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Research Article

A Generalized Nonlinear Gronwall-Bellman Inequality with Maxima in Two Variables

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This paper deals with a generalized form of nonlinear retarded Gronwall-Bellman type integral inequality in which the maximum of the unknown function of two variables is involved. This form includes both a nonconstant term outside the integrals and more than one distinct nonlinear integrals. Requiring neither monotonicity nor separability of given functions, we apply a technique of monotonization to estimate the unknown function. Our result can be used to weaken conditions for some known results. We apply our result to a boundary value problem of a partial differential equation with maxima for uniqueness.

1. Introduction

The Gronwall-Bellman inequality [1, 2] plays an important role in the study of existence, uniqueness, boundedness, stability, invariant manifolds, and other qualitative properties of solutions of differential equations and integral equations. There can be found a lot of its generalizations in various cases from literatures (see, e.g., [3–18]). In 1956, Bihari [3] discussed the integral inequality

$$u(t) \le c + \int_0^t f(s)\,\omega(u(s))\,ds, \quad t \ge 0, \tag{1}$$

where $c > 0$ is a constant, f is a continuous and nonnegative function, and ω is a continuous and nondecreasing positive function. Replacing t by a function $b(t)$ in (1), Lipovan [4] investigated the retarded integral inequality

$$u(t) \le c + \int_{t_0}^t f(s)\,\omega(u(s))\,ds + \int_{b(t_0)}^{b(t)} g(s)\,\omega(u(s))\,ds, \quad t_0 \le t < t_1. \tag{2}$$

Their results were further generalized by Agarwal et al. [5] to the inequality

$$u(t) \le a(t) + \sum_{i=1}^{n} \int_{b_i(t_0)}^{b_i(t)} f_i(t,s)\,\omega_i(u(s))\,ds, \quad t_0 \le t < t_1, \tag{3}$$

where the constant c is replaced with a function $a(t)$, b_i's are continuously differentiable and nondecreasing functions, and ω_i's are continuous and nondecreasing positive functions such that

$$\omega_1 \propto \omega_2 \propto \cdots \propto \omega_n, \tag{4}$$

that is, each ratio ω_{i+1}/ω_i is also nondecreasing on $A \subseteq \mathbb{R}\backslash\{0\}$, called in [6] that ω_{i+1} is *stronger nondecreasing* than ω_i. On the basis of this work, Wang [7] considered the inequality of two variables

$$u^p(x,y) \le a(x,y) + \sum_{i=1}^{n} \int_{b_i(x_0)}^{b_i(x)} \int_{c_i(y_0)}^{c_i(y)} f_i(x,y,s,t)\,\omega_i(u(s,t))\,dt\,ds, \tag{5}$$

where the functions a, f_i, and ω_i are not required to be monotone, and those ω_i's are not required to be stronger monotone than the one after the next as shown in (4). This inequality belongs to both the case of multivariables, to which great attentions [7–11] have been paid, and to the case that the left-hand side is a composition of the unknown function with a known function, in which Ou-Iang's idea [19] was applied [11–14]. He applied a technique of monotonization to construct a sequence of functions, made each function possess stronger monotonization than the previous one, and gave an estimate for the unknown function u.

On the other aspect, many problems in the control theory can be modeled in the form of differential equations with the maxima of the unknown function [20–22]. In connection with the development of the theory of differential equations with maxima (see, e.g., [20, 21, 23]) and partial differential equations with maxima [24, 25], a new type of integral inequalities with maxima is required, respectively. There have been given some results for integral inequalities containing the maxima of the unknown function [23, 26–28]. Concretely, in 2012, Bohner et al. [26] discussed the following system of integral inequalities:

$$\begin{aligned}\varphi(u(t)) \le a(t) &+ \sum_{i=1}^{m} \int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(s)\, u^p(s)\, \omega_i(u(s))\, ds \\ &+ \sum_{j=m+1}^{m+n} \int_{\alpha_j(t_0)}^{\alpha_j(t)} f_j(s)\, u^p(s) \\ &\times \omega_j\left(\max_{\xi\in[s-h,s]} g(u(\xi))\right) ds, \\ u(t) &\le \psi(t),\end{aligned} \tag{6}$$

where a, f_i's, ω_i's, φ, and ψ are nonnegative continuous functions and α_i's are nonnegative continuously differentiable and nondecreasing functions. They required that $a(t) \ge 1$, φ is C^1 on $\mathbb{R}_+ := [0, +\infty)$ and increasing such that $\varphi(tx) \ge t\varphi(x)$ for $0 \le t \le 1$, and ω_i satisfies the following: (i) $\omega_i \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ is an increasing function, and (ii) $\omega_i(tx) \ge t\omega_i(x)$ for all $0 \le t \le 1$ and $x > 0$. Bainov and Hristova [23] considered the following system:

$$\begin{aligned}u(x,y) \le a(x,y) &+ \int_{x_0}^{x}\int_{y_0}^{y} f(s,t)\, u^p(s,t)\, dt\, ds \\ &+ \int_{\alpha(x_0)}^{\alpha(x)}\int_{y_0}^{y} h(s,t)\left(\max_{\xi\in[s-h,s]} u^p(\xi,t)\right) dt\, ds, \\ u(x,y) &\le \psi(x,y),\end{aligned} \tag{7}$$

where $a(x, y)$ is nonnegative and nondecreasing in both of its arguments, f, h, and ψ are continuous and nonnegative functions, and $p \in (0, 1]$.

In this paper, we consider the system of integral inequalities as follows:

$$\begin{aligned}&\varphi(u(x,y)) \\ &\quad\le a(x,y) \\ &\qquad+ \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} f_i(x,y,s,t)\, \omega_i(u(s,t))\, dt\, ds \\ &\qquad+ \sum_{j=n+1}^{n+m} \int_{\alpha_j(x_0)}^{\alpha_j(x)}\int_{\beta_j(y_0)}^{\beta_j(y)} f_j(x,y,s,t) \\ &\qquad\qquad\times \omega_j\left(\max_{\xi\in[s-h,s]} g(u(\xi,t))\right) dt\, ds, \\ &\qquad\qquad (x,y) \in [x_0,x_1)\times[y_0,y_1), \\ &u(x,y) \le \psi(x,y), \quad (x,y) \in [\alpha_*(x_0)-h, x_0]\times[y_0,y_1),\end{aligned} \tag{8}$$

where a, f_i's, ω_i's, and g are continuous and nonnegative functions, α_i's and β_i's are nonnegative continuously differentiable and nondecreasing functions, and $\alpha_*(x_0) := \min_{1\le i\le m+n}\alpha_i(x_0)$. As required in previous works [27–29], we suppose that $0 \le \alpha_i(t) \le t$, $0 \le \beta_i(t) \le t$, $h > 0$ is constant. In this paper, we require neither monotonicity of a, ω_i's, f_i's, and g nor $a(x, y) \ge 1$. We monotonize those ω_i's to make a sequence of functions in which each one possesses stronger monotonicity than the previous one so as to give an estimation for the unknown function. We can use our result to discuss inequalities (6) and (7), giving the stronger results under weaker conditions. We finally apply the obtained result to a boundary value problem of a partial differential equation with maxima for uniqueness.

2. Main Result

Consider system (8) of integral inequalities with $x_0 < x_1$ and $y_0 < y_1$ in $\mathbb{R}_+ := [0, \infty)$. Let $\Lambda := [x_0, x_1) \times [y_0, y_1)$, $\Omega := [\alpha_*(x_0) - h, x_0] \times [y_0, y_1)$. Suppose that

(H_1) $\alpha_i : [x_0, x_1) \to \mathbb{R}_+$ ($i = 1, 2, \ldots, m + n$) and $\beta_i : [y_0, y_1) \to [y_0, y_1)$, $i = 1, 2, \ldots, m + n$, are nondecreasing such that $\alpha_i(x) \le x$ on $[x_0, x_1)$, $\beta_i(y) \le y$ on $[y_0, y_1)$ and $\beta_i(y_0) = y_0$;

(H_2) all f_i's ($i = 1, 2, \ldots, m + n$) are continuous and nonnegative functions on $\Lambda \times [\alpha_*(x_0), x_1) \times [y_0, y_1)$;

(H_3) $g, \varphi : \mathbb{R}_+ \to \mathbb{R}_+$ and $\psi : [\alpha_*(x_0) - h, x_1) \to \mathbb{R}_+$ are continuous, and φ is strictly increasing such that $\lim_{t\to\infty}\varphi(t) = +\infty$;

(H_4) all ω_i's ($i = 1, 2, \ldots, m + n$) are continuous on $\mathbb{R}_+$ and positive on $(0, +\infty)$;

(H_5) $a(x, y)$ is a continuous and nonnegative function on Λ.

For those ω_i's given in (H_4), define $\widetilde{\omega}_i(t)$, $i = 1, \ldots, m+n$, inductively by

$$\widetilde{\omega}_1(t) := \max_{\tau\in[0,t]} \{\omega_1(\tau)\},$$
$$\widetilde{\omega}_{i+1}(t) := \max_{\tau\in[0,t]} \left\{ \frac{\omega_{i+1}(\tau)}{\widetilde{\omega}_i(\tau)+\epsilon_i} \right\} \widetilde{\omega}_i(t), \tag{9}$$

for $i = 1, 2, \ldots, m-1$ and

$$\widetilde{\omega}_{m+1}(t) := \max_{\tau\in[0,t]} \left\{ \frac{\widehat{\omega}_{m+1}\left(\max_{s\in[0,\tau]}\{g(s)\}\right)}{\widetilde{\omega}_m(\tau)+\epsilon_m} \right\} \widetilde{\omega}_m(t),$$
$$\widetilde{\omega}_{j+1}(t) := \max_{\tau\in[0,t]} \left\{ \frac{\widehat{\omega}_{j+1}\left(\max_{s\in[0,\tau]}\{g(s)\}\right)}{\widetilde{\omega}_j(\tau)+\epsilon_j} \right\} \widetilde{\omega}_j(t), \tag{10}$$

for $j = m+1, \ldots, m+n$, where $\widehat{\omega}_j(t) := \max_{\tau\in[0,t]}\{\omega_j(\tau)\}$ for $j = m+1, \ldots, m+n$, $\epsilon_i := \varepsilon_1$ if $\widetilde{\omega}_i(0) = 0$ or $:= 0$ if $\widetilde{\omega}_i(0) \neq 0$ for $i = 1, 2, \ldots, m+n-1$, and $\varepsilon_1 > 0$ be a given very small constant.

Theorem 1. *Suppose that (H_1)–(H_5) hold, $\max_{s\in[\alpha_*(x_0)-h,x_0]}\psi(s,y) \leq \varphi^{-1}(a(x_0,y))$ for all $y \in [y_0, y_1)$ and $u \in C(\Omega, \mathbb{R}_+)$ satisfies the system (8) of integral inequalities. Then,*

$$u(x,y) \leq \varphi^{-1}\left(W_{m+n}^{-1}\left(\Omega_{m+n}(x,y)\right)\right), \tag{11}$$

for all $(x,y) \in [x_0, X_1) \times [y_0, Y_1)$, where

$$\Omega_i(x,y) := W_i(r_i(x,y)) + \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} \max_{(\iota,\xi)\in[x_0,x]\times[y_0,y]} f_i(\iota,\xi,s,t)\,dt\,ds, \tag{12}$$

W_i^{-1} is the inverse of the function

$$W_i(u) := \int_{u_i}^{u} \frac{ds}{\widetilde{\omega}_i(\varphi^{-1}(s))}, \quad u \geq u_i,\ i = 1, 2, \ldots, m+n, \tag{13}$$

$u_i > 0$ is a given constant, $\widetilde{\omega}_i$ is defined just before the theorem, and $r_i(x,y)$ is defined recursively by

$$r_1(x,y) = \max_{(\iota,\xi)\in[x_0,x]\times[y_0,y]} a(\iota,\xi),$$
$$r_{i+1}(x,y) = W_i^{-1}(\Omega_i(x,y)), \tag{14}$$

for $i = 1, 2, \ldots, m+n-1$, and $X_1 \in [x_0, x_1), Y_1 \in [y_0, y_1)$ are chosen such that

$$\Omega_i(X_1, Y_1) \leq \int_{u_i}^{\infty} \frac{ds}{\widetilde{\omega}_i(\varphi^{-1}(s))}, \tag{15}$$

for $i = 1, 2, \ldots, m+n$.

For the special choice that $n = m = 1$, $\omega_1(s) = s^p$, $\omega_2(s) = s$, $f_1(x,y,s,t) = f(s,t)$, $f_2(x,y,s,t) = h(s,t)$, $\alpha_1(s) = s$, $\alpha_2(s) = \alpha(s)$, $g(s) = s^p$, and $\beta_1(s) = \beta_2(s) = s$, where α is a nonnegative continuously differentiable and nondecreasing function, Theorem 1 gives an estimate for the unknown u in the system (7). we require neither the monotonicity of a nor the monotonicity of ω_i. Obviously, Lemma 2 and Theorem 1 are applicable to more general forms than Corollary 2.3.4 in [23]. Even if $\omega_i(s)$ is enlarged to $\max_{1\leq i\leq m+n}\omega_i(s)$ such that (8) is changed into the form of (2.1) in [29], where $m = n = 1$, our theorem gives a better estimate. For example, the system of inequalities

$$\begin{gathered} u(x,y) \leq 3 + 4\int_0^x \int_0^y ts\sqrt{u(s,t)+1}\,dt\,ds + 4\int_0^{\sqrt{x}} \int_0^y ts\left(\max_{\xi\in[s-h,s]} u(\xi,t)+1\right)dt\,ds, \\ (x,y) \in [0,x_1) \times [0,y_1), \\ u(x,y) \leq x+3, \quad (x,y) \in [-h,0] \times [0,y_1), \end{gathered} \tag{16}$$

implies that

$$\begin{gathered} u(x,y) \leq 3 + 4\int_0^x \int_0^y ts(u(s,t)+1)\,dt\,ds + 4\int_0^{\sqrt{x}} \int_0^y ts\left(\max_{\xi\in[s-h,s]} u(\xi,t)+1\right)dt\,ds, \\ (x,y) \in [0,x_1) \times [0,y_1), \\ u(x,y) \leq x+3, \quad (x,y) \in [-h,0] \times [0,y_1), \end{gathered} \tag{17}$$

by enlarging $\sqrt{s+1}$ to $s+1$. Applying Theorem 1, we obtain

$$u(x,y) \leq \frac{\left(x^2y^2+4\right)^2}{4} e^{xy^2}, \quad (x,y) \in [0,x_1) \times [0,y_1). \tag{18}$$

On the other hand, Theorem 2.2 of [29] gives from (17) that

$$u(t) \le 4e^{(x^2y^2+xy^2)}, \quad (x,y) \in [0,x_1)\times[0,y_1). \tag{19}$$

Clearly, (18) is sharper than (19) for large x and y.

In order to prove Theorem 1, we need the following lemma.

Lemma 2. *Suppose that*

(C1) $\alpha_i : [x_0,x_1) \to \mathbb{R}_+$ $(i = 1,2,\ldots,m+n)$ *and* $\beta_i : [y_0,y_1) \to \mathbb{R}_+$ $(i=1,2,\ldots,m+n)$ *are nondecreasing such that* $\alpha_i(x) \le x$ *on* $[\alpha_*(x_0),x_1)$ *and* $\beta_i(y) \le y$ *on* $[y_0,y_1)$ *and* $\beta_i(y_0) = y_0$;

(C2) $\psi \in C([\alpha_*(x_0)-h,x_1)\times[y_0,y_1),\mathbb{R}_+)$, $b_i \in C([\alpha_*(x_0),x_1)\times[y_0,y_1),\mathbb{R}_+)$ *for* $i = 1,2,\ldots,m+n$;

(C3) *all* h_i*'s* $(i = 1,2,\ldots,m+n)$ *are continuous and nondecreasing on* $\mathbb{R}_+$ *and positive on* $(0,+\infty)$ *such that* $h_1 \propto h_2 \propto \ldots \propto h_{m+n}$;

(C4) $a(x,y)$ *is continuously differentiable in* x *and* y*, nonnegative on* $[\alpha_*(x_0),x_1)\times[y_0,y_1)$*, and* $\max_{s\in[\alpha_*(x_0)-h,x_1]}\psi(s,y) \le a(x_0,y)$ *for all* $y \in [y_0,y_1)$.

If $u \in C([\alpha_*(x_0)-h,x_1)\times[y_0,y_1),\mathbb{R}_+)$ *satisfies the system of inequalities as follows:*

$$\begin{aligned} u(x,y) &\le a(x,y) + \sum_{i=1}^{m}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)\,h_i(u(s,t))\,dt\,ds \\ &\quad + \sum_{j=m+1}^{m+n}\int_{\alpha_j(x_0)}^{\alpha_j(x)}\int_{\beta_j(y_0)}^{\beta_j(y)} h_j(s,t)\,h_j\left(\max_{\xi\in[s-h,s]}u(\xi,t)\right)dt\,ds, \\ &\qquad (x,y)\in\Lambda, \\ u(x,y) &\le \psi(x,y), \quad (x,y)\in\Omega, \end{aligned} \tag{20}$$

then

$$\begin{aligned} u(x,y) \le H_{m+n}^{-1}\Bigg(& H_{m+n}(\gamma_{m+n}(x,y)) \\ &+ \int_{\alpha_{m+n}(x_0)}^{\alpha_{m+n}(x)}\int_{\beta_{m+n}(y_0)}^{\beta_{m+n}(y)} h_{m+n}(s,t)\,dt\,ds\Bigg), \end{aligned} \tag{21}$$

for all $(x,y) \in [x_0,X^*)\times[y_0,Y^*)$*, where* H_i^{-1} *is the inverse of the function*

$$H_i(u) := \int_{u_i}^{u}\frac{dx}{\omega_i(x)}, \quad u \ge u_i,\ i = 1,2,\ldots,m+n, \tag{22}$$

$u_i > 0$ *is a given constant, and* $\gamma_i(x,y)$ *is defined recursively by*

$$\begin{aligned} \gamma_1(x,y) &:= a(x_0,y_0) + \int_{x_0}^{x}\left|a_x(t,y)\right|dt + \int_{y_0}^{y}\left|a_y(x,s)\right|ds, \\ \gamma_{i+1}(x,y) &:= H_i^{-1}\left(H_i(\gamma_i(x,y)) + \int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(t_0)}^{\beta_i(t)} b_i(s,t)\,dt\,ds\right) \end{aligned} \tag{23}$$

for $i = 1,2,\ldots,m+n-1$*, and* $x_0 \le X^* < x_1$, $y_0 \le Y^* < y_1$ *are chosen such that*

$$H_i(\gamma_i(X^*,Y^*)) + \int_{\alpha_i(x_0)}^{\alpha_i(X^*)}\int_{\beta_i(y_0)}^{\beta_i(Y^*)} b_i(s,t)\,dt\,ds \le \int_{u_i}^{\infty}\frac{ds}{h_i(s)}, \tag{24}$$

for $i = 1,2,\ldots,m+n$.

Proof. From (23), we see that $\gamma_1(x,y)$ is nondecreasing on Λ, $a(x,y) \le \gamma_1(x,y)$, and $\max_{s\in[\alpha_*(x_0)-h,x_1]}\psi(s,y) \le a(x_0,y) \le \gamma_1(x_0,y)$ for $y \in [y_0,y_1)$. It implies from (20) that

$$\begin{aligned} u(x,y) \le \gamma_1(x,y) &+ \sum_{i=1}^{m}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)\,h_i(u(s,t))\,dt\,ds \\ &+ \sum_{j=m+1}^{m+n}\int_{\alpha_j(x_0)}^{\alpha_j(x)}\int_{\beta_j(y_0)}^{\beta_j(y)} b_j(s,t) \\ &\qquad \times h_j\left(\max_{\xi\in[s-h,s]}u(\xi,t)\right)dt\,ds \end{aligned} \tag{25}$$

for all $(x,y) \in \Lambda$. Let

$$z(x,y) := \begin{cases} \gamma_1(x,y) + \displaystyle\sum_{i=1}^{m}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)\,h_i(u(s,t))\,dt\,ds & \\ \quad + \displaystyle\sum_{j=m+1}^{m+n}\int_{\alpha_j(x_0)}^{\alpha_j(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} b_j(s,t)\,h_j\left(\max_{\xi\in[s-h,s]}u(\xi,t)\right)dt\,ds, & (x,y)\in\Lambda, \\ \gamma_1(x_0,y), & (x,y)\in\Omega. \end{cases} \tag{26}$$

Clearly, $z(x, y)$ is nondecreasing in x. Then, we have

$$u(x,y) \le z(x,y), \quad (x,y) \in [\alpha_*(x_0) - h, \xi) \times [y_0, \eta),$$
$$\max_{\xi\in[s-h,s]} u(\xi,y) \le \max_{\xi\in[s-h,s]} z(\xi,y) = z(s,y),$$
$$(s,y) \in [\alpha_*(x_0), x_1) \times [y_0, y_1). \tag{27}$$

From (25), (27), and (28) and the definition of $z(x, y)$ on Λ, we get

$$z(x,y) \le \gamma_1(x,y) + \sum_{i=1}^{m} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t) h_i(u(s,t))\, dt\, ds$$
$$+ \sum_{j=m+1}^{m+n} \int_{\alpha_j(x_0)}^{\alpha_j(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} b_j(s,t) h_j(z(s,t))\, dt\, ds. \tag{28}$$

Applying Theorem 1 of [7] to the case that $f_i(x, y, s, t) = b_i(s, t)$, $a(x, y) = \gamma_1(x, y)$, $p = 1$, and $\omega_i(t) = h_i(t)$, $i = 1, 2, \ldots, m + n$, we obtain (21) from (28). This completes the proof. □

Proof of Theorem 1. First of all, we monotonize some given functions f_i, ω_i, g, and a in the system (8) of integral inequalities. Let

$$\tilde{g}(t) := \max_{\tau\in[0,t]} \{g(\tau)\}, \quad t \ge 0,$$
$$\tilde{a}(x,y) := \max_{(\tau,\xi)\in[x_0,x]\times[y_0,y]} \{a(\tau,\xi)\}, \tag{29}$$
$$(x,y) \in [x_0, x_1) \times [y_0, y_1).$$

From (13), we see that the function W_i is strictly increasing, and therefore its inverse W_i^{-1} is well defined, continuous, and increasing in its domain. The sequence $\{\tilde{\omega}_i(t)\}$, defined by $\omega_i(s)$, consists of nondecreasing nonnegative functions on $\mathbb{R}_+$ and satisfies

$$\omega_i(t) \le \tilde{\omega}_i(t), \quad i = 1, 2, \ldots, m,$$
$$\omega_i(t) \le \hat{\omega}_i(t), \quad i = m+1, \ldots, m+n, \tag{30}$$
$$\hat{\omega}_i(\tilde{g}(t)) \le \tilde{\omega}_i(t), \quad i = m+1, \ldots, m+n.$$

Moreover,

$$\tilde{\omega}_i \propto \tilde{\omega}_{i+1}, \quad i = 1, 2, \ldots, m+n, \tag{31}$$

because the ratios $\tilde{\omega}_{i+1}/\tilde{\omega}_i$, $i = 1, 2, \ldots, m + n$, are all nondecreasing. Furthermore, let

$$\tilde{f}_i(x,y,s,t) := \max_{(\iota,\xi)\in[x_0,x]\times[y_0,y]} f_i(\iota,\xi,s,t), \tag{32}$$

which is nondecreasing in x and y for each fixed t and s and satisfies $\tilde{f}_i(x, y, s, t) \ge f_i(x, y, s, t) \ge 0$ for all $i = 1, 2, \ldots, m + n$. The monotonicity of $\tilde{g}$ implies that

$$\max_{\xi\in[s-h,s]} g(u(\xi,y)) \le \max_{\xi\in[s-h,s]} \tilde{g}(u(\xi,y)) \le \tilde{g}\left(\max_{\xi\in[s-h,s]} u(\xi,y)\right) \tag{33}$$

for $(s, y) \in [\alpha_*(x_0) - h, x_1) \times [y_0, y_1)$. From (8) and the definition of $\tilde{f}_i(x, y, s, t)$, we obtain

$$\varphi(u(x,y)) \le \tilde{a}(x,y) + \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} \tilde{f}_i(x,y,s,t) \times \omega_i(u(s,t))\, dtds$$
$$+ \sum_{j=n+1}^{n+m} \int_{\alpha_j(x_0)}^{\alpha_j(x)} \int_{\beta_j(y_0)}^{\beta_j(y)} \tilde{f}_j(x,y,s,t) \times \omega_j\left(\max_{\xi\in[s-h,s]} g(u(\xi,t))\right) dt\, ds, \quad (x,y) \in \Lambda,$$
$$u(x,y) \le \psi(x,y), \quad (x,y) \in \Omega. \tag{34}$$

Concerning (34), we consider the auxiliary system of inequalities

$$\varphi(u(x,y)) \le \tilde{a}(X,Y) + \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} \tilde{f}_i(X,Y,s,t) \times \omega_i(u(s,t))\, dt\, ds$$
$$+ \sum_{j=1}^{m} \int_{\alpha_j(x_0)}^{\alpha_j(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} \tilde{f}_j(X,Y,s,t) \times \omega_j\left(\max_{\xi\in[s-h,s]} g(u(\xi,t))\right) dt\, ds, \tag{35}$$
$$(x,y) \in [x_0, X) \times [y_0, Y),$$
$$u(x,y) \le \psi(x,y), \quad (x,y) \in \Omega,$$

where $x_0 \le X \le X_1$ and $y_0 \le Y \le Y_1$ are chosen arbitrarily, and claim

$$u(x,y) \le \varphi^{-1}\left(W_{m+n}^{-1}\left(\tilde{\Omega}_i(X,Y,x,y)\right)\right), \tag{36}$$

for all $x_0 \le x \le \min\{X, X_2\}$, $y_0 \le y \le \min\{Y, Y_2\}$, where

$$\tilde{\Omega}_i(X,Y,x,y) := W_i(\tilde{r}_i(X,Y,x,y)) + \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} \tilde{f}(X,Y,s,t)\, dt\, ds, \tag{37}$$

$i = 1, 2, \ldots, m + n$, $\tilde{r}_i(X, Y, x, y)$ is defined inductively by

$$\tilde{r}_1(X,Y,x,y) := \tilde{a}(X,Y),$$
$$\tilde{r}_{i+1}(X,Y,x,y) := W_i^{-1}\left(W_i\left(\tilde{\Omega}_i(X,Y,x,y)\right)\right), \tag{38}$$

for $i = 1, 2, \ldots, m+n-1$, and X_2, Y_2 are chosen such that

$$\widetilde{\Omega}_i(X, Y, X_2, Y_2) \le \int_{u_i}^{\infty} \frac{ds}{\widetilde{\omega}_i(\varphi^{-1}(s))}, \tag{39}$$

for $i = 1, 2, \ldots, m+n$.

Notice that we may take $X_2 = X_1$ and $Y_2 = Y_1$. In fact, the monotonicity that $\widetilde{r}_i(X, Y, x, y)$ and $\widetilde{f}_i(X, Y, x, y)$ are both nondecreasing in X and Y for fixed x, y. Furthermore, it is easy to check that $\widetilde{r}_{i+1}(X, Y, X, Y) = r_i(X, Y)$, for $i = 1, 2, \ldots, m+n$. If X_2, Y_2 are replaced with X_1, Y_1, respectively, on the left side of (39), we get from (15) that

$$\begin{aligned}
\widetilde{\Omega}_i(X, Y, X_1, Y_1) &= W_i(\widetilde{r}_i(X, Y, X_1, Y_1)) \\
&\quad + \int_{\alpha_i(x_0)}^{\alpha_i(X_1)} \int_{\beta_i(y_0)}^{\beta_i(Y_1)} \widetilde{f}(X, Y, s, t)\, dt\, ds \\
&\le W_i(\widetilde{r}_i(X_1, Y_1, X_1, Y_1)) \\
&\quad + \int_{\alpha_i(x_0)}^{\alpha_i(X_1)} \int_{\beta_i(y_0)}^{\beta_i(Y_1)} \widetilde{f}(X_1, Y_1, s, t)\, dt\, ds \\
&= r_i(X_1, Y_1) \\
&\quad + \int_{\alpha_i(x_0)}^{\alpha_i(X_1)} \int_{\beta_i(y_0)}^{\beta_i(Y_1)} \widetilde{f}(X_1, Y_1, s, t)\, dt\, ds \\
&= \Omega_i(X_1, Y_1) \\
&\le \int_{u_i}^{\infty} \frac{ds}{\widetilde{\omega}_i(\varphi^{-1}(s))}.
\end{aligned} \tag{40}$$

Thus, it means that we can take $X_2 = X_1$, $Y_2 = Y_1$.

Now, we prove (36) by induction. From (33), (35), and the definitions of $\widetilde{g}(t)$, $\widetilde{\omega}_i(t)$, and $\widehat{\omega}_i(t)$, we obtain

$$\begin{aligned}
&\varphi(u(x, y)) \\
&\quad \le \widetilde{a}(X, Y) \\
&\qquad + \sum_{i=1}^{m} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{y_0}^{y} \widetilde{f}_i(X, Y, s, t)\, \omega_i(u(s, t)) \\
&\qquad + \sum_{j=m+1}^{m+n} \int_{\alpha_j(x_0)}^{\alpha_j(x)} \int_{y_0}^{y} \widetilde{f}_j(X, Y, s, t) \\
&\qquad\qquad \times \widehat{\omega}_j\left(\widetilde{g}\left(\max_{\xi \in [s-h, s]} u(\xi, t)\right)\right) dt\, ds \\
&\quad \le \widetilde{a}(X, Y) \\
&\qquad + \sum_{i=1}^{m} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} \widetilde{f}_i(X, Y, s, t)\, \widetilde{\omega}_i(u(s))\, ds \\
&\qquad + \sum_{j=m+1}^{m+n} \int_{\alpha_j(x_0)}^{\alpha_j(x)} \int_{\beta_j(y_0)}^{\beta_j(y)} \widetilde{f}_j(X, Y, s, t) \\
&\qquad\qquad \times \widetilde{\omega}_j\left(\max_{\xi \in [s-h, s]} u(\xi)\right) dt\, ds,
\end{aligned} \tag{41}$$

for all $(x, y) \in [x_0, X) \times [y_0, Y)$, where $x_0 \le X \le X_1$ and $y_0 \le Y \le Y_1$ are chosen arbitrarily. Since $\max_{s \in [\alpha_*(x_0)-h, x_0]} \psi(s, y) \le \varphi^{-1}(a(x_0, y))$ and $a(x_0, y) \le \widetilde{a}(x_0, y) \le \widetilde{a}(X, Y)$, we have $\max_{s \in [\alpha_*(x_0)-h, x_0]} \psi(s, y) \le \varphi^{-1}(\widetilde{a}(X, Y))$. Define a function $z(x, y) : [\alpha_*(x_0) - h, X) \times [y_0, Y) \to \mathbb{R}_+$ by

$$z(x, y) = \begin{cases}
\widetilde{a}(X, Y) + \sum_{i=1}^{m} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} \widetilde{f}_i(X, Y, s, t)\, \widetilde{\omega}_i(u(s, t))\, dt\, ds \\
\quad + \sum_{j=m+1}^{m+n} \int_{\alpha_j(x_0)}^{\alpha_j(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} \widetilde{f}_j(s, t)\, \widetilde{\omega}_j\left(\max_{\xi \in [s-h, s]} u(\xi, t)\right) dt\, ds, & (x, y) \in [x_0, X) \times [y_0, Y), \\
\widetilde{a}(X, Y), & (x, y) \in [\alpha_*(x_0) - h, x_0] \times [y_0, Y).
\end{cases} \tag{42}$$

Clearly, $z(x, y)$ is nondecreasing in x. By (41) and the definition of $z(x, y)$, we have

$$\begin{aligned}
&u(x, y) \le \varphi^{-1}(z(x, y)), \\
&\quad (x, y) \in [\alpha_*(x_0) - h, X) \times [y_0, Y).
\end{aligned} \tag{43}$$

Then noting that $z(x, y)$ is nondecreasing and $\varphi(t)$ is strictly increasing, from (43), we obtain

$$\begin{aligned}
\max_{\xi \in [s-h, s]} u(\xi, y) &\le \max_{\xi \in [s-h, s]} \varphi^{-1}(z(\xi, y)) \le \max_{\xi \in [s-h, s]} \varphi^{-1}(z(s, y)) \\
&\le \varphi^{-1}\left(\max_{\xi \in [s-h, s]} z(\xi, y)\right), \\
&\qquad (s, y) \in [\alpha_*(x_0), X) \times [y_0, Y).
\end{aligned} \tag{44}$$

It follows from (43), (44), and the definition of $z(x, y)$ that

$$
\begin{aligned}
z(x,y) \le{} & \tilde{a}(X,Y) \\
& + \sum_{i=1}^{m} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} \tilde{f}_i(X,Y,s,t) \\
& \quad \times \tilde{\omega}_i\left(\varphi^{-1}(z(s,t))\right) dt\, ds \\
& + \sum_{j=m+1}^{m+n} \int_{\alpha_j(x_0)}^{\alpha_j(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} \tilde{f}_i(X,Y,s,t) \\
& \quad \times \tilde{\omega}_j\left(\varphi^{-1}\left(\max_{\xi\in[s-h,s]} z(\xi,t)\right)\right) dt ds, \\
& (x,y) \in [x_0, X) \times [y_0, Y], \\
z(x,y) \le{} & \tilde{a}(X,Y), \quad (x,y) \in [\alpha_*(x_0) - h, x_0] \times [y_0, Y].
\end{aligned} \tag{45}
$$

In order to demonstrate the basic condition of monotonicity, let $h(t) := \varphi^{-1}(t)$, which is clearly a continuous and nondecreasing function on $\mathbb{R}_+$. Thus, each $\tilde{\omega}_i(h(t))$ is continuous and nondecreasing on $\mathbb{R}_+$ and satisfies $\tilde{\omega}_i(h(t)) > 0$ for $t > 0$. Moreover, since $\tilde{\omega}_i(t) \propto \tilde{\omega}_{i+1}(t)$, $\tilde{\omega}_{i+1}(h(t))/\tilde{\omega}_i(h(t))$ is also continuous and nondecreasing on $\mathbb{R}_+$ and positive on $(0, +\infty)$, implying that $\tilde{\omega}_i(h(t)) \propto \tilde{\omega}_{i+1}(h(t))$, for $i = 1, 2, \ldots, m+n-1$. By Lemma 2 and (45),

$$
\begin{aligned}
z(x,y) \le W_{m+n}^{-1} \Bigg(& W\left(\tilde{r}_{m+n}(X,Y,x,y)\right) \\
& + \int_{\alpha_{m+n}(x_0)}^{\alpha_{m+n}(x)} \int_{\beta_{m+n}(y_0)}^{\beta_{m+n}(y)} \tilde{f}_{m+n}(X,Y,s,t)\, dt\, ds \Bigg),
\end{aligned} \tag{46}
$$

for $x_0 \le x < X_2$ and $y_0 \le y < Y_2$. It follows from (43) and (46) that

$$
\begin{aligned}
& u(x,y) \\
& \le \varphi^{-1}\Bigg(W_{m+n}^{-1}\Bigg(W\left(\tilde{r}_{m+n}(X,Y,x,y)\right) \\
& \qquad + \int_{\alpha_{m+n}(x_0)}^{\alpha_{m+n}(x)} \int_{\beta_{m+n}(y_0)}^{\beta_{m+n}(y)} \tilde{f}_{m+n}(X,Y,s,t)\, dt\, ds\Bigg)\Bigg),
\end{aligned} \tag{47}
$$

for $x_0 \le x < X_2$ and $y_0 \le y < Y_2$. This proves the claimed (36).

Taking $x = X$, $y = Y$, $X_2 = X_1$, and $Y_2 = Y_1$ in (36), we have

$$
\begin{aligned}
& u(X,Y) \\
& \le \varphi^{-1}\Bigg(W_{m+n}^{-1}\Bigg(W_{m+n}\left(\tilde{r}_{m+n}(X,Y,X,Y)\right) \\
& \qquad + \int_{\alpha_{m+n}(x_0)}^{\alpha_{m+n}(X)} \int_{\beta_{m+n}(y_0)}^{\beta_{m+n}(Y)} \tilde{f}_{m+n}(X,Y,s,t)\, dt\, ds\Bigg)\Bigg),
\end{aligned} \tag{48}
$$

for all $x_0 \le X < X_1$, $y_0 \le Y < Y_1$. It is easy to verify $\tilde{r}_{m+n}(X,Y,X,Y) = r_{m+n}(X,Y)$. Thus, (48) can be written

$$
\begin{aligned}
& u(X,Y) \\
& \le \varphi^{-1}\Bigg(W_{m+n}^{-1}\Bigg(W_{m+n}\left(r_{m+n}(X,Y)\right) \\
& \qquad + \int_{\alpha_{m+n}(x_0)}^{\alpha_{m+n}(X)} \int_{\beta_{m+n}(y_0)}^{\beta_{m+n}(Y)} \tilde{f}_{m+n}(X,Y,s,t)\, dt\, ds\Bigg)\Bigg).
\end{aligned} \tag{49}
$$

Since X, Y are arbitrary, replacing X and Y with x and y, respectively, we get

$$
\begin{aligned}
& u(x,y) \\
& \le \varphi^{-1}\Bigg(W_{m+n}^{-1}\Bigg(W_{m+n}\left(r_{m+n}(x,y)\right) \\
& \qquad + \int_{\alpha_{m+n}(x_0)}^{\alpha_{m+n}(x)} \int_{\beta_{m+n}(y_0)}^{\beta_{m+n}(y)} \tilde{f}_{m+n}(x,y,s,t)\, dt\, ds\Bigg)\Bigg),
\end{aligned} \tag{50}
$$

for all $(x, y) \in [x_0, X_1) \times [y_0, Y_1)$. This completes the proof. □

3. Applications

In this section, we apply our result to prove the boundedness of solutions for a differential equation with the maxima.

Consider a system of partial differential equations with maxima

$$
\begin{aligned}
& \frac{\partial^2 z(x,y)}{\partial x \partial y} = F\left(x, y, z(x,y), \max_{s\in[\alpha(x),\beta(x)]} \omega(z(s,y))\right), \\
& \qquad (x,y) \in [x_0, x_1) \times [y_0, y_1) \\
& z(x,y) = \psi(x,y), \quad (x,y) \in [\beta(x_0) - h, x_0] \times [y_0, y_1), \\
& z(x, y_0) = f(x), \quad z(x_0, y) = g(y), \quad x \ge x_0,\ y \ge y_0,
\end{aligned} \tag{51}
$$

where $F \in C([x_0, x_1) \times [y_0, y_1) \times \mathbb{R}^2, \mathbb{R})$, $\omega \in C([0, \infty), \mathbb{R}_+)$, $\alpha, \beta \in C^1([x_0, x_1), \mathbb{R}_+)$ are nondecreasing such that $\alpha(x) \le x$,

$\beta(x) \le x$, and $0 < \beta(x) - \alpha(x) \le h$ (h is a positive constant) for $x \in [x_0, x_1]$, $\psi \in C([\beta(x_0) - h, x_0] \times [y_0, y_1)$, and $f \in C([x_0, x_1), \mathbb{R})$, $g \in C([y_0, y_1), \mathbb{R})$ satisfy that $f(x_0) = g(y_0)$ and $g(y) = \psi(x_0, y)$, for all $y \in [y_0, y_1)$.

Equation (51) is more general than the equation considered in Section 2.4 of [23]. The following result gives an estimate for its solutions.

Corollary 3. *Suppose that functions F and ψ in (51) satisfy*

$$\left|F(x, y, s, t)\right| \le h_1(x, y)\mu_1(|s|) + h_2(x, y)\mu_2(|t|),$$
$$\left|\psi(x, y)\right| \le \left|g(y)\right|, \quad \forall (x, y) \in [\beta(x_0) - h, x_0] \times [y_0, y_1), \tag{52}$$

where $h_i \in C([x_0, x_1) \times [y_0, y_1), \mathbb{R}_+)$ and $\mu_i \in C(\mathbb{R}_+, (0, \infty))$, $i = 1, 2$. Then, any solution $z(x, y)$ of (51) has the estimate

$$\left|z(x, y)\right| \le Q_2^{-1}\left(Q_2(\gamma(x, y)) + \int_{x_0}^{x}\int_{y_0}^{y} h_2(s, t)\,dt\,ds\right), \tag{53}$$

for all $(x, y) \in [x_0, X_1) \times [y_0, Y_1)$, where

$$\gamma(x, y) := Q_1^{-1}\left(Q_1(\gamma_1(x, y)) + \int_{x_0}^{x}\int_{y_0}^{y} h_1(s, t)\,dt\,ds\right),$$
$$\gamma_1(x, y) := \max_{(\xi,\eta)\in[x_0,x]\times[y_0,y]}\left(\left|f(\xi) + g(\eta) - f(x_0)\right|\right),$$
$$Q_2(u) := \int_{u_2}^{u}\frac{ds}{\left\{\max_{\tau\in[0,s]}\left\{\tilde{\mu}_2(\tilde{\omega}(\tau))/\max_{\tau_1\in[0,\tau]}\{\mu_1(\tau_1)\}\right\}\max_{\tau\in[0,s]}\{\mu_1(\tau)\}\right\}},$$
$$Q_1(u) := \int_{u_1}^{u}\frac{ds}{\max_{\tau\in[0,s]}\{\mu_1(\tau)\}},$$
$$\tilde{\omega}(t) := \max_{s\in[0,t]}\{\omega(s)\}, \qquad \tilde{\mu}_2(t) := \max_{s\in[0,t]}\{\mu_2(s)\}, \text{ and} \tag{54}$$

X_1, Y_1 are given as in Theorem 1, and constants $u_1 > 0$, $u_2 > 0$ are given arbitrarily.

Proof. From (51), we obtain

$$\begin{aligned} z(x, y) &= f(x) + g(y) - f(x_0) \\ &\quad + \int_{x_0}^{x}\int_{y_0}^{y} F\left(s, t, z(s, t), \max_{\xi\in[\alpha(s),\beta(s)]}\omega(z(\xi, t))\right)dt\,ds, \\ &\qquad (x, y) \in \Lambda, \\ z(x, y) &= \psi(x, y), \\ &\quad (x, y) \in (x, y) \in [\beta(x_0) - h, x_0] \times [y_0, y_1). \end{aligned} \tag{55}$$

From (52) and (55), we get

$$\begin{aligned} &\left|z(x, y)\right| \\ &\le \left|f(x) + g(y) - f(x_0)\right| \\ &\quad + \int_{x_0}^{x}\int_{y_0}^{y} h_1(s, t)\mu_1(|z(s, t)|)\,dt\,ds \\ &\quad + \int_{x_0}^{x}\int_{y_0}^{y} \mu_2\left(\left|\max_{\xi\in[\alpha(s),\beta(s)]}\omega(z(\xi, t))\right|\right)dt\,ds \\ &\le \left|f(x) + g(y) - f(x_0)\right| \\ &\quad + \int_{x_0}^{x}\int_{y_0}^{y} h_1(s, t)\mu_1(|z(s, t)|)\,dtds \\ &\quad + \int_{x_0}^{x}\int_{y_0}^{y} h_2(s, t)\tilde{\mu}_2\left(\max_{\xi\in[\alpha(s),\beta(s)]}\tilde{\omega}\left(\left|z(\xi, t)\right|\right)\right)dtds, \\ &\qquad (x, y) \in \Lambda, \\ &\left|z(x, y)\right| \le \left|\psi(x, y)\right|, \\ &\quad (x, y) \in [\beta(x_0) - h, x_0] \times [y_0, y_1). \end{aligned} \tag{56}$$

Set $v(x, y) = |z(x, y)|$ for $(x, y) \in [\beta(x_0) - h, x_1) \times [y_0, y_1)$. Noting that $\max_{\xi\in[\alpha(s),\beta(s)]} z(\xi, y) \le \max_{\xi\in[\beta(s)-h,\beta(s)]} z(\xi, y)$, from (56), we get

$$\begin{aligned} v(x, y) &\le \gamma_1(x, y) + \int_{x_0}^{x}\int_{y_0}^{y} h_1(s, t)\mu_1(v(s, t))\,dt\,ds \\ &\quad + \int_{\beta(x_0)}^{\beta(x)}\int_{y_0}^{y} h_2\left(\beta^{-1}(\eta), t\right)\left(\beta^{-1}(\eta)\right)'\tilde{\mu}_2 \\ &\qquad \times \left(\max_{\xi\in[\eta-h,\eta]}\tilde{\omega}(v(\xi, t))\right)dt\,d\eta \\ v(x, y) &\le \left|\psi(x, y)\right|, \quad (x, y) \in [\beta(x_0) - h, x_0] \times [y_0, y_1). \end{aligned} \tag{57}$$

Applying Theorem 1 to specified $m = n = 1$, $\varphi(u) = u$, $f_1(x, y, s, t) = h_1(s, t)$, $\alpha_1(t) = t$, and $\alpha_2(t) = \beta(t)$, $\beta_i(t) = t$, $i = 1, 2$, $f_2(s, t) = h_2(\beta^{-1}(s), t)(\beta^{-1}(s))'$, and $\omega_i(u) = \mu_i(u)$, $i = 1, 2$, we obtain (53) from (57). □

Next, we discuss the uniqueness of solutions for system (51).

Corollary 4. *Suppose that $g(s) = s$ and*

$$\begin{aligned} &\left|F(x, y, s_1, t_1) - F(x, y, s_2, t_2)\right| \\ &\quad \le h_1(x, y)\mu_1\left(\left|x_1 - x_2\right|\right) + h_2(x, y)\mu_2\left(\left|y_1 - y_2\right|\right), \end{aligned} \tag{58}$$

for all $(x, y) \in [x_0, x_1] \times [y_0, y_1]$ and all $x_i, y_i \in \mathbb{R}$ ($i = 1, 2$), where $h_i \in C([x_0, x_1] \times [y_0, y_1], \mathbb{R}_+)$ and $\mu_i \in C([\mathbb{R}, \mathbb{R}_+)$ are *both nondecreasing such that* $\mu_i(0) = 0$, $\mu_i(u) > 0$ *for* $u > 0$, μ_2/μ_1 *is also nondecreasing, and* $\int_0^1 ds/\mu_i(s) = +\infty$, $i = 1, 2$. *Then, system* (51) *has at most one solution on* $[x_0, x_1)\times[y_0, y_1)$.

Proof. $g(s) = s$. From (51), we get

$$\frac{\partial^2 z(x,y)}{\partial x\,\partial y} = F\left(x, y, z(x,y), \max_{s\in[\alpha(x),\beta(x)]} z(s,y)\right),$$
$$(x,y) \in [x_0, x_1) \times [y_0, y_1),$$
$$z(x,y) = \psi(x,y), \quad (x,y) \in [\alpha(x_0) - h, x_0] \times [y_0, y_1],$$
$$z(x, y_0) = f(x), z(x_0, y) = g(y), \quad x \ge x_0,\ y \ge y_0. \tag{59}$$

Assume that (59) has two different solutions $u(x,y)$ and $v(x,y)$. From the equivalent integral equation system (55), we have

$$\begin{aligned}
&|u(x,y) - v(x,y)| \\
&\le \int_{x_0}^{x}\int_{y_0}^{y} h_1(s,t)\,\mu_1(|u(s,t) - v(s,t)|)\,dt\,ds \\
&\quad + \int_{x_0}^{x}\int_{y_0}^{y} h_2(s,t)\,\mu_2\left(\left|\max_{\xi\in[\alpha(s),\beta(s)]} u(\xi,t) - \max_{\xi\in[\alpha(s),\beta(s)]} v(\xi,t)\right|\right) dt\,ds \\
&\le \int_{x_0}^{x}\int_{y_0}^{y} h_1(s,t)\,\mu_1(|u(s) - v(s)|)\,dt\,ds \\
&\quad + \int_{x_0}^{x}\int_{y_0}^{y} h_2(s,t)\,\mu_2 \times \left(\max_{\xi\in[\alpha(s),\beta(s)]} |u(\xi,t) - v(\xi,t)|\right) ds,
\end{aligned} \tag{60}$$

for all $(x,y) \in [x_0,x_1] \times [y_0,y_1]$. The continuity of the function $u(x,y)$ implies that for any fixed points $s \in [x_0,x]$ and $t \in [y_0,y]$ there exists a point $\eta \in [\alpha(s),\beta(s)]$ such that the inequality $\max_{\xi\in[\alpha(s),\beta(s)]} u(\xi,t) = u(\eta,t)$ holds, and therefore

$$\begin{aligned}
&\left|\max_{\xi\in[\alpha(s),\beta(s)]} u(\xi,t) - \max_{\xi\in[\alpha(s),\beta(s)]} v(\xi,t)\right| \\
&= \left|u(\eta,t) - \max_{\xi\in[\alpha(s),\beta(s)]} v(\xi,t)\right| \\
&\le |u(\eta,t) - v(\eta,t)| \le \max_{\xi\in[\alpha(s),\beta(s)]} |u(\xi,t) - v(\xi,t)|.
\end{aligned} \tag{61}$$

Hence,

$$\begin{aligned}
&|u(x,y) - v(x,y)| \\
&\le \int_{x_0}^{x}\int_{y_0}^{y} h_1(s,t)\,\mu_1(|u(s,y) - v(s,y)|)\,dt\,ds \\
&\quad + \int_{x_0}^{x}\int_{y_0}^{y} h_2(s,t)\,\mu_2\left(\max_{\xi\in[\alpha(s),\beta(s)]} |u(\xi,t) - v(\xi,t)|\right) ds.
\end{aligned} \tag{62}$$

Let

$$\phi(x,y) := |u(x,y) - v(x,y)|,$$
$$(x,y) \in [\beta(x_0) - h, x_1] \times [y_0, y_1]. \tag{63}$$

Because $\max_{\xi\in[\alpha(s),\beta(s)]} u(\xi,y) \le \max_{\xi\in[\beta(s)-h,\beta(s)]} u(\xi,y)$, from (62), we obtain

$$\begin{aligned}
\phi(x,y) \le{}& \int_{x_0}^{x}\int_{y_0}^{y} h_1(s,t)\,\mu_1(\phi(s,t))\,dt\,ds \\
&+ \int_{\beta(x_0)}^{\beta(x)}\int_{y_0}^{y} h_2(\beta^{-1}(\eta),t)(\beta^{-1}(\eta))' \\
&\times \mu_2\left(\max_{\xi\in[\eta-h,\eta]} \phi(\xi,t)\right) dt\,d\eta, \\
&(x,y) \in [x_0,x_1) \times [y_0,y_1), \\
\phi(x,y) \le 0,&\quad (x,y) \in [\beta(x_0) - h, x_0] \times [y_0, y_1).
\end{aligned} \tag{64}$$

Applying Theorem 1 to specified $m = n = 1$, $\varphi(u) = u$, $f_1(s,t) = h_1(s,t)$, $\alpha_1(t) = t$, $\alpha_2(t) = \beta(t)$, $f_2(s,t) = h_2(\beta^{-1}(t),s)(\beta^{-1}(t))'$, $g(t) = t$, $a(x,y) = 0$, and $\omega_i(t) = \mu_i(t)$, $i = 1,2$, from (64), we obtain

$$\phi(x,y) \le \widehat{Q}_2^{-1}\left(\widehat{Q}_2(\overline{\gamma}_2(s,t)) + \int_{x_0}^{x}\int_{y_0}^{y} h_2(s,t)\,dt\,ds\right), \tag{65}$$

for all $(x,y) \in [x_0, X_1] \times [y_0, Y_1]$, where

$$\widehat{Q}_1(u) := \int_1^u \frac{ds}{\mu_1(s)}, \qquad \widehat{Q}_2(u) := \int_1^u \frac{ds}{\mu_2(\tau)}, \tag{66}$$

$$\overline{r}_1(x,y) := 0, \tag{67}$$

$$\overline{r}_2(x,y) := \widehat{Q}_1^{-1}\left(\widehat{Q}_1(\overline{r}_1(x,y)) + \int_{x_0}^{x}\int_{y_0}^{y} h_1(s,t)\,dt\,ds\right). \tag{68}$$

By the definition of $\widehat{Q}_i$ and properties of μ_i, noting that $\int_0^1 ds/\mu_i(s) = +\infty$ $(i = 1,2)$, we obtain

$$\lim_{u\to 0^+} \widehat{Q}_i(u) = -\infty, \qquad \lim_{u\to -\infty} \widehat{Q}_i^{-1}(u) = 0, \quad i = 1,2. \tag{69}$$

Since $\int_{x_0}^{x}\int_{y_0}^{y} h_1(s,t)dtds$ is finite on a finite interval, $[x_0,x_1]$ and $[y_0,y_1]$, by (67), we obtain

$$\widehat{Q}_1(\overline{r}_1(x,y)) + \int_{x_0}^{x}\int_{y_0}^{y} h_1(s,t)\,dt\,ds = -\infty. \tag{70}$$

Thus, we obtain $\overline{\gamma}_2(x, y) = 0$ from (68), (69), and (70) immediately. Similarly, noting that $\int_{x_0}^{x}\int_{y_0}^{y} h_2(s,t)dtds$ is finite on finite interval, $[x_0, x_1]$ and $[y_0, y_1]$, from (69), we obtain

$$\widehat{Q}_2(\overline{r}_2(x, y)) + \int_{x_0}^{x}\int_{y_0}^{y} h_2(s,t)\,dt\,ds = -\infty. \quad (71)$$

Thus, we conclude from (65), (69), and (71) that $|u(x, y) - v(x, y)| \leq 0$, which implies that $u(x, y) = v(x, y)$, for all $(x, y) \in [x_0, X_1) \times [y_0, Y_1)$, where X_1, Y_1 are given as in Theorem 1. The uniqueness is proved. □

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