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Research Article

Constructing the Lyapunov Function through Solving Positive Dimensional Polynomial System

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We propose an approach for constructing Lyapunov function in quadratic form of a differential system. First, positive polynomial system is obtained via the local property of the Lyapunov function as well as its derivative. Then, the positive polynomial system is converted into an equation system by adding some variables. Finally, numerical technique is applied to solve the equation system. Some experiments show the efficiency of our new algorithm.

1. Introduction

Analysis of the stability of dynamical systems plays a very important role in control system analysis and design. For linear systems, it is easy to verify the stability of equilibria. For nonlinear dynamical systems, proving stability of equilibria of nonlinear systems is more complicated than linear systems. One can use the Lyapunov function at the *equilibria* to determine the stability.

For an autonomous polynomial system of differential equations, how to compute the Lyapunov function at *equilibria* is a basic problem. In [1, 2], the author transformed the problem of computing the Lyapunov function into a quantifier elimination problem. The disadvantage of the method is that the computation complexity of quantifier elimination is doubly exponential in the number of total variables. In order to avoid this problem, She et al. [3] propose a symbolic method; they first construct a special *semialgebraic* system using the local properties of a Lyapunov function as well as its derivative and solving these inequations using cylindrical algebraic decomposition (CAD) introduced by Collins in [4]. The algorithm in [5] uses semidefinite programming to search for Lyapunov function. There are also other algorithms, see [6, 7] for more details.

In this paper, we suppose Lyapunov function has quadratic form and some coefficients of Lyapunov function are unknown numbers. Some positive polynomials are obtained using the technique mentioned in [3] first, then a positive dimensional polynomial system is constructed by adding some new variables. The parameter in Lyapunov function is computed through solving the real root of the positive dimensional system using the numerical method.

The rest of this paper is organized as follows: Definitions and preliminaries about the Lyapunov function and the asymptotic stability analysis of differential system are given in Section 2. Section 3 reviews some methods for solving the real root of positive dimensional polynomial system. The new algorithm to compute the Lyapunov function and some experiments are shown in Section 4. In Section 5, some examples are given to illustrates the efficiency of our algorithm. Finally, Section 6 draws a conclusion of this paper.

2. Stability Analysis of Differential Equations

In this section, some preliminaries on the stability analysis of differential equations are presented.

In this paper, we consider the following differential equations:

$$\begin{aligned} \dot{x}_1 &= f_1(\mathbf{x}) \\ \dot{x}_2 &= f_2(\mathbf{x}) \\ &\vdots \\ \dot{x}_n &= f_n(\mathbf{x}), \end{aligned} \tag{1}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $f_i \in \mathbb{R}[\mathbf{x}]$, and $x_i = x_i(t)$, $\dot{x}_i = dx_i/dt$. A point $\overline{\mathbf{x}} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)$ in the n-dimensional real Euclidean space $\mathbb{R}^n$ is called an *equilibrium* of differential system (1) if $f_i(\overline{\mathbf{x}}) = 0$ for all $i \in \{1, 2, \ldots, n\}$. Without loss of generality, we suppose the origin is an *equilibrium* of the given system in this paper.

In general, there exists two techniques to analyze the stability of an *equilibrium*: the Lyapunov's first method with the technique of linearization which considers the eigenvalues of the Jacobian matrix at *equilibrium*.

Theorem 1. *Let $J_F(\overline{\mathbf{x}})$ denote the Jacobian matrix of system $\{f_1, \ldots, f_n\}$ at point $\overline{\mathbf{x}}$. If all the eigenvalues of $J_F(\overline{\mathbf{x}})$ have negative real parts, then $\overline{\mathbf{x}}$ is asymptotically stable. If the matrix $J_F(\overline{\mathbf{x}})$ has at least one eigenvalue with positive real part, then $\overline{\mathbf{x}}$ is unstable.*

For a small system, it is easy to obtain the eigenvalues of the matrix $J_F(\overline{\mathbf{x}})$; then one can analyze the stability of the equilibrium using Theorem 1. For a high-dimensional system, solving the characteristic polynomial to get the exact zeros is a difficult problem. Indeed, to answer the question on stability of an *equilibrium*, we only need to know whether all the eigenvalues have negative real parts or not. Therefore, the theorem of Routh-Hurwitz [8] serves to determine whether all the roots of a polynomial have negative real parts.

Another method to determine asymptotic stability is to check if there exists a Lyapunov function at the point $\overline{\mathbf{x}}$, which is defined in the following.

Definition 2. Given a differential system and a neighborhood $\mathbf{U}$ of the *equilibrium*, a Lyapunov function with respect to the differential system is a continuously differential function $F : \mathbf{U} \to \mathbb{R}$ such that

(1) : $F(\mathbf{0}) = 0$ and $F(\overline{\mathbf{x}}) > 0$ whenever $\overline{\mathbf{x}} \neq \mathbf{0}$;

(2) : $(d/dt)F(\mathbf{0}) = 0$ and $(d/dt)F(\overline{\mathbf{x}}) < 0$ whenever $\overline{\mathbf{x}} \neq \mathbf{0}$.

3. Solving the Real Roots of Positive Dimensional Polynomial System

Solving polynomial system has been one of the central topics in computer algebra. It is required and used in many scientific and engineering applications. Indeed, we only care about the real roots of a polynomial system arising from many practical problems. For zero dimensional system, homotopy continuation method [9, 10] is a global convergence algorithm. For positive dimensional system, computing real roots of this system is a difficult and extremely important problem.

Due to the importance of this problem, many approaches have been proposed. The most popular algorithm which solves this problem is CAD; another is the so-called critical point methods, such as Seidenberg's approach of computing critical points of the distance function [11]. The algorithm in [12] uses the idea of Seidenberg to compute the real root of a positive dimensional defined by a signal polynomial; and extends it to a random polynomial system in [13]. Actually, these algorithms depend on symbolic computations, so they are restricted to small size systems because of the high complexity of the symbolic computation. In order to avoid this problem, homotopy method has been used to compute real root of polynomial system in [14, 15].

Recently, Wu and Reid [16] propose a new approach, which is different from the critical point technique. In order to facilitate the description of this algorithm, we suppose polynomial system $g = \{g_1, g_2, \ldots, g_k\}$; the system has k polynomials, n variables, and $k < n$. First, $n - k$ hyperplanes $h = \{h_1, \ldots, h_{n-k}\}$ in $\mathbb{R}[\mathbf{x}]$ are chosen randomly. Note that $\{g_1, \ldots, g_k, h_1, \ldots, h_{n-k}\}$ is a square system; then witness points are computed by homotopy method and verified by the following theorem.

Theorem 3 (see [17]). *Let $f(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}^n$ be a polynomial system, and $\overline{\mathbf{x}} \in \mathbb{R}^n$. Let $\mathbb{IR}$ be the set of real intervals, and $\mathbb{IR}^n$ and $\mathbb{IR}^{n\times n}$ be the set of real interval vectors and real interval matrices, respectively. Given $\mathbf{X} \in \mathbb{IR}^n$ with $0 \in \mathbf{X}$ and $M \in \mathbb{IR}^{n\times n}$ satisfies $\nabla f_i(\overline{\mathbf{x}} + \mathbf{X}) \subseteq M_i$, for $i = 1, 2, \ldots, n$. Denote by I_n the identity matrix and assume*

$$-F_{\mathbf{x}}^{-1}(\overline{\mathbf{x}}) F(\overline{\mathbf{x}}) + (I_n - F_{\mathbf{x}}(\overline{\mathbf{x}}) M) \quad \mathbf{X} \subseteq \operatorname{int}(\mathbf{X}), \tag{2}$$

where $F_{\mathbf{x}}(\overline{\mathbf{x}})$ is the Jacobian matrix of $F(\mathbf{x})$ at $\overline{\mathbf{x}}$. Then there is a unique $\hat{\mathbf{x}} \in X$ such that $f(\hat{\mathbf{x}}) = 0$. Moreover, every matrix $\overline{M} \in M$ is nonsingular, and the Jacobian matrix $F_{\mathbf{x}}(\overline{\mathbf{x}})$ is nonsingular.

There may exist some components which have no intersection with these random hyperplanes. Some points on these components must be the solutions of the Lagrange optimization problem:

$$f = 0, \quad \sum_{i=1}^{k} \lambda_i \nabla f_i = \mathbf{n}. \tag{3}$$

Here $\mathbf{n}$ is a random vector in $\mathbb{R}^n$. The system has $n + k$ equations and $n+k$ variables; thus we can find real points through solving system (3).

4. Algorithm for Computing the Lyapunov Function

In this section, we will present an algorithm for constructing the Lyapunov function. Our idea is to compute positive polynomial system which satisfies the definition of Lyapunov function first. Then we solve the polynomial system deduced from the positive polynomial system using homotopy algorithm; at this step, we use the famous package hom4ps2 [18].

Given a quadratic polynomial $F(\mathbf{x})$, the following theorem gives a sufficient condition for the polynomial to be a Lyapunov function.

Theorem 4 (see [3]). *Let $F(\mathbf{x})$ be a quadratic polynomial, for a given differential system; if $F(\mathbf{x})$ satisfies the fact that $Hess(F)|_{\bar{\mathbf{x}}=0}$ is positive definite and $Hess((d/dt)F)|_{\bar{\mathbf{x}}=0}$ is negative definite, then $F(\mathbf{x})$ is a Lyapunov function.*

By the theory of linear algebra, one knows that the symmetric matrix $Hess(F)|_{\bar{\mathbf{x}}=0}$ is positive definite if and only if all its eigenvalues are positive, and $Hess((d/dt)F)|_{\bar{\mathbf{x}}=0}$ is negative definite if and only if all its eigenvalues are negative.

Let

$$h = s^n + t_{n-1}s^{n-1} + \cdots + t_0 \tag{4}$$

be a characteristic polynomial of a matrix; the following theorem deduced from the Descartes' rule of signs [19] can be used to determine whether h has only positive roots or not.

Theorem 5 (see [3]). *Suppose all the roots of a real polynomial h are real; then its roots are all positive if and only if for all $1 \le i \le n, (-1)^i t_{n-i} > 0$.*

Combine Theorems 4 and 5, finding that the Lyapunov function in quadratic form can be converted into solving the real root of some positive polynomial system, denoting it by

$$\text{Inequ} = \{g_1 > 0,\ g_2 > 0, \ldots,\ g_n > 0\}. \tag{5}$$

Suppose we have obtained the positive polynomial system as in (5), and denote the variable in the system by $\mathbf{a}$. In order to obtain one value of $\mathbf{a}$ using numerical technique, we first convert the positive equation into equation. A simple ideal is to add new variable set $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, and construct the equation system as follows:

$$ps = \{g_1 - x_1^2, g_2 - x_2^2, \ldots, g_n - x_n^2\}. \tag{6}$$

If we find one real point $(\bar{\mathbf{a}}, \bar{\mathbf{x}})$ of system (6) such that there has nonzero element in $\bar{\mathbf{x}}$, then it is easy to see that the point $\bar{\mathbf{a}}$ satisfies

$$\{g_1(\bar{\mathbf{a}}) > 0, g_2(\bar{\mathbf{a}}) > 0, \ldots, g_n(\bar{\mathbf{a}}) > 0\}, \tag{7}$$

which means the differential system exists a Lyapunov function at the *equilibrium*.

Note that the number of variable is more than the number of equation in system (6); then the system ps must be a positive dimensional polynomial system.

Recall the algorithm mentioned in Section 3; all of the algorithms obtain at least one real point in each connect component, and they use Theorem 3 to verify the existence of real root which deduces the low efficiency. However, in this paper, we only need one real point of system (6) to ensure the establishment of these inequalities in (7), so we verify the establishment of these inequalities using the residue of inequalities at the real part of every approximate real root of the system (6).

In the following we propose an algorithm to determine if there exists a Lyapunov function at the *equilibrium*.

Algorithm 6. Input: a differential system as defined in (1) and a tolerance ϵ.

Output: a Lyapunov function or UNKNOW.

(1) Construct the positive polynomial.

(2) Convert the positive polynomial system into positive dimensional system defined in system (6).

(3) We choose n random point $(\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \ldots, \hat{\mathbf{x}}_n)$ and n random vector $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$; then construct n hyperplane in $\mathbb{R}^n$ through $\hat{\mathbf{x}}_i$ with normal $\mathbf{v}_i$ for $i = 1, 2, \ldots, n$. Denote the set of this hyperplane by ps_2.

(4) Let $ps = \{ps_1, ps_2\}$, and solve the square system using homotopy continuation algorithm, denoting solution of ps by *roots*.

(5) for $s = 1 : length(roots)$

   (a) if the norm of imaginary part of $roots\{s\}$ is smaller than ϵ, then substitute the real part of $roots\{s\}$ into $\{g_1, \ldots, g_n\}$, and denote the value by $\{v_1, v_2, \ldots, v_n\}$. If $v_i > 0$ for all $i \in \{1, 2, \ldots, n\}$, then return the real part of $roots\{s\}$ and break the program.

(6) End for.

(7) Construct polynomial system $ps_3 = \sum_{i=1}^n \lambda_i \nabla f_i = \mathbf{v}$, where λ_i is new variable and $\mathbf{v}$ are chosen from $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ randomly.

(8) Solve $\{ps_1, ps_3\}$ using homotopy continuation algorithm, denote its solution by *roots*, and go to Step 4.

(9) return UNKNOW.

In the following, we present a simple example to illustrate our algorithm.

Example 7. This is an example from [20]

$$\begin{aligned} \dot{x} &= -x + 2y^3 - 2y^4 \\ \dot{y} &= -x - y + xy. \end{aligned} \tag{8}$$

Let Lyapunov function $F(x, y) = x^2 + axy + by^2$.

Step 1. We obtain the positive polynomial using Theorems 4 and 5 as follows:

$$\begin{gathered} [2b + 2 > 0, -a^2 + 4b > 0, \\ 2a + 4b + 4 > 0, 4a^2 + 4b^2 - 16b > 0]. \end{gathered} \tag{9}$$

Step 2. Convert system (9) into the following system:

$$ps_1 = \begin{cases} 2b + 2 - x_1^2 = 0 \\ -a^2 + 4b - x_2^2 = 0 \\ 2a + 4b + 4 - x_3^2 = 0 \\ 4a^2 + 4b^2 - 16b - x_4^2 = 0. \end{cases} \tag{10}$$

Step 3. Construct two hyperplanes $\{h_1, h_2\}$ in $\mathbb{R}^6$ randomly, where

$$\begin{aligned} h_1 &= 0.09713178123584754a + 0.04617139063115394b \\ &\quad + 0.27692298496089x_1 + 0.8234578283272926x_2 \\ &\quad + 0.694828622975817x_3 + 0.3170994800608605x_4 \\ &\quad + 0.9502220488383549, \\ h_2 &= 0.3815584570930084a + 0.4387443596563982b \\ &\quad + 0.03444608050290876x_1 + 0.7655167881490024x_2 \\ &\quad + 0.7951999011370632x_3 + 0.1868726045543786x_4 \\ &\quad + 0.4897643957882311. \end{aligned} \tag{11}$$

Step 4. Compute the roots of the augmented system $\{ps_1 = 0, h_1 = 0, h_2 = 0\}$ using homotopy method, and we find the system has only 16 roots.

Step 5. We obtain the first approximate real root of the system

$$\begin{aligned} \bar{\mathbf{x}} = [&-2.407604610156789, 4.633115716668555, \\ &3.356520733339377, 3.568739680591174, \\ &-4.209186815331512, -5.909266734956268]. \end{aligned} \tag{12}$$

Substituting $a = -2.407604610156789$, $b = 4.633115716668555$ into the left of the positive polynomial in (9), we obtain the following result:

$$[11.26623143, 12.73590291, 17.71725365, 34.91943333]. \tag{13}$$

This ensure the establishment of inequality in (9).

Thus,

$$\begin{aligned} F(x, y) = x^2 &+ 4.633115716668555y^2 \\ &- 2.407604610156789xy \end{aligned} \tag{14}$$

is a Lyapunov function.

If the random hyperplanes $\{h_1, h_2\}$ are as follows:

$$\begin{aligned} h_1 &= -3a - b + x_1 + 2x_2 - 2x_3 - 2x_4 - 3, \\ h_2 &= 3a - 3b - x_1 - 2x_2 + x_3 + 2x_4 - 2, \end{aligned} \tag{15}$$

we find that polynomial system $\{h_1 = 0, h_2 = 0, ps = 0\}$ has no real root; then we go to Step 7 in Algorithm 6 and obtain the following system:

$$ps_3 = \begin{cases} -2\lambda_2 a + 2\lambda_3 + 8\lambda_4 a - 1 = 0 \\ 2\lambda_1 + 4\lambda_2 + 4\lambda_3 + \lambda_4(8b - 16) - 3 = 0 \\ -2\lambda_1 x_1 + 1 = 0 \\ -2\lambda_2 x_2 + 2 = 0 \\ -2\lambda_3 x_3 - 2 = 0 \\ -2\lambda_4 x_4 - 3 = 0. \end{cases} \tag{16}$$

Solving the system $\{ps_1 = 0, ps_3 = 0\}$, we find the first approximate real root and substitute the value of $a = 1.3053335232048229$, $b = 0.4314538107033688$ into the left of the positive polynomial in (9) and we obtain the following result:

$$\begin{aligned} [&2.862907621406738, 0.021919636011159, \\ &8.336482289223121, 0.656931019037197]. \end{aligned} \tag{17}$$

This ensures the establishment of inequality in (9).

Thus,

$$\begin{aligned} F(x, y) = x^2 &+ 0.4314538107033688y^2 \\ &+ 1.3053335232048229xy \end{aligned} \tag{18}$$

is a Lyapunov function.

5. Experiments

In this section, some examples are given to illustrate the efficiency of our algorithm.

Example 8. This is an example from [7]

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= -4x - 3y - 2z + x^2 y + x^2 z. \end{aligned} \tag{19}$$

We assume that $F(x, y, z) = x^2 + y^2 + z^2 + axy + bxz + cyz$. Algorithm 6 returns a Lyapunov function

$$\begin{aligned} F(x, y, z) = x^2 + y^2 + z^2 &+ 1.370502803658027xy \\ &+ 0.655753434727512xz \\ &+ 0.632220465746607yz, \end{aligned} \tag{20}$$

at Step 4 using only 1.085175 s. If the algorithm does not terminate at Step 4, it returns

$$\begin{aligned} F(x, y, z) = x^2 + y^2 + z^2 &+ 0.566986159377122xy \\ &+ 1.934844270891010xz \\ &+ 0.065341301862036yz, \end{aligned} \tag{21}$$

using about 21.285095 s.

Example 9. This is an example from a classic ODE's textbook:

$$\begin{aligned} \dot{x} &= -x - 3y + 2y + yz, \\ \dot{y} &= 3x - y - z + xz, \\ \dot{z} &= -2x + y - z + xy. \end{aligned} \tag{22}$$

Assume that $F(x, y, z) = x^2 + axy + xz + cy^2 + dyz + ez^2$. With about 2.4 s, we got a real root for the parameters that form the coefficients of F. Indeed, this point was obtained from Step 4. If there is no real point at Step 4, this program returns one real root using about 267 s, which is also more efficient than 1800 s in [3].

Example 10. This is another example from an ODE's textbook:

$$\dot{x} = -x + y + xz^2 - x^3,$$
$$\dot{y} = x - y + z^2 - y^3, \quad (23)$$
$$\dot{z} = -yz - z^2.$$

Assume that $F = x^2 + bxz + cy^2 + dyz + ez^2$. For this program, our algorithm stops at Step 3, using about 1.24475 s. In [3], they use about 840 s.

6. Conclusion

For a differential system, based on the technique of computing real root of positive dimensional polynomial system, we present a numerical method to compute the Lyapunov function at *equilibria*. According to the relationship between the positive dimensional system and the Lyapunov function, we know we just need only one real root of this system, so we convert the algorithm into two steps. At each step, rather than using interval Newton's method to verify the existence of real root, we use the residue of the positive polynomial system at approximate real root to verify the correctness of the positive polynomial system.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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