{12}^* f_2(\xi_2,\eta_2)\right|^q d\eta_2 d\xi_2\right)^{1/q} ds_2 dt_2.$$
$$\le \frac{1}{pq}(x_1y_1)^{1/q}\left(\int_0^{x_1}\int_0^{y_1}\left(\int_0^{s_1}\int_0^{t_1}\left|D_{12}^* f_1(\xi_1,\eta_1)\right|^p d\eta_1 d\xi_1\right) dt_1 ds_1\right)^{1/p} \times(x_2y_2)^{1/p}\left(\int_0^{x_2}\int_0^{y_2}\left(\int_0^{s_2}\int_0^{t_2}\left|D_{12}^* f_2(\xi_2,\eta_2)\right|^p d\eta_2 d\xi_2\right) dt_2 ds_2\right)^{1/p}$$
$$= \frac{1}{pq}(x_1y_1)^{1/q}(x_2y_2)^{1/p}\left(\int_0^{x_1}\int_0^{y_1}(x_1-s_1)(y_1-t_1)\left|D_{12}^* f_1(s_1,t_1)\right|^p dt_1 ds_1\right)^{1/p} \times\left(\int_0^{x_2}\int_0^{y_2}(x_2-s_2)(y_2-t_2)\left|D_{12}^* f_2(s_2,t_2)\right|^q dt_2 ds_2\right)^{1/q}. \tag{25}$$

This completes the proof. □

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