

## Research Article

# Stability and Convergence of an Effective Finite Element Method for Multiterm Fractional Partial Differential Equations

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A finite element method (FEM) for multiterm fractional partial differential equations (MT-FPDEs) is studied for obtaining a numerical solution effectively. The weak formulation for MT-FPDEs and the existence and uniqueness of the weak solutions are obtained by the well-known Lax-Milgram theorem. The Diethelm fractional backward difference method (DFBDM), based on quadrature for the time discretization, and FEM for the spatial discretization have been applied to MT-FPDEs. The stability and convergence for numerical methods are discussed. The numerical examples are given to match well with the main conclusions.

## 1. Introduction

In recent years, the numerical treatment and supporting analysis of fractional order differential equations has become an important research topic that offers great potential. The FEMs for fractional partial differential equations have been studied by many authors (see [1–3]). All of these papers only considered single-term fractional equations, where they only had one fractional differential operator. In this paper, we consider the MT-FPDEs, which include more than one fractional derivative. Some authors also considered solving linear problems with multiterm fractional derivatives (see [4, 5]). This motivates us to consider their effective numerical solutions for such MT-FPDEs, which have been proposed in [6, 7].

Let  $\Omega = (0, X)^d$ , where  $d \geq 1$  is the space dimension. We consider the MT-FPDEs with the Caputo time fractional derivatives as follows:

$$P({}^C D_t)u(t, x) - \Delta_x u(t, x) = f(t, x), \quad t \in [0, T], \quad x \in \Omega, \quad (1)$$

$$u(0, x) = u_0(x), \quad x \in \Omega, \quad (2)$$

$$u(t, x) = 0, \quad t \in [0, T], \quad x \in \partial\Omega, \quad (3)$$

where the operator  $P({}^C D_t)u(t, x)$  is defined as

$$P({}^C D_t)u(t, x) = \left( {}^C D_t^\alpha + \sum_{i=1}^s a_i {}^C D_t^{\alpha_i} \right) u(t, x), \quad (4)$$

with  $0 < \alpha_s < \alpha_{s-1} < \dots < \alpha_1 < \alpha < 1$  and  $\{a_i > 0\}_{i=1}^s$ . Here  ${}^C D_t^\alpha u(t, x)$  denotes the left Caputo fractional derivative with respect to the time variable  $t$  and  $\Delta_x$  denotes the Laplace operator with respect to the space variable  $x$ .

Some numerical methods have been considered for solving the multiterm fractional differential equations. In [8], Liu et al. investigate some effective numerical methods for time fractional wave-diffusion and diffusion equations:

$${}^C D_t^\alpha u(t, x) - k \Delta_x u(t, x) = f(t, x), \quad 0 < x < L, \quad t > 0, \quad (5)$$

where  $k$  and  $L$  are arbitrary positive constants and  $f(t, x)$  is a sufficiently smooth function. The authors consider the implicit finite difference methods (FDMs) and prove that it is unconditionally stable. The error estimate of the FDM is  $O(\Delta t + \Delta t^{2-\alpha} + \Delta x)$ , where  $\Delta t$  and  $\Delta x$  are the time and space step size, respectively. They also investigate the fractional predictor-corrector methods (FPCMs) of the Adams-Moulton methods for multiterm time fractional differential equations (1) with order  $\{\alpha_i\}_{i=1, \dots, s}$  by solving the equivalent

Volterra integral equations. The error estimate of the FPCM is  $O(\Delta t + \Delta t^{1+\min\{\alpha_i\}} + \Delta x^2)$ . In recent years, there are some articles for the predictor-correction method for initial-value problems (see [9–14]). For the application of the FDMs, there have been many research articles as follows. In [15–20], Simos et al. investigate the numerical methods for solving the Schrödinger equation. In [21–24], the Runge-Kutta methods are considered and applied to get the numerical solution of orbital problems. For long-time integration, the Newton-Cotes formulae are considered in [25–27].

In [28], Badr investigate the FEM for linear multiterm fractional differential equations with one variable as follows:

$${}_0^C D_t^{1+\alpha} u(t) + \sum_{i=1}^s A_i(x) D^{\alpha_i} u(x) = f(t), \quad \alpha \leq n, \quad (6)$$

$$\alpha_i < n - 1, \quad 0 < t < 1,$$

where  $A_i(x)$  are known functions. The author gives the details of the modified Galerkin method for the above equations and makes the numerical example for checking the numerical method. In [29], Ford et al. consider the FEM for (5) with singular fractional order and obtain the error estimate  $O(\Delta t^{2-\alpha} + \Delta x^2)$ . In this paper, we follow the work in [29] and consider the FEM for solving MT-FPDEs (1)–(3). Then, we prove the stability and convergence of the FEM for MT-FPDEs and make the error estimate.

The paper is organized as follows. In Section 2, the weak formulation of the MT-FPDEs is given and the existence and uniqueness results for such problems are proved. In Section 3, we consider the convergence rate of time discretization of MT-FPDEs, based on the Diethelm fractional backward difference method (DFBDM). In Section 4, we propose an FEM based on the weak formulation and carry out the error analysis. In Section 5, the stability of this method is proven. Finally, the numerical examples are considered for matching well with the main conclusions.

## 2. Existence and Uniqueness

Let  $\Gamma(\cdot)$  denote the gamma function. For any positive integer  $n$  and  $n - 1 < \alpha < n$ , the Caputo derivative and the Riemann-Liouville derivative are, respectively, defined as follows [30].

(i) The left Caputo derivatives:

$${}_0^C D_t^\alpha v(t) := \frac{1}{\Gamma(n - \alpha)} \int_0^t \frac{1}{(t - \tau)^{\alpha - n + 1}} \left( \frac{d^n}{d\tau^n} v(\tau) \right) d\tau. \quad (7)$$

(ii) The left Riemann-Liouville derivatives:

$${}_0^R D_t^\alpha v(t) := \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_0^t \frac{v(\tau)}{(t - \tau)^{\alpha - n + 1}} d\tau. \quad (8)$$

(iii) The right Riemann-Liouville derivatives:

$${}_t^R D_T^\alpha v(t) := \frac{(-1)^n}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_t^T \frac{v(\tau)}{(\tau - t)^{\alpha - n + 1}} d\tau. \quad (9)$$

Let  $C^\infty(0, T)$  denote the space of infinitely differentiable functions on  $(0, T)$  and  $C_0^\infty(0, T)$  denote the space of infinitely differentiable functions with compact support in  $(0, T)$ . We use the expression  $A \lesssim B$  to mean that  $A \leq cB$  when  $c$  is a positive real number and use the expression  $A \cong B$  to mean that  $A \lesssim B \lesssim A$ . Let  $L_2(\mathcal{Q})$  be the space of measurable functions whose square is the Lebesgue integrable in  $\mathcal{Q}$  which may denote a domain  $\mathcal{Q} = I \times \Omega$ ,  $I$  or  $\Omega$ . Here time domain  $I := (0, T)$  and space domain  $\Omega := (0, X)$ . The inner product and norm of  $L_2(\mathcal{Q})$  are defined by

$$(u, v)_{L_2(\mathcal{Q})} := \int_{\mathcal{Q}} uv d\mathcal{Q}, \quad \|u\|_{L_2(\mathcal{Q})} := (u, u)_{L_2(\mathcal{Q})}^{1/2}, \quad (10)$$

$$\forall u, v \in L_2(\mathcal{Q}).$$

For any real  $\sigma > 0$ , we define the spaces  ${}^l H_0^\sigma(\mathcal{Q})$  and  ${}^r H_0^\sigma(\mathcal{Q})$  to be the closure of  $C_0^\infty(\mathcal{Q})$  with respect to the norms  $\|v\|_{{}^l H_0^\sigma(\mathcal{Q})}$ , and  $\|v\|_{{}^r H_0^\sigma(\mathcal{Q})}$  respectively, where

$$\|v\|_{{}^l H_0^\sigma(\mathcal{Q})} := \left( \|v\|_{L_2(\mathcal{Q})}^2 + |v|_{{}^l H_0^\sigma(\mathcal{Q})}^2 \right)^{1/2},$$

$$|v|_{{}^l H_0^\sigma(\mathcal{Q})} := \left\| {}_0^R D_t^\sigma v \right\|_{L_2(\mathcal{Q})}, \quad (11)$$

$$\|v\|_{{}^r H_0^\sigma(\mathcal{Q})} := \left( \|v\|_{L_2(\mathcal{Q})}^2 + |v|_{{}^r H_0^\sigma(\mathcal{Q})}^2 \right)^{1/2},$$

$$|v|_{{}^r H_0^\sigma(\mathcal{Q})} := \left\| {}_t^R D_T^\sigma v \right\|_{L_2(\mathcal{Q})}.$$

In the usual Sobolev space  $H_0^\sigma(\mathcal{Q})$ , we also have the definition

$$\|v\|_{H_0^\sigma(\mathcal{Q})} := \left( \|v\|_{L_2(\mathcal{Q})}^2 + |v|_{H_0^\sigma(\mathcal{Q})}^2 \right)^{1/2}, \quad (12)$$

$$|v|_{H_0^\sigma(\mathcal{Q})} := \frac{\left( {}_0^R D_t^\sigma v, {}_t^R D_T^\sigma v \right)_{L_2(\mathcal{Q})}}{\cos(\pi\sigma)}.$$

From [3], for  $\sigma > 0$ ,  $\sigma \neq n - 1/2$ , the spaces  ${}^l H_0^\sigma(\mathcal{Q})$ ,  ${}^r H_0^\sigma(\mathcal{Q})$ , and  $H_0^\sigma(\mathcal{Q})$  are equal, and their seminorms are all equivalent to  $|\cdot|_{H_0^\sigma(\mathcal{Q})}$ . We first recall the following results.

**Lemma 1** (see [3]). *Let  $0 < \theta < 2$ ,  $\theta \neq 1$ . Then for any  $w, v \in H_0^{\theta/2}(0, T)$ , then*

$$\left( {}_0^R D_t^\theta w, v \right)_{L_2(0, T)} = \left( {}_0^R D_t^{\theta/2} w, {}_t^R D_T^{\theta/2} v \right)_{L_2(0, T)}. \quad (13)$$

From [3], we define the following space:

$$B^{\alpha/2}(I \times \Omega) = H^{\alpha/2}(I, L_2(\Omega)) \cap H^{\alpha/2}(I, L_2(\Omega)) \cap \dots$$

$$\cap H^{\alpha/2}(I, L_2(\Omega)) \cap L_2(I, H_0^1(\Omega))$$

$$= H^{\alpha/2}(I, L_2(\Omega)) \cap L_2(I, H_0^1(\Omega)). \quad (14)$$

Here  $B^{\alpha/2}(I \times \Omega)$  is a Banach space with respect to the following norm:

$$\|v\|_{B^{\alpha/2}(I \times \Omega)} = \left( \|v\|_{H^{\alpha/2}(I, L_2(\Omega))}^2 + \|v\|_{L_2(I, H_0^{\alpha/2}(\Omega))}^2 \right)^{1/2}, \quad (15)$$

where  $H^{\alpha/2}(I, L_2(\Omega)) := \{v; \|v(t, \cdot)\|_{L_2(\Omega)} \in H^{\alpha/2}(I)\}$ , endowed with the norm

$$\|v\|_{H^{\alpha/2}(I, L_2(\Omega))} := \left\| \|v(t, \cdot)\|_{L_2(\Omega)} \right\|_{H^{\alpha/2}(I)}. \quad (16)$$

Based on the relation equation between the left Caputo and the Riemann-Liouville derivative in [31], we can translate the Caputo problem to the Riemann-Liouville problem. Then, we consider the weak formulation of (1) as follows. For  $f \in B^{\alpha/2}(I \times \Omega)'$ , find  $u(t, x) \in B^{\alpha/2}(I \times \Omega)$  such that

$$\mathcal{A}(u, v) = \mathcal{F}(v), \quad v \in B^{\alpha/2}(I \times \Omega), \quad (17)$$

where the bilinear form is, by Lemma 1,

$$\begin{aligned} \mathcal{A}(u, v) := & \left( {}_0^R D_t^{\alpha/2} u, {}_t^R D_T^{\alpha/2} v \right)_{L_2(I \times \Omega)} \\ & + \sum_{i=1}^s a_i \left( {}_0^R D_t^{\alpha_i/2} u, {}_t^R D_T^{\alpha_i/2} v \right)_{L_2(I \times \Omega)} \\ & + (\nabla_x u, \nabla_x v)_{L_2(I \times \Omega)}, \end{aligned} \quad (18)$$

and the functional is  $\mathcal{F}(v) := (\bar{f}, v)_{L_2(I \times \Omega)}$ ,  $\bar{f}(t, x) := f(t, x) + (u_0(x)t^{-\alpha}/\Gamma(1-\alpha)) + \sum_{i=1}^s a_i(u_0(x)t^{-\alpha_i}/\Gamma(1-\alpha_i))$ .

Based on the main results in Subsection 3.2 in [32], we can prove the following existence and uniqueness theorem.

**Theorem 2.** Assume that  $0 < \alpha < 1$  and  $\bar{f} \in B^{\alpha/2}(I \times \Omega)'$ . Then the system (17) has a unique solution in  $B^{\alpha/2}(I \times \Omega)$ . Furthermore,

$$\|u\|_{B^{\alpha/2}(I \times \Omega)} \leq \|\bar{f}\|_{B^{\alpha/2}(I \times \Omega)'}. \quad (19)$$

*Proof.* The existence and uniqueness of the solution of (17) is guaranteed by the well-known Lax-Milgram theorem. The continuity of the bilinear form  $\mathcal{A}$  and the functional  $\mathcal{F}$  is obvious. Now we need to prove the coercivity of  $\mathcal{A}$  in the space  $B^{\alpha/2}(I \times \Omega)$ . From the equivalence of  ${}^L H_0^\alpha(I \times \Omega)$ ,  ${}^R H_0^\alpha(I \times \Omega)$  and  $H_0^\alpha(I \times \Omega)$ , for all  $u, v \in B^{\alpha/2}(I \times \Omega)$ , using the similar proof process in [32], we obtain

$$\begin{aligned} \mathcal{A}(v, v) \geq & \left( {}_0^R D_t^{\alpha/2} v, {}_0^R D_t^{\alpha/2} v \right)_{L_2(I \times \Omega)} \\ & + \sum_{i=1}^s a_i \left( {}_0^R D_t^{\alpha_i/2} v, {}_0^R D_t^{\alpha_i/2} v \right)_{L_2(I \times \Omega)} \\ & + (\nabla_x v, \nabla_x v)_{L_2(I \times \Omega)} \geq \|v\|_{B^{\alpha/2}(I \times \Omega)}^2. \end{aligned} \quad (20)$$

Then we take  $v = u$  in (17) to get  $\|u\|_{B^{\alpha/2}(I \times \Omega)}^2 \leq (\bar{f}, u)_{L_2(I \times \Omega)}$  by the Schwarz inequality and the Poincaré inequality.  $\square$

### 3. Time Discretization and Convergence

In this section, we consider DFBDM for the time discretization of (1)–(3), which is introduced in [33] for fractional ordinary differential equations. We can obtain the convergence order for the time discretization for the MT-FPDEs. Let  $A = -\Delta_x$ ,  $D(A) = H_0^1(\Omega) \cap H^2(\Omega)$ . Let  $u(t)$ ,  $f(t)$ , and  $u(0)$  denote the one-variable functions as  $u(t, \cdot)$ ,  $f(t, \cdot)$ , and  $u(0, \cdot)$ , respectively. Then (1) can be written in the abstract form, for  $0 < t < T$ ,  $0 < \alpha_s < \dots < \alpha_1 < \alpha < 1$ , with initial value  $u(0) = u_0$ . Now we have

$${}_0^R D_t^\alpha [u - u_0](t) + \sum_{i=1}^s a_i {}_i^R D_t^{\alpha_i} [u - u_0](t) + Au(t) = f(t). \quad (21)$$

Let  $0 = t_0 < t_1 < \dots < t_N = T$  be a partition of  $[0, T]$ . Then, for fixed  $t_j$ ,  $j = 1, 2, \dots, N$ , we have

$${}_0^R D_t^\alpha [u - u_0](t_j) = \frac{t_j^{-\alpha}}{\Gamma(-\alpha)} \int_0^1 g(\tau) \tau^{-1-\alpha} d\tau, \quad (22)$$

where  $g(\tau) = u(t_j - t_j\tau) - u_0$ . Here, the integral is a Hadamard finite-part integral in [33] and [34].

Now, for every  $j$ , we replace the integral by a first-degree compound quadrature formula with equispaced nodes  $0, (1/j), (2/j), \dots, 1$  and obtain

$$\int_0^1 g(\tau) \tau^{-1-\alpha} d\tau = \sum_{k=0}^j \alpha_{kj}^{(\alpha)} g\left(\frac{k}{j}\right) + R_j^{(\alpha)}(g), \quad (23)$$

where the weights  $\alpha_{kj}^{(\alpha)}$  are

$$\begin{aligned} & \alpha(1-\alpha)j^{-\alpha}\alpha_{kj}^{(\alpha)} \\ & = \begin{cases} -1, & \text{for } k=0, \\ 2k^{1-\alpha} - (k-1)^{1-\alpha} - (k+1)^{1-\alpha}, & \text{for } k=1, 2, \dots, j-1, \\ (\alpha-1)k^{-\alpha} - (k-1)^{1-\alpha} + k^{1-\alpha}, & \text{for } k=j, \end{cases} \end{aligned} \quad (24)$$

and the remainder term  $R_j^{(\alpha)}(g)$  satisfies  $\|R_j^{(\alpha)}(g)\| \leq \gamma_\alpha j^{\alpha-2} \sup_{0 \leq t \leq T} \|g''(t)\|$ , where  $\gamma_\alpha > 0$  is a constant.

Thus, for  $\omega_{kj}^{(\alpha)} = j^{-\alpha}\alpha_{kj}^{(\alpha)}/\Gamma(-\alpha)$ , we have

$$\begin{aligned} {}_0^R D_t^\alpha [u - u_0](t_j) = & \Delta t^{-\alpha} \sum_{k=0}^j \omega_{kj}^{(\alpha)} (u(t_j - t_k) - u(0)) \\ & + \frac{t_j^{-\alpha}}{\Gamma(-\alpha)} R_j^{(\alpha)}(g). \end{aligned} \quad (25)$$

Let  $t = t_j$ , we can write (21) as

$$\begin{aligned} &\Delta t^{-\alpha} \sum_{k=0}^j \omega_{kj}^{(\alpha)} (u(t_j - t_k) - u(0)) \\ &+ \sum_{i=1}^s a_i \Delta t^{-\alpha_i} \sum_{k=0}^j \omega_{kj}^{(\alpha_i)} (u(t_j - t_k) - u(0)) + Au(t_j) \\ &= f(t_j) - \frac{t_j^{-\alpha}}{\Gamma(-\alpha)} R_j^{(\alpha)}(g) - \sum_{i=1}^s a_i \frac{t_j^{-\alpha_i}}{\Gamma(-\alpha_i)} R_j^{(\alpha_i)}(g), \\ & \hspace{15em} j = 1, 2, 3, \dots \end{aligned} \tag{26}$$

Denote  $U^j$  as the approximation of  $u(t_j)$  and  $f_j = f(t_j)$ . We obtain the following equation:

$$\begin{aligned} &\Delta t^{-\alpha} \sum_{k=0}^j \omega_{kj}^{(\alpha)} (U^{j-k} - U^0) \\ &+ \sum_{i=1}^s a_i \Delta t^{-\alpha_i} \sum_{k=0}^j \omega_{kj}^{(\alpha_i)} (U^{j-k} - U^0) + AU^j = f_j. \end{aligned} \tag{27}$$

**Lemma 3** (see [34]). For  $0 < \alpha < 1$ , let the sequence  $\{d_j\}_{j=1,2,\dots}$  be given by  $d_1 = 1$  and  $d_j = 1 + \alpha(1-\alpha)j^{-\alpha} \sum_{k=1}^{j-1} \alpha_{kj}^{(\alpha)} d_{j-k}$ . Then,  $1 \leq d_j \leq (\sin(\pi\alpha)/\pi\alpha(1-\alpha))j^\alpha$ , for  $j = 1, 2, 3, \dots$

Let  $e^j = u(t_j) - U^j$  denote the error in  $t_j$ . Then we have the following error estimate.

**Theorem 4.** Let  $U^j$  and  $u(t_j)$  be the solutions of (27) and (21), respectively. Then one has  $\|U^j - u(t_j)\| \leq \Delta t^{2-\alpha}$ .

*Proof.* Subtracting (27) from (26), we obtain the error equation

$$\begin{aligned} &\Delta t^{-\alpha} \sum_{k=0}^j \omega_{kj}^{(\alpha)} (e^{j-k} - e^0) + \sum_{i=1}^s a_i \Delta t^{-\alpha_i} \sum_{k=0}^j \omega_{kj}^{(\alpha_i)} (e^{j-k} - e^0) + Ae^j \\ &= -\frac{t_j^{-\alpha}}{\Gamma(-\alpha)} R_j^{(\alpha)}(g) - \sum_{i=1}^s a_i \frac{t_j^{-\alpha_i}}{\Gamma(-\alpha_i)} R_j^{(\alpha_i)}(g). \end{aligned} \tag{28}$$

Note that  $e^0 = u(0) - U^0 = 0$ . Denote

$$\begin{aligned} e^j &= \left( \alpha_{0j}^{(\alpha)} + \sum_{i=1}^s a_i \Delta t^{\alpha-\alpha_i} \frac{\Gamma(-\alpha)}{\Gamma(-\alpha_i)} \alpha_{0j}^{(\alpha_i)} + At^\alpha \Gamma(-\alpha) \right)^{-1} \\ &\times \left( \sum_{k=1}^j \alpha_{kj}^{(\alpha)} e^{j-k} + \sum_{i=1}^s a_i \Delta t^{\alpha-\alpha_i} \frac{\Gamma(-\alpha)}{\Gamma(-\alpha_i)} \sum_{k=1}^j \alpha_{kj}^{(\alpha_i)} e^{j-k} \right. \\ &\left. - R_j^{(\alpha)}(g) - \sum_{i=1}^s a_i \frac{\Gamma(-\alpha)}{\Gamma(-\alpha_i)} t_j^{\alpha-\alpha_i} R_j^{(\alpha_i)}(g) \right). \end{aligned} \tag{29}$$

Let  $\|\cdot\|$  denote the  $L_2$ -norm, then we have

$$\begin{aligned} \|e^j\| &\leq \left\| \left( \alpha_{0j}^{(\alpha)} + \sum_{i=1}^s a_i \Delta t^{\alpha-\alpha_i} \frac{\Gamma(-\alpha)}{\Gamma(-\alpha_i)} \alpha_{0j}^{(\alpha_i)} + At^\alpha \Gamma(-\alpha) \right)^{-1} \right\| \\ &\times \left( \sum_{k=1}^j \alpha_{kj}^{(\alpha)} \|e^{j-k}\| + \sum_{i=1}^s a_i \Delta t^{\alpha-\alpha_i} \frac{\Gamma(-\alpha)}{\Gamma(-\alpha_i)} \sum_{k=1}^j \alpha_{kj}^{(\alpha_i)} \|e^{j-k}\| \right. \\ &\left. + \|R_j^{(\alpha)}(g)\| + \sum_{i=1}^s a_i \frac{\Gamma(-\alpha)}{\Gamma(-\alpha_i)} t_j^{\alpha-\alpha_i} \|R_j^{(\alpha_i)}(g)\| \right). \end{aligned} \tag{30}$$

Note that  $A$  is a positive definite elliptic operator with all of eigenvalues  $\lambda > 0$ . Since  $\alpha_{0j}^{(\alpha)} < 0$  and  $\Gamma(-\alpha) < 0$ , we have

$$\begin{aligned} &\left\| \left( \alpha_{0j}^{(\alpha)} + \sum_{i=1}^s a_i \Delta t^{\alpha-\alpha_i} \frac{\Gamma(-\alpha)}{\Gamma(-\alpha_i)} \alpha_{0j}^{(\alpha_i)} + At^\alpha \Gamma(-\alpha) \right)^{-1} \right\| \\ &= \sup_{\lambda > 0} \left\| \left( \alpha_{0j}^{(\alpha)} + \sum_{i=1}^s a_i \Delta t^{\alpha-\alpha_i} \frac{\Gamma(-\alpha)}{\Gamma(-\alpha_i)} \alpha_{0j}^{(\alpha_i)} + \lambda t^\alpha \Gamma(-\alpha) \right)^{-1} \right\| \\ &\leq \left( -\alpha_{0j}^{(\alpha)} - \sum_{i=1}^s a_i \Delta t^{\alpha-\alpha_i} \frac{\Gamma(-\alpha)}{\Gamma(-\alpha_i)} \alpha_{0j}^{(\alpha_i)} \right)^{-1}. \end{aligned} \tag{31}$$

Hence,

$$\begin{aligned} \|e^j\| &\leq \alpha(1-\alpha)j^{-\alpha} \sum_{k=1}^j \alpha_{kj}^{(\alpha)} \|e^{j-k}\| \\ &+ \sum_{i=1}^s \alpha_i(1-\alpha_i)j^{-\alpha_i} \sum_{k=1}^j \alpha_{kj}^{(\alpha_i)} \|e^{j-k}\| \\ &+ \alpha(1-\alpha)\gamma_\alpha n^{-2} \sup_{0 \leq t \leq T} \|u''\| \\ &+ \sum_{i=1}^s \alpha_i(1-\alpha_i)\gamma_{\alpha_i} n^{-2} \sup_{0 \leq t \leq T} \|u''\|. \end{aligned} \tag{32}$$

Denote  $d_1 = 1$  and

$$\begin{aligned} d_j &= 1 + \alpha(1-\alpha)j^{-\alpha} \sum_{k=1}^{j-1} \alpha_{kj}^{(\alpha)} d_{j-k}, \quad j = 2, 3, \dots, n, \\ d_j^i &= 1 + \alpha_i(1-\alpha_i)j^{-\alpha_i} \sum_{k=1}^{j-1} \alpha_{kj}^{(\alpha_i)} d_{j-k}, \quad j = 2, 3, \dots, n, \end{aligned} \tag{33}$$

where  $i = 1, 2, \dots, s$ . By induction and Lemma 3, then we have

$$\begin{aligned} \|e^j\| &\leq \alpha(1-\alpha)n^{-2} \sup_{0 \leq t \leq T} \|u''(t)\| \cdot d_j \\ &\quad + \sum_{i=1}^s \alpha_i(1-\alpha_i)n^{-2} \sup_{0 \leq t \leq T} \|u''(t)\| \cdot d_j^i \\ &\leq n^{-2} \sup_{0 \leq t \leq T} \|u''(t)\| \frac{\sin(\pi\alpha)}{\pi} j^\alpha \\ &\quad + \sum_{i=1}^s n^{-2} \sup_{0 \leq t \leq T} \|u''(t)\| \frac{\sin(\pi\alpha_i)}{\pi} j^{\alpha_i} \\ &\leq \Delta t^{2-\alpha} + \sum_{i=1}^s \Delta t^{2-\alpha_i}. \quad \square \end{aligned} \tag{34}$$

#### 4. Space Discretization and Convergence

In this section, we will consider the space discretization for MT-FPDEs (1) and show the complete process and details of numerical scheme. The variational form of (1) is to find  $u(t, \cdot) \in H_0^1(\Omega)$ , such that, for all  $v \in H_0^1(\Omega)$ ,

$$\begin{aligned} & \left( {}^R_0 D_t^\alpha u(t, x), v \right)_{L_2(\Omega)} + \sum_{i=1}^s a_i \left( {}^R_0 D_t^{\alpha_i} u(t, x), v \right)_{L_2(\Omega)} \\ & + \left( \nabla_x u, \nabla_x v \right) = \left( \bar{f}(t, x), v \right)_{L_2(\Omega)}. \end{aligned} \tag{35}$$

Let  $h$  denote the maximal length of intervals in  $\Omega$  and let  $r$  be any nonnegative integer. We denote the norm in  $H^r(\Omega)$  by  $\|\cdot\|_{H^r(\Omega)}$ . Let  $S_h \subset H_0^1$  be a family of finite element spaces with the accuracy of order  $r \geq 2$ , that is,  $S_h$  consists of continuous functions on the closure  $\bar{\Omega}$  of  $\Omega$  which are polynomials of degree at most  $r-1$  in each interval and which vanish outside  $\Omega_h$ , such that for small  $h$ ,  $v \in H^b(\Omega) \cap H_0^1(\Omega)$ ,

$$\begin{aligned} \inf_{\chi \in S_h} \left( \|v - \chi\|_{L_2(\Omega)} + h \|\nabla_x(v - \chi)\|_{L_2(\Omega)} \right) &\leq Ch^b \|v\|_{H^b(\Omega)}, \\ 1 &\leq b \leq r. \end{aligned} \tag{36}$$

The semidiscrete problem of (1) is to find the approximate solution  $u_h(t) = u_h(t, \cdot) \in S_h$  and  $\bar{f}(t) = \bar{f}(t, \cdot)$  for each  $t$  such that

$$\begin{aligned} & \left( {}^R_0 D_t^\alpha u_h(t), \chi \right)_{L_2(\Omega)} + \sum_{i=1}^s a_i \left( {}^R_0 D_t^{\alpha_i} u_h(t), \chi \right)_{L_2(\Omega)} \\ & + \left( \nabla_x u_h(t), \nabla_x \chi \right)_{L_2(\Omega)} = \left( \bar{f}(t), \chi \right)_{L_2(\Omega)}, \quad \forall \chi \in S_h. \end{aligned} \tag{37}$$

Let  $U^N = u_h(t_N, x)$ . After the time discretization, we have

$$\begin{aligned} & \Delta t^{-\alpha} \sum_{k=0}^N \omega_{kN}^{(\alpha)} (U^N, \chi)_{L_2(\Omega)} + \sum_{i=1}^s a_i \Delta t^{-\alpha_i} \sum_{k=0}^N \omega_{kN}^{(\alpha_i)} (U^N, \chi)_{L_2(\Omega)} \\ & + \left( \nabla_x U^N, \nabla_x \chi \right)_{L_2(\Omega)} = \left( \bar{f}(t), \chi \right)_{L_2(\Omega)}, \quad \forall \chi \in S_h. \end{aligned} \tag{38}$$

In terms of the basis  $\{\psi_m\}_{m=1}^{M-1} \subseteq S_h$ , choosing  $\chi = \psi_m$ , writing

$$u_h(t_N, x) = \sum_{j=1}^{M-1} U_j^N \psi_j(x), \tag{39}$$

and inserting it into (38), one obtains

$$\begin{aligned} & \sum_{j=1}^{M-1} \Delta t^{-\alpha} \sum_{k=1}^N \omega_{kN}^{(\alpha)} U_j^{N-k} (\psi_j, \psi_m)_{L_2(\Omega)} \\ & + \sum_{i=1}^s a_i \sum_{j=1}^{M-1} \Delta t^{-\alpha_i} \sum_{k=1}^N \omega_{kN}^{(\alpha_i)} U_j^{N-k} (\psi_j, \psi_m)_{L_2(\Omega)} \\ & + \sum_{j=1}^{M-1} U_j^N (\nabla_x \psi_j, \nabla_x \psi_m)_{L_2(\Omega)} = \left( \bar{f}, \psi_m \right)_{L_2(\Omega)}, \\ & m = 1, 2, \dots, M-1. \end{aligned} \tag{40}$$

Let  $U_N = (U_1^N, U_2^N, \dots, U_{M-1}^N)^T$ . From (40), we obtain a vector equation

$$\begin{aligned} & \Psi_1 \left( \Delta t^{-\alpha} \omega_{0N}^{(\alpha)} U_N + \sum_{i=1}^s a_i \Delta t^{-\alpha_i} \omega_{0N}^{(\alpha_i)} U_N \right) + \Psi_2 U_N \\ & = \Psi_1 \left( F_N - \Delta t^{-\alpha} \sum_{k=1}^N \omega_{kN}^{(\alpha)} U_{N-k} - \sum_{i=1}^s a_i \Delta t^{-\alpha_i} \sum_{k=1}^N \omega_{kN}^{(\alpha_i)} U_{N-k} \right), \end{aligned} \tag{41}$$

where initial condition is  $U_0 = u(0, x)$ ,  $\Psi_1 := \{(\psi_j, \psi_m)_{L_2(\Omega)}\}_{j,m=1}^{M-1}$  is the mass matrix,  $\Psi_2$  is stiffness matrix as  $\Psi_2 := \{(\nabla_x \psi_j, \nabla_x \psi_m)_{L_2(\Omega)}\}_{j,m=1}^{M-1}$ , and  $F_N := (\bar{f}_1, \dots, \bar{f}_{M-1})^T$  is a vector valued function. Then, we can obtain the solution  $U_N$  at  $t = t_N$ .

Let  $R_h : H^1(\Omega) \rightarrow S_h$  be the elliptic projection, defined by  $(\nabla_x R_h u, \nabla_x \chi)_{L_2(\Omega)} = (\nabla_x u, \nabla_x \chi)_{L_2(\Omega)}$ , for all  $\chi \in S_h$ .

**Lemma 5** (see [35]). *Assume that (36) holds, then with  $R_h$  and  $v \in H^b(\Omega) \cap H_0^1(\Omega)$ , we have  $\|R_h v - v\|_{L_2(\Omega)} + h \|\nabla_x(R_h v - v)\|_{L_2(\Omega)} \leq Ch^b \|v\|_{H^b(\Omega)}$  for  $1 \leq b \leq r$ .*

In virtue of the standard error estimate for the FEM of MT-FPDEs, one has the following theorem which can be proved easily by Lemma 5 and the similar proof in [35].

**Theorem 6.** *For  $0 < \alpha_s < \dots < \alpha_1 < \alpha < 1$ , let  $u_h \in S_h$  and  $u(t, \cdot) \in H_0^1(\Omega)$  be, respectively, the solutions of (37) and (1), then  $\|u - u_h\|_{L_2(\Omega)} \leq h^2 \|u\|_{L_2(\Omega)}$ .*

#### 5. Stability of the Numerical Method

In this section, we analyze the stability of the FEM for MT-FPDEs (1)–(3). Now we do some preparations before proving the stability of the method. Based on the definition of coefficients  $\omega_{kj}^{(\alpha)}$  in Section 3, we can obtain the following lemma easily.

**Lemma 7.** For  $0 < \alpha < 1$ , the coefficients  $\omega_{kj}^{(\alpha)}$ ,  $(k = 1, \dots, j)$  satisfy the following properties:

$$(i) \omega_{0j}^{(\alpha)} > 0 \text{ and } \omega_{kj}^{(\alpha)} < 0 \text{ for } k = 1, 2, \dots, j,$$

$$(ii) \Gamma(2 - \alpha) \sum_{k=1}^j \omega_{kj}^{(\alpha)} = (1 - \alpha)j^{-\alpha} + 1.$$

Now we report the stability theorem of this FEM for MT-FPDEs in this section as follows.

**Theorem 8.** The FEM defined as in (38) is unconditionally stable.

*Proof.* In (38), let  $\chi(\cdot) = U^j(\cdot)$  at  $t = t_j$  and the right hand  $\bar{f} = 0$ . We have

$$\begin{aligned} & \Delta t^{-\alpha} \left\{ \frac{1}{\Gamma(2 - \alpha)} (U^j, U^j)_{L_2(\Omega)} + \sum_{k=1}^{j-1} \omega_{kj}^{(\alpha)} (U^{j-k}, U^j)_{L_2(\Omega)} \right. \\ & \quad \left. + \omega_{jj}^{(\alpha)} (U^0, U^j)_{L_2(\Omega)} \right\} \\ & + \sum_{i=1}^s a_i \Delta t^{-\alpha_i} \left\{ \frac{1}{\Gamma(2 - \alpha_i)} (U^j, U^j)_{L_2(\Omega)} \right. \\ & \quad \left. + \sum_{k=1}^{j-1} \omega_{kj}^{(\alpha_i)} (U^{j-k}, U^j)_{L_2(\Omega)} + \omega_{jj}^{(\alpha_i)} (U^0, U^j)_{L_2(\Omega)} \right\} \\ & + (\nabla_x U^j, \nabla_x U^j)_{L_2(\Omega)} = 0. \end{aligned} \quad (42)$$

Using Cauchy-Schwarz inequality,  $\pm(U^{j-k}, U^j) \leq (1/2)(\|U^{j-k}\|_{L_2(\Omega)}^2 + \|U^j\|_{L_2(\Omega)}^2)$  for  $k = 0, 1, 2, \dots, N$  and Lemma 7, we get

$$\begin{aligned} & \left( \frac{\Delta t^{-\alpha}}{2\Gamma(2 - \alpha)} (1 + (1 - \alpha)j^{-\alpha}) \right. \\ & \quad \left. + \sum_{i=1}^s a_i \frac{\Delta t^{-\alpha_i}}{2\Gamma(2 - \alpha_i)} (1 + (1 - \alpha_i)j^{-\alpha_i}) \right) \|U^j\|_{L_2(\Omega)}^2 + \|\nabla_x U^j\|_{L_2(\Omega)}^2 \\ & \leq \frac{\Delta t^{-\alpha}}{2} \left[ -\sum_{k=1}^{j-1} \omega_{kj}^{(\alpha)} \|U^{j-k}\|_{L_2(\Omega)}^2 - \omega_{jj}^{(\alpha)} \|U^0\|_{L_2(\Omega)}^2 \right] \\ & \quad + \sum_{i=1}^s a_i \frac{\Delta t^{-\alpha_i}}{2} \left[ -\sum_{k=1}^{j-1} \omega_{kj}^{(\alpha_i)} \|U^{j-k}\|_{L_2(\Omega)}^2 - \omega_{jj}^{(\alpha_i)} \|U^0\|_{L_2(\Omega)}^2 \right]. \end{aligned} \quad (43)$$

We prove the stability of (37) by induction. Since when  $j = 1$ , we have

$$\begin{aligned} & \left( \frac{\Delta t^{-\alpha}}{2\Gamma(2 - \alpha)} (1 + (1 - \alpha)) \right. \\ & \quad \left. + \sum_{i=1}^s a_i \frac{\Delta t^{-\alpha_i}}{2\Gamma(2 - \alpha_i)} (1 + (1 - \alpha_i)) \right) \|U^1\|_{L_2(\Omega)}^2 \\ & \leq \left( \frac{\Delta t^{-\alpha}}{2\Gamma(2 - \alpha)} (1 - (1 - \alpha)) \right. \\ & \quad \left. + \sum_{i=1}^s a_i \frac{\Delta t^{-\alpha_i}}{2\Gamma(2 - \alpha_i)} (1 - (1 - \alpha_i)) \right) \|U^0\|_{L_2(\Omega)}^2. \end{aligned} \quad (44)$$

The induction basis  $\|U^1\|_{L_2(\Omega)} \leq \|U^0\|_{L_2(\Omega)}$  is presupposed. For the induction step, we have  $\|U^j\|_{L_2(\Omega)} \leq \|U^{j-1}\|_{L_2(\Omega)} \leq \dots \leq \|U^0\|_{L_2(\Omega)}$ . Then using this result, by Lemma 7, we obtain

$$\begin{aligned} & \left( \frac{\Delta t^{-\alpha} (1 + (1 - \alpha)(j + 1)^{-\alpha})}{2\Gamma(2 - \alpha)} \right. \\ & \quad \left. + \sum_{i=1}^s a_i \frac{\Delta t^{-\alpha_i}}{2\Gamma(2 - \alpha_i)} (1 + (1 - \alpha_i)(j + 1)^{-\alpha_i}) \right) \|U^{j+1}\|_{L_2(\Omega)}^2 \\ & \leq \left( \frac{\Delta t^{-\alpha} (1 - (1 - \alpha)(j + 1)^{-\alpha})}{2\Gamma(2 - \alpha)} \right. \\ & \quad \left. + \sum_{i=1}^s a_i \frac{\Delta t^{-\alpha_i}}{2\Gamma(2 - \alpha_i)} (1 - (1 - \alpha_i)(j + 1)^{-\alpha_i}) \right) \|U^0\|_{L_2(\Omega)}^2. \end{aligned} \quad (45)$$

Here  $0 < 1 - \alpha < 1$ . After squaring at both sides of the above inequality, we obtain  $\|U^{j+1}\|_{L_2(\Omega)} \leq \|U^0\|_{L_2(\Omega)}$ .  $\square$

## 6. Numerical Experiments

In this section, we present the numerical examples of MT-FPDEs to demonstrate the effectiveness of our theoretical analysis. The main purpose is to check the convergence behavior of numerical solutions with respect to  $\Delta t$  and  $\Delta x$ , which have been shown in Theorem 4 and Theorem 6. It is noted that the method in [29] is a special case of the method in our paper for fractional partial differential equation with single fractional order. So, we just need to compare FEM in our paper with other existing methods in [8, 28].

*Example 9.* For  $t \in [0, T]$ ,  $x \in (0, 1)$ , consider the MT-FPDEs with two variables as follows:

$$\begin{aligned} & {}_0^C D_t^\alpha u(t, x) + {}_0^C D_t^\beta u(t, x) - \partial_x^2 u(t, x) = f(t, x), \\ & u(0, x) = 0, \quad x \in (0, 1), \\ & u(t, 0) = u(t, 1) = 0, \quad t \in [0, T], \end{aligned} \quad (46)$$



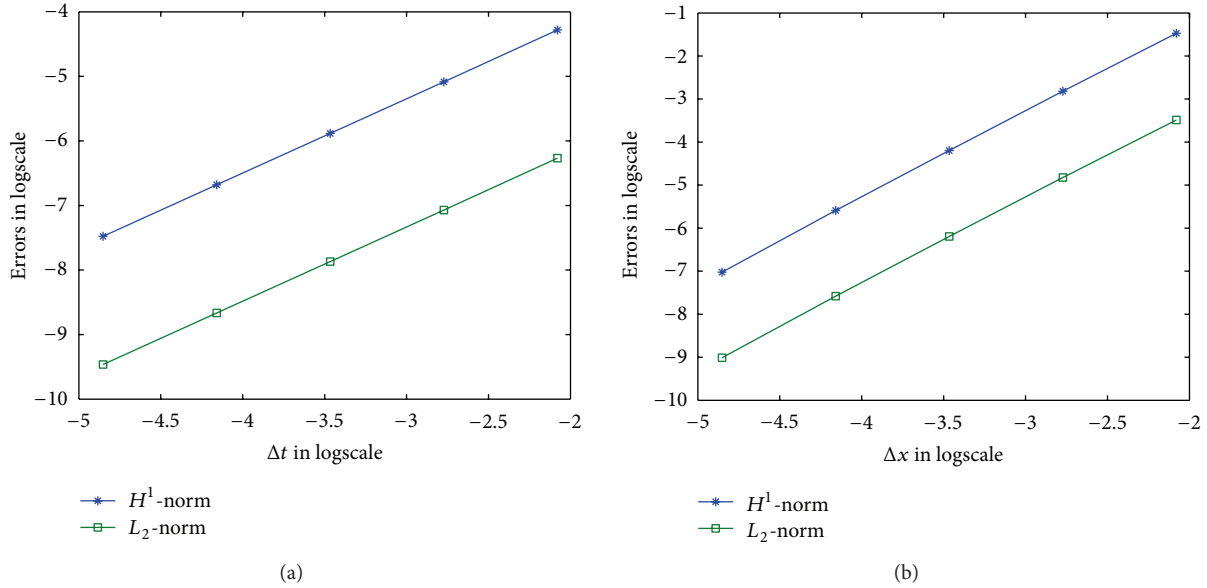


FIGURE 1:  $H^1$ -norm and  $L_2$ -norm of errors for (46) with  $\alpha = 0.9, \beta = 0.5, \Delta x = 0.001$  (a), and  $\Delta t = 0.001$  (b).

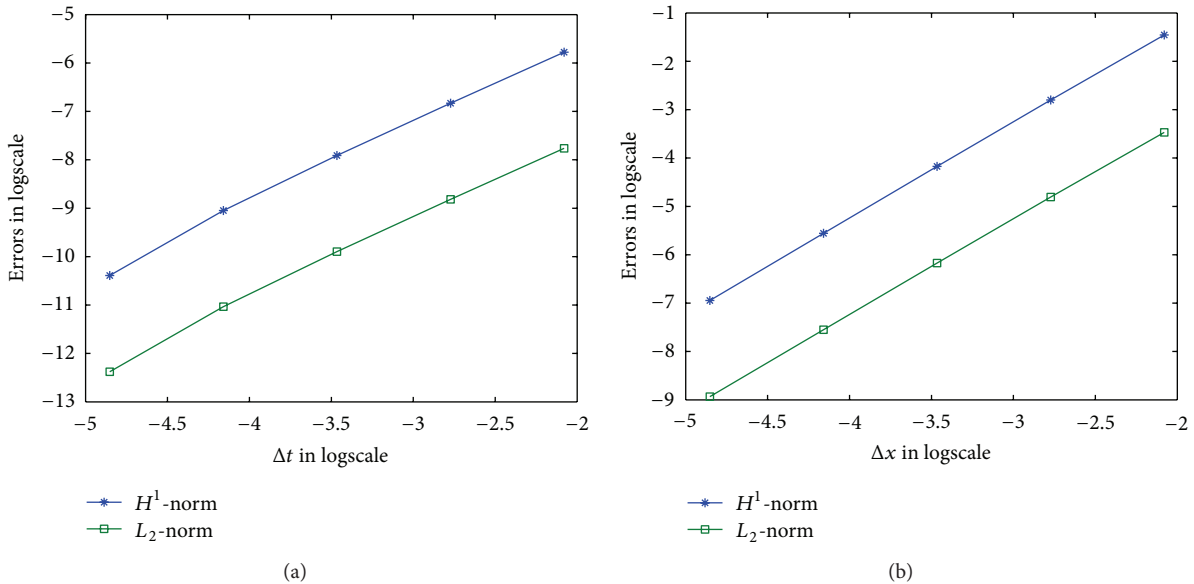


FIGURE 2:  $H^1$ -norm and  $L_2$ -norm of errors for (46) with  $\alpha = 0.5, \beta = 0.25, \Delta x = 0.001$  (a), and  $\Delta t = 0.001$  (b).

where the right-side function  $f(t, x) = (2t^{2-\alpha}/\Gamma(3 - \alpha)) \sin(2\pi x) + (2t^{2-\beta}/\Gamma(3 - \beta)) \sin(2\pi x) + 4\pi^2 \sin(2\pi x)t^2$ . The exact solution is  $u(t, x) = t^2 \sin(2\pi x)$ .

We use this example to check the convergence rate (c. rate) and CPU time (CPUT) of numerical solutions with respect to the fractional orders  $\alpha$  and  $\beta$ .

In the first test, we fix  $T = 1, \alpha = 0.9$  and  $\beta = 0.5$  and choose  $\Delta x = 0.001$  which is small enough such that the space discretization errors are negligible as compared with the time errors. Choosing  $\Delta t = 1/2^i$  ( $i = 2, 4, \dots, 7$ ), we report that the convergence rate of FDM in time is nearly 1.15 in Table 1, which matches well with the result of Theorem 4. On the

other hand, Table 2 shows that an approximate convergence rate is 2, by fixing  $\Delta t = 0.001$  and choosing  $\Delta x = 1/2^i$  ( $i = 2, \dots, 6$ ), which matches well with the result of Theorem 6. In the second test, we give the convergence rate when  $\alpha = 0.5, \beta = 0.25$  for  $\Delta t$  in Table 3, and  $\Delta x$  in Table 4, respectively. We also report the  $L_2$ -norm and  $H^1$ -norm of errors in Figures 1 and 2, respectively.

Fixing  $\Delta x = 0.001, \alpha = 0.9$ , and  $\beta = 0.3$  in (46), we compare the error and CPUT calculated by the FEM in this paper with the FDM in [8] and the FPCM in [8]. From Table 5, it can be seen that the FEM in this paper is computationally effective.

TABLE 1: Convergence rate in time for (46) with  $\alpha = 0.9$  and  $\beta = 0.5$ .

$\Delta x$	$\Delta t$	$H^1$ -norm	$L_2$ -norm	c. rate	CPUT (seconds)
0.001	1/4	$1.3815 \times 10^{-2}$	$1.8960 \times 10^{-3}$		0.214
0.001	1/16	$6.1890 \times 10^{-3}$	$8.4939 \times 10^{-4}$	1.1585	0.357
0.001	1/32	$2.7858 \times 10^{-3}$	$3.8234 \times 10^{-4}$	1.1516	0.736
0.001	1/64	$1.2571 \times 10^{-3}$	$1.7252 \times 10^{-4}$	1.1481	1.438
0.001	1/128	$5.6567 \times 10^{-4}$	$7.7635 \times 10^{-5}$	1.1520	2.922

TABLE 2: Convergence rate in space for (46) with  $\alpha = 0.9$  and  $\beta = 0.5$ .

$\Delta t$	$\Delta x$	$H^1$ -norm	$L_2$ -norm	c. rate	CPUT (seconds)
0.001	1/4	0.2294	$3.0611 \times 10^{-2}$		20.35
0.001	1/16	$5.9763 \times 10^{-2}$	$8.0381 \times 10^{-3}$	1.9291	21.85
0.001	1/32	$1.5067 \times 10^{-2}$	$2.0442 \times 10^{-3}$	1.9753	26.68
0.001	1/64	$3.7356 \times 10^{-3}$	$5.0966 \times 10^{-4}$	2.0039	32.72
0.001	1/128	$8.9129 \times 10^{-4}$	$1.2198 \times 10^{-4}$	2.0629	41.03

TABLE 3: Convergence rate in time for (46) with  $\alpha = 0.5$  and  $\beta = 0.25$ .

$\Delta x$	$\Delta t$	$H^1$ -norm	$L_2$ -norm	c. rate	CPUT (seconds)
0.001	1/4	$3.0985 \times 10^{-3}$	$4.2525 \times 10^{-4}$		0.218
0.001	1/16	$1.0789 \times 10^{-3}$	$1.4807 \times 10^{-4}$	1.5221	0.413
0.001	1/32	$3.6702 \times 10^{-4}$	$5.0372 \times 10^{-5}$	1.5556	0.921
0.001	1/64	$1.1772 \times 10^{-4}$	$1.6156 \times 10^{-5}$	1.6406	1.855
0.001	1/128	$3.0704 \times 10^{-5}$	$4.2139 \times 10^{-6}$	1.6388	3.783

TABLE 4: Convergence rate in space for (46) with  $\alpha = 0.5$  and  $\beta = 0.25$ .

$\Delta t$	$\Delta x$	$H^1$ -norm	$L_2$ -norm	c. rate	CPUT (seconds)
0.001	1/4	$2.3325 \times 10^{-1}$	$3.1119 \times 10^{-2}$		23.73
0.001	1/16	$6.0836 \times 10^{-2}$	$8.1823 \times 10^{-3}$	1.9272	26.29
0.001	1/32	$1.5382 \times 10^{-2}$	$2.0870 \times 10^{-3}$	1.9711	33.49
0.001	1/64	$3.8569 \times 10^{-3}$	$5.2622 \times 10^{-4}$	1.9877	41.68
0.001	1/128	$9.6378 \times 10^{-4}$	$1.3189 \times 10^{-4}$	1.9963	55.24

TABLE 5: Comparison of error and CPUT for (46) with  $\alpha = 0.9$  and  $\beta = 0.3$ .

$\Delta x$	$\Delta t$	FEM		FDM [8]		FPCM [8]	
		Error	CPUT	Error	CPUT	Error	CPUT
0.001	1/4	$3.7056 \times 10^{-3}$	0.238	$5.8723 \times 10^{-3}$	0.897	$2.2027 \times 10^{-2}$	6.16
0.001	1/8	$1.6794 \times 10^{-3}$	0.481	$2.6751 \times 10^{-3}$	1.837	$8.7467 \times 10^{-3}$	16.63
0.001	1/16	$7.6528 \times 10^{-4}$	0.962	$1.2159 \times 10^{-3}$	3.512	$3.4693 \times 10^{-3}$	30.11
0.001	1/32	$3.5009 \times 10^{-4}$	1.335	$5.5190 \times 10^{-4}$	7.001	$1.3765 \times 10^{-3}$	52.71
0.001	1/64	$1.6027 \times 10^{-4}$	2.703	$2.4997 \times 10^{-4}$	14.45	$5.4564 \times 10^{-4}$	106.49

TABLE 6: Comparison of error, convergence rate, and CPUT for (47) with  $\alpha = 0.5$  and  $\beta = 0.3$ .

$\Delta t$	DFBDM (Section 3)			FEM2 [28]		
	Error	c. rate	CPUT	Error	c. rate	CPUT
1/4	$9.1975 \times 10^{-4}$		0.000864	$3.6606 \times 10^{-3}$		0.001862
1/8	$3.3037 \times 10^{-4}$	1.4772	0.001986	$7.8173 \times 10^{-3}$	2.2274	0.004902
1/16	$1.1375 \times 10^{-4}$	1.5382	0.004649	$1.6210 \times 10^{-4}$	2.2697	0.051816
1/32	$3.5112 \times 10^{-5}$	1.6958	0.012112	$3.2629 \times 10^{-4}$	2.3127	0.518130



*Example 10.* Consider the following multiterm fractional differential problem:

$${}_0^C D_t^\alpha u(t) + t^{-0.2} {}_0^C D_t^\beta u(t) = f(t), \quad u(0) = 0, \quad (47)$$

where  $f(t) = (4t^{1.5}/\Gamma(2.5)) + (12t^2/\Gamma(3.5))$ . For  $\alpha = 0.5$  and  $\beta = 0.3$ , the exact solution is  $u(t) = t^2 + t^{2.5}$ .

For the problem (47), our method in this paper is just the DFBDM in Section 3. Therefore, we only need to compare M1 with the FEM in [28] (FEM2). In Table 6, although the convergence rate of FEM2 is higher than that of DFBDM, the error and CPUT of DFBDM are smaller than those of FEM2.

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