

## Research Article

# Stability of a Functional Differential System with a Finite Number of Delays

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The paper is devoted to the study of asymptotic properties of a real two-dimensional differential system with unbounded nonconstant delays. The sufficient conditions for the stability and asymptotic stability of solutions are given. Used methods are based on the transformation of the considered real system to one equation with complex-valued coefficients. Asymptotic properties are studied by means of Lyapunov-Krasovskii functional. The results generalize some previous ones, where the asymptotic properties for two-dimensional systems with one or more constant delays or one nonconstant delay were studied.

## 1. Introduction

There are a lot of papers dealing with the stability and asymptotic behaviour of  $n$ -dimensional real vector equations with delay. Among others we should mention the recent results [1–13]. Since the plane has special topological properties different from those of  $n$ -dimensional space, where  $n \geq 3$  or  $n = 1$ , it is interesting to study asymptotic behaviour of two-dimensional systems by using tools which are typical and effective for two-dimensional systems. The convenient tool is the combination of the method of complexification and the method of Lyapunov-Krasovskii functional. Using these techniques we obtain new and easy applicable results on stability, asymptotic stability, or boundedness of solutions of real two-dimensional differential system

$$\begin{aligned}x'(t) &= A(t)x(t) \\ &+ \sum_{k=1}^m B_k(t)x(\theta_k(t)) \\ &+ h(t, x(t), x(\theta_1(t)), \dots, x(\theta_m(t))),\end{aligned}\quad (1)$$

where  $\theta_k(t)$  are real functions,  $A(t) = (a_{ij}(t))$ ,  $B_k(t) = (b_{ijk}(t))$  ( $i, j = 1, 2; k = 1, \dots, m$ ) are real square matrices, and

$h(t, x, y) = (h_1(t, x, y_1, \dots, y_m), h_2(t, x, y_1, \dots, y_m))$  is a real vector function,  $x = (x_1, x_2)$ ,  $y_k = (y_{1k}, y_{2k})$ . It is supposed that the functions  $\theta_k, a_{ij}$  are locally absolutely continuous on  $[t_0, \infty)$ ,  $b_{ijk}$  are locally Lebesgue integrable on  $[t_0, \infty)$ , and the function  $h$  satisfies Carathéodory conditions on  $[t_0, \infty) \times \mathbb{R}^{2(m+1)}$ .

Delayed differential equations recently gain more importance in applications in science and real world. They can be found in applications in medicine (control of drug therapies and neurological, physiological, and epidemiological models), biology (predator-prey models and blowflies lifecycle), chemistry (chemical kinetics), physics (private communication and signal masking), and engineering (machining operation on a lathe). Equation (1) represents a generalization of many of these models. Particularly, (1) in this general form has an application in modeling of multiple regenerative effect in tool chatter. Obtained results on stability give the possibility to find the best spindle speeds and depth-of-cut for the machines for chatter-free high-productivity operation. For more details, see [14].

The main idea of the investigation, the combination of the method of complexification and the method of Lyapunov-Krasovskii functional, was introduced for ordinary differential equations in the paper by Ráb and Kalas [15].

The principle was transferred to differential equations with delay by Kalas and Baráková [16]. The results for several constant delays can be found in papers by Rebenda [17, 18]. Differential equations with one nonconstant delay are studied by Kalas [19] and Rebenda [20].

We extend such type of results to differential equations with a finite number of nonconstant delays. We introduce the transformation of the considered real system to one equation with complex-valued coefficients. We present sufficient conditions for stability and asymptotic stability of a solution and the conditions under which all solutions tend to zero. The applicability of the results is demonstrated with an example.

At the end of this introduction we append an overview of notations used in the paper and the transformation of the real system to one equation with complex-valued coefficients.

Consider the following:

- $\mathbb{R}$ : set of all real numbers,
- $\mathbb{R}_+$ : set of all positive real numbers,
- $\mathbb{R}_+^0$ : set of all nonnegative real numbers,
- $\mathbb{R}_-$ : set of all negative real numbers,
- $\mathbb{R}_-^0$ : set of all nonpositive real numbers,
- $\mathbb{C}$ : set of all complex numbers,
- $\mathcal{C}$ : class of all continuous functions  $[-r, 0] \rightarrow \mathbb{C}$ ,
- $AC_{loc}(I, M)$ : class of all locally absolutely continuous functions  $I \rightarrow M$ ,
- $L_{loc}(I, M)$ : class of all locally Lebesgue integrable functions  $I \rightarrow M$ ,
- $K(I \times \Omega, M)$ : class of all functions  $I \times \Omega \rightarrow M$  satisfying Carathéodory conditions on  $I \times \Omega$ ,
- $\operatorname{Re} z$ : real part of  $z$ ,
- $\operatorname{Im} z$ : imaginary part of  $z$ ,
- $\bar{z}$ : complex conjugate of  $z$ .

Introducing complex variables  $z = x_1 + ix_2, w_1 = y_{11} + iy_{12}, \dots, w_m = y_{m1} + iy_{m2}$ , we can rewrite system (1) into an equivalent equation with complex-valued coefficients:

$$z'(t) = a(t)z(t) + b(t)\bar{z}(t) + \sum_{k=1}^m [A_k(t)z(\theta_k(t)) + B_k(t)\bar{z}(\theta_k(t))] + g(t, z(t), z(\theta_1(t)), \dots, z(\theta_m(t))), \quad (2)$$

where  $\theta_k \in AC_{loc}(J, \mathbb{R})$  for  $k = 1, \dots, m, A_k, B_k \in L_{loc}(J, \mathbb{C}), a, b \in AC_{loc}(J, \mathbb{C}), g \in K(J \times \mathbb{C}^{m+1}, \mathbb{C}), J = [t_0, \infty)$ .

The relations between the functions are as follows:

$$\begin{aligned} a(t) &= \frac{1}{2}(a_{11}(t) + a_{22}(t)) + \frac{i}{2}(a_{21}(t) - a_{12}(t)), \\ b(t) &= \frac{1}{2}(a_{11}(t) - a_{22}(t)) + \frac{i}{2}(a_{21}(t) + a_{12}(t)), \\ A_k(t) &= \frac{1}{2}(b_{11k}(t) + b_{22k}(t)) + \frac{i}{2}(b_{21k}(t) - b_{12k}(t)), \\ B_k(t) &= \frac{1}{2}(b_{11k}(t) - b_{22k}(t)) + \frac{i}{2}(b_{21k}(t) + b_{12k}(t)), \\ g(t, z, w_1, \dots, w_m) &= h_1\left(t, \frac{1}{2}(z + \bar{z}), \frac{1}{2i}(z - \bar{z}), \frac{1}{2}(w_1 + \bar{w}_1), \dots, \frac{1}{2i}(w_m - \bar{w}_m)\right) \\ &\quad + ih_2\left(t, \frac{1}{2}(z + \bar{z}), \frac{1}{2i}(z - \bar{z}), \frac{1}{2}(w_1 + \bar{w}_1), \dots, \frac{1}{2i}(w_m - \bar{w}_m)\right). \end{aligned} \quad (3)$$

Conversely, putting

$$\begin{aligned} a_{11}(t) &= \operatorname{Re}[a(t) + b(t)], \\ a_{12}(t) &= \operatorname{Im}[b(t) - a(t)], \\ a_{21}(t) &= \operatorname{Im}[a(t) + b(t)], \\ a_{22}(t) &= \operatorname{Re}[a(t) - b(t)], \\ b_{11k}(t) &= \operatorname{Re}[A_k(t) + B_k(t)], \\ b_{12k}(t) &= \operatorname{Im}[B_k(t) - A_k(t)], \\ b_{21k}(t) &= \operatorname{Im}[A_k(t) + B_k(t)], \\ b_{22k}(t) &= \operatorname{Re}[A_k(t) - B_k(t)], \end{aligned} \quad (4)$$

$$\begin{aligned} h_1(t, x, y_1, \dots, y_m) &= \operatorname{Re} g(t, x_1 + ix_2, y_{11} + iy_{12}, \dots, y_{m1} + iy_{m2}), \\ h_2(t, x, y_1, \dots, y_m) &= \operatorname{Im} g(t, x_1 + ix_2, y_{11} + iy_{12}, \dots, y_{m1} + iy_{m2}), \end{aligned}$$

equation (2) can be written in real form (1) as well.

## 2. Preliminaries

We consider (2) in the case when

$$\liminf_{t \rightarrow \infty} (|\operatorname{Im} a(t)| - |b(t)|) > 0 \quad (5)$$

and study the behavior of solutions of (2) under this assumption. This situation corresponds to the case when the equilibrium 0 of the autonomous homogeneous system

$$x' = Ax, \quad (6)$$

where  $\mathbf{A}$  is supposed to be regular constant matrix, is a centre or a focus.

This case is included in the case  $\liminf_{t \rightarrow \infty} (|a(t)| - |b(t)|) > 0$  considered in [21], but in this special case, we are able to derive more useful results as we will see later in an example. The idea is based on the well-known result that the condition  $|a| > |b|$  in an autonomous equation  $z' = az + b\bar{z}$  ensures that zero is a focus, a centre, or a node while under the condition  $|\operatorname{Im} a| > |b|$  zero can be just a focus or a centre. Details are found in [15].

A simple example shows that, in some cases, the results of this paper can be applied more suitably than those given in [21].

Regarding (5) and since the delay functions  $\theta_k$  satisfy  $\lim_{t \rightarrow \infty} \theta_k(t) = \infty$ , there are numbers  $T_1 \geq t_0, T \geq T_1$ , and  $\mu > 0$  such that

$$\begin{aligned} |\operatorname{Im} a(t)| &> |b(t)| + \mu \quad \text{for } t \geq T_1, \\ t \geq \theta_k(t) &\geq T_1 \quad \text{for } t \geq T \quad (k = 1, \dots, m). \end{aligned} \tag{7}$$

Denote

$$\begin{aligned} \tilde{\gamma}(t) &= \operatorname{Im} a(t) + \sqrt{(\operatorname{Im} a(t))^2 - |b(t)|^2} \operatorname{sgn}(\operatorname{Im} a(t)), \\ \tilde{c}(t) &= -ib(t). \end{aligned} \tag{8}$$

Notice that, unlike the function  $\gamma$  introduced in [21], the previously defined function  $\tilde{\gamma}$  need not be positive.

Since  $|\tilde{\gamma}(t)| > |\operatorname{Im} a(t)|$  and  $|\tilde{c}(t)| = |b(t)|$ , the inequality

$$|\tilde{\gamma}(t)| > |\tilde{c}(t)| + \mu \tag{9}$$

is valid for  $t \geq T_1$ . It can be easily verified that  $\tilde{\gamma}, \tilde{c} \in AC_{\text{loc}}([T_1, \infty), \mathbb{C})$ .

For the rest of this section, denote that

$$\begin{aligned} \tilde{\vartheta}(t) &= \frac{\operatorname{Re}(\tilde{\gamma}(t)\tilde{\gamma}'(t) - \tilde{c}(t)\tilde{c}'(t)) - |\tilde{\gamma}(t)\tilde{c}'(t) - \tilde{\gamma}'(t)\tilde{c}(t)|}{\tilde{\gamma}^2(t) - |\tilde{c}(t)|^2}. \end{aligned} \tag{10}$$

The stability and asymptotic stability are studied under the following assumptions.

- (i) The numbers  $T_1 \geq t_0, T \geq T_1$ , and  $\mu > 0$  are such that (7) holds.
- (ii) There exist functions  $\tilde{\kappa}, \tilde{\kappa}_k, \tilde{\varrho} : [T, \infty) \rightarrow \mathbb{R}$  such that

$$\begin{aligned} &|\tilde{\gamma}(t)g(t, z, w_1, \dots, w_m) \\ &+ \tilde{c}(t)\bar{g}(t, z, w_1, \dots, w_m)| \\ &\leq \tilde{\kappa}(t)|\tilde{\gamma}(t)z + \tilde{c}(t)\bar{z}| \\ &+ \sum_{k=1}^m \tilde{\kappa}_k(t)|\tilde{\gamma}(\theta_k(t))w_k \\ &+ \tilde{c}(\theta_k(t))\bar{w}_k| + \tilde{\varrho}(t), \end{aligned} \tag{11}$$

for  $t \geq T, z, w_k \in \mathbb{C} (k = 1, \dots, m)$ , where  $\tilde{\kappa}, \tilde{\varrho} \in L_{\text{loc}}([T, \infty), \mathbb{R})$ .

- (iii)  $\tilde{\beta} \in AC_{\text{loc}}([T, \infty), \mathbb{R}_+^0)$  is a function satisfying

$$\theta'_k(t)\tilde{\beta}(t) \geq \tilde{\lambda}_k(t) \quad \text{a.e. on } [T, \infty), \tag{12}$$

where  $\tilde{\lambda}_k$  is defined for  $t \geq T$  by

$$\begin{aligned} \tilde{\lambda}_k(t) &= \tilde{\kappa}_k(t) + (|A_k(t)| + |B_k(t)|) \\ &\times \frac{|\tilde{\gamma}(t)| + |\tilde{c}(t)|}{|\tilde{\gamma}(\theta_k(t))| - |\tilde{c}(\theta_k(t))|}. \end{aligned} \tag{13}$$

- (iv) There exists a function  $\tilde{\Lambda} \in L_{\text{loc}}([T, \infty), \mathbb{R})$  which satisfies the inequalities  $\tilde{\beta}'(t) \leq \tilde{\Lambda}(t)\tilde{\beta}(t), \tilde{\Theta}(t) \leq \tilde{\Lambda}(t)$  for almost all  $t \in [T, \infty)$ , where the function  $\tilde{\Theta}$  is defined by

$$\tilde{\Theta}(t) = \operatorname{Re} a(t) + \tilde{\vartheta}(t) + \tilde{\kappa}(t) + m\tilde{\beta}(t). \tag{14}$$

If  $A_k, B_k, \tilde{\kappa}_k, \theta'_k$  are locally absolutely continuous on  $[T, \infty)$  and  $\tilde{\lambda}_k(t) \geq 0, \theta'_k(t) > 0$  on  $[T, \infty)$ , the choice  $\tilde{\beta}(t) = \max_{k=1, \dots, m} [\tilde{\lambda}_k(t)(\theta'_k(t))^{-1}]$  is admissible in (iii).

From the assumption (i), it follows that

$$\begin{aligned} |\tilde{\vartheta}| &\leq \frac{|\operatorname{Re}(\tilde{\gamma}\tilde{\gamma}' - \tilde{c}\tilde{c}')| + |\tilde{\gamma}\tilde{c}' - \tilde{\gamma}'\tilde{c}|}{\tilde{\gamma}^2 - |\tilde{c}|^2} \\ &\leq \frac{(|\tilde{\gamma}'| + |\tilde{c}'|)(|\tilde{\gamma}| + |\tilde{c}|)}{\tilde{\gamma}^2 - |\tilde{c}|^2} \\ &= \frac{|\tilde{\gamma}'| + |\tilde{c}'|}{|\tilde{\gamma}| - |\tilde{c}|} \leq \frac{1}{\mu} (|\tilde{\gamma}'| + |\tilde{c}'|); \end{aligned} \tag{15}$$

hence, the function  $\tilde{\vartheta}$  is locally Lebesgue integrable on  $[T, \infty)$ . Moreover, if  $\tilde{\beta} \in AC_{\text{loc}}([T, \infty), \mathbb{R}_+)$  and  $\tilde{\kappa} \in L_{\text{loc}}([T, \infty))$ , then we can choose

$$\tilde{\Lambda}(t) = \max\left(\tilde{\Theta}(t), \frac{\tilde{\beta}'(t)}{\tilde{\beta}(t)}\right), \tag{16}$$

in (iv).

Finally, if  $\tilde{\varrho}(t) \equiv 0$  in (ii), then (2) has the trivial solution  $z(t) \equiv 0$ . Notice that in this case the condition (ii) implies that the functions  $\tilde{\kappa}(t), \tilde{\kappa}_k(t)$  are nonnegative on  $[T, \infty)$  for  $k = 1, \dots, m$ , and due to this,  $\tilde{\lambda}_k(t) \geq 0$  on  $[T, \infty)$ . The case  $\tilde{\varrho}(t) < 0$  is omitted since it can be replaced by  $\tilde{\varrho}(t) \equiv 0$ .

### 3. Main Results

The aim is to generalize the results for ordinary differential equations published in [15] as well as the results contained in [16] (one constant delay), [18] (a finite number of constant delays), and [20] (one nonconstant delay). In the proof of the crucial theorem, we use the following auxiliary result.

**Lemma 1.** Let  $a_1, a_2, b_1, b_2 \in \mathbb{C}$  and  $|a_2| > |b_2|$ . Then

$$\operatorname{Re} \frac{a_1 z + b_1 \bar{z}}{a_2 z + b_2 \bar{z}} \leq \frac{\operatorname{Re}(a_1 \bar{a}_2 - b_1 \bar{b}_2) + |a_1 b_2 - a_2 b_1|}{|a_2|^2 - |b_2|^2}, \quad (17)$$

for  $z \in \mathbb{C}, z \neq 0$ .

The proof of Lemma 1 can be found, for example, in [15, page 131] or [17, page 101].

**Theorem 2.** Let the conditions (i), (ii), (iii), and (iv) hold and  $\tilde{q}(t) \equiv 0$ .

(a) If

$$\limsup_{t \rightarrow \infty} \int^t \tilde{\Lambda}(s) ds < \infty, \quad (18)$$

then the trivial solution of (2) is stable on  $[T, \infty)$ .

(b) If

$$\lim_{t \rightarrow \infty} \int^t \tilde{\Lambda}(s) ds = -\infty, \quad (19)$$

then the trivial solution of (2) is asymptotically stable on  $[T, \infty)$ .

*Proof.* Choose arbitrary  $t_1 \geq T$ . Let  $z(t)$  be any solution of (2) satisfying the condition  $z(t) = z_0(t)$  for  $t \in [T_1, t_1]$ , where  $z_0(t)$  is a continuous complex-valued initial function defined on  $t \in [T_1, t_1]$ . Consider the Lyapunov functional

$$V(t) = U(t) + \tilde{\beta}(t) \sum_{k=1}^m \int_{\theta_k(t)}^t U(s) ds, \quad (20)$$

where

$$U(t) = |\tilde{\gamma}(t) z(t) + \tilde{c}(t) \bar{z}(t)|. \quad (21)$$

To simplify the computations, denote that  $w_k(t) = z(\theta_k(t))$  and write the functions of variable  $t$  without brackets, for example,  $z$  instead of  $z(t)$ .

From (20) we get

$$\begin{aligned} V' &= U' + \tilde{\beta}' \sum_{k=1}^m \int_{\theta_k(t)}^t U(s) ds + m \tilde{\beta} |\tilde{\gamma} z + \tilde{c} \bar{z}| \\ &\quad - \sum_{k=1}^m \theta_k' \tilde{\beta} |\tilde{\gamma}(\theta_k) w_k + \tilde{c}(\theta_k) \bar{w}_k|, \end{aligned} \quad (22)$$

for almost all  $t \geq t_1$  for which  $z(t)$  is defined and  $U'(t)$  exists.

Denote that  $\mathcal{X} = \{t \geq t_1 : z(t) \text{ exists, } U(t) \neq 0\}$  and  $\mathcal{M} = \{t \geq t_1 : z(t) \text{ exists, } U(t) = 0\}$ . It is clear that the derivative  $U'(t)$  exists for almost all  $t \in \mathcal{X}$ ; hence, we focus on the set  $\mathcal{M}$ .

In view of (9) we have  $z(t) = 0$  for  $t \in \mathcal{M}$ . For almost all  $t \in \mathcal{M}$ , we compute

$$\begin{aligned} U'_{\pm}(t) &= \lim_{\tau \rightarrow t_{\pm}} \frac{U(\tau) - U(t)}{\tau - t} \\ &= \lim_{\tau \rightarrow t_{\pm}} \frac{U(\tau)}{\tau - t} \\ &= \lim_{\tau \rightarrow t_{\pm}} \frac{|\tilde{\gamma}(\tau) [z(\tau) - z(t)] - \tilde{c}(\tau) [\bar{z}(\tau) - \bar{z}(t)]|}{\tau - t} \\ &= \pm |\tilde{\gamma}(t) z'(t) + \tilde{c}(t) \bar{z}'(t)| \\ &= \pm |\tilde{\gamma}(t) g^*(t) + \tilde{c}(t) \bar{g}^*(t)|, \end{aligned} \quad (23)$$

where

$$\begin{aligned} g^*(t) &= \sum_{k=1}^m (A_k(t) w_k(t) + B_k(t) \bar{w}_k(t)) \\ &\quad + g(t, 0, w_1(t), \dots, w_m(t)). \end{aligned} \quad (24)$$

Hence,  $U$  has one-sided derivatives a.e. in  $\mathcal{M}$ . According to [22, Chapter IX., Theorem (1.1)] or [23], the set of all  $t$  such that  $U'_+(t) \neq U'_-(t)$  can be at most countable; thus, the derivative  $U'$  exists for almost all  $t \in \mathcal{M}$ , and for these  $t$ ,  $U'(t) = 0$ .

In particular, the derivative  $U'$  exists for almost all  $t \geq t_1$  for which  $z(t)$  is defined; thus, (22) holds for almost all  $t \geq t_1$  for which  $z(t)$  is defined.

Now return the attention to the set  $\mathcal{X}$ . For almost all  $t \in \mathcal{X}$ , it holds that  $UU' = U(\sqrt{(\tilde{\gamma}z + \tilde{c}\bar{z})(\tilde{\gamma}\bar{z} + \tilde{c}z)})' = \operatorname{Re}[(\tilde{\gamma}\bar{z} + \tilde{c}z)(\tilde{\gamma}'z + \tilde{\gamma}z' + \tilde{c}'\bar{z} + \tilde{c}\bar{z}')]'$ . As  $z(t)$  is a solution of (2), we have

$$\begin{aligned} UU' &= \operatorname{Re} \left\{ (\tilde{\gamma}\bar{z} + \tilde{c}z) \right. \\ &\quad \times \left[ \tilde{\gamma}'z + \tilde{c}'\bar{z} \right. \\ &\quad \left. \left. + \tilde{\gamma} \left( az + b\bar{z} + \sum_{k=1}^m (A_k w_k + B_k \bar{w}_k) + g \right) \right. \right. \\ &\quad \left. \left. + \tilde{c} \left( \bar{a}\bar{z} + \bar{b}z + \sum_{k=1}^m (\bar{A}_k \bar{w}_k + \bar{B}_k w_k) + \bar{g} \right) \right] \right\} \\ &= \operatorname{Re} \left\{ (\tilde{\gamma}\bar{z} + \tilde{c}z) \right. \\ &\quad \times \left[ \tilde{\gamma}'z + \tilde{c}'\bar{z} + (\tilde{\gamma}a + \tilde{c}\bar{b})z \right. \\ &\quad \left. \left. + (\tilde{\gamma}b + \tilde{c}\bar{a})\bar{z} \right] \right\} \end{aligned}$$

$$\begin{aligned}
 & + \tilde{\gamma} \left( \sum_{k=1}^m (A_k w_k + B_k \bar{w}_k) + g \right) \\
 & + \tilde{c} \left( \sum_{k=1}^m (\bar{A}_k \bar{w}_k + \bar{B}_k w_k) + \bar{g} \right) \Bigg\}, \tag{25}
 \end{aligned}$$

for almost all  $t \in \mathcal{K}$ .

Short computation gives  $(\tilde{\gamma}a + \tilde{c}\bar{b})\tilde{c} = (\tilde{\gamma}b + \tilde{c}\bar{a})\tilde{\gamma}$ , and from this we get

$$\begin{aligned}
 UU' & \leq \operatorname{Re} \left\{ (\tilde{\gamma}\bar{z} + \tilde{c}z) (\tilde{\gamma}a + \tilde{c}\bar{b}) \left( z + \frac{\tilde{c}}{\tilde{\gamma}}\bar{z} \right) \right\} \\
 & + \operatorname{Re} \left\{ (\tilde{\gamma}\bar{z} + \tilde{c}z) \left[ \tilde{\gamma} \sum_{k=1}^m (A_k w_k + B_k \bar{w}_k) \right. \right. \\
 & \quad \left. \left. + \tilde{c} \sum_{k=1}^m (\bar{A}_k \bar{w}_k + \bar{B}_k w_k) \right] \right\} \\
 & + \operatorname{Re} [(\tilde{\gamma}\bar{z} + \tilde{c}z) (\tilde{\gamma}g + \tilde{c}\bar{g})] \\
 & + \operatorname{Re} [(\tilde{\gamma}\bar{z} + \tilde{c}z) (\tilde{\gamma}'z + \tilde{c}'\bar{z})] \\
 & \leq U^2 \operatorname{Re} \left( a + \frac{\tilde{c}}{\tilde{\gamma}}\bar{b} \right) \\
 & + U (|\tilde{\gamma}| + |\tilde{c}|) \left( \sum_{k=1}^m |A_k w_k + B_k \bar{w}_k| \right) \\
 & + U |\tilde{\gamma}g + \tilde{c}\bar{g}| + U^2 \operatorname{Re} \frac{\tilde{\gamma}'z + \tilde{c}'\bar{z}}{\tilde{\gamma}z + \tilde{c}\bar{z}}, \tag{26}
 \end{aligned}$$

for almost all  $t \in \mathcal{K}$ .

Applying Lemma 1 to the last term, we obtain

$$\operatorname{Re} \frac{\tilde{\gamma}'z + \tilde{c}'\bar{z}}{\tilde{\gamma}z + \tilde{c}\bar{z}} \leq \tilde{\vartheta}. \tag{27}$$

Using this inequality together with (13), assumption (ii), and the relation  $\operatorname{Re} (a + (\tilde{c}/\tilde{\gamma})\bar{b}) = \operatorname{Re} a$ , we obtain

$$\begin{aligned}
 UU' & \leq U^2 (\operatorname{Re} a + \tilde{\vartheta} + \tilde{\kappa}) \\
 & + U \sum_{k=1}^m (\tilde{\kappa}_k |\tilde{\gamma}(\theta_k) w_k + \tilde{c}(\theta_k) \bar{w}_k|) \\
 & + U (|\tilde{\gamma}| + |\tilde{c}|) \\
 & \times \left( \sum_{k=1}^m \frac{|A_k| |w_k| + |B_k| |\bar{w}_k|}{|\tilde{\gamma}(\theta_k)| - |\tilde{c}(\theta_k)|} (|\tilde{\gamma}(\theta_k)| - |\tilde{c}(\theta_k)|) \right) \\
 & \leq U^2 (\operatorname{Re} a + \tilde{\vartheta} + \tilde{\kappa})
 \end{aligned}$$

$$\begin{aligned}
 & + U \left\{ \sum_{k=1}^m \left[ \tilde{\kappa}_k + (|A_k| + |B_k|) \frac{|\tilde{\gamma}| + |\tilde{c}|}{|\tilde{\gamma}(\theta_k)| - |\tilde{c}(\theta_k)|} \right] \right. \\
 & \quad \left. \times |\tilde{\gamma}(\theta_k) w_k + \tilde{c}(\theta_k) \bar{w}_k| \right\} \\
 & \leq U^2 (\operatorname{Re} a + \tilde{\vartheta} + \tilde{\kappa}) \\
 & + U \sum_{k=1}^m \tilde{\lambda}_k |\tilde{\gamma}(\theta_k) w_k + \tilde{c}(\theta_k) \bar{w}_k|, \tag{28}
 \end{aligned}$$

for almost all  $t \in \mathcal{K}$ .

Consequently,

$$U' \leq U (\operatorname{Re} a + \tilde{\vartheta} + \tilde{\kappa}) + \sum_{k=1}^m \tilde{\lambda}_k |\tilde{\gamma}(\theta_k) w_k + \tilde{c}(\theta_k) \bar{w}_k|, \tag{29}$$

for almost all  $t \in \mathcal{K}$ .

Recalling that  $U'(t) = 0$  for almost all  $t \in \mathcal{M}$ , we can see that inequality (29) is valid for almost all  $t \geq t_1$  for which  $z(t)$  is defined.

From (29) we have

$$\begin{aligned}
 V' & \leq U (\operatorname{Re} a + \tilde{\vartheta} + \tilde{\kappa} + m\tilde{\beta}) \\
 & + \sum_{k=1}^m (\tilde{\lambda}_k - \theta'_k \tilde{\beta}) |\tilde{\gamma}(\theta_k) w_k + \tilde{c}(\theta_k) \bar{w}_k| \\
 & + \tilde{\beta}' \sum_{k=1}^m \int_{\theta_k(t)}^t |\tilde{\gamma}(s) z(s) + \tilde{c}(s) \bar{z}(s)| ds. \tag{30}
 \end{aligned}$$

As  $\tilde{\beta}(t)$  fulfills condition (12), we obtain

$$\begin{aligned}
 V'(t) & \leq U(t) \tilde{\Theta}(t) \\
 & + \tilde{\beta}'(t) \sum_{k=1}^m \int_{\theta_k(t)}^t |\tilde{\gamma}(s) z(s) + \tilde{c}(s) \bar{z}(s)| ds, \tag{31}
 \end{aligned}$$

Hence,

$$V'(t) - \tilde{\Lambda}(t) V(t) \leq 0, \tag{32}$$

for almost all  $t \geq t_1$  for which the solution  $z(t)$  exists.

Notice that, with respect to (9),

$$V(t) \geq (|\tilde{\gamma}(t)| - |\tilde{c}(t)|) |z(t)| \geq \mu |z(t)|, \tag{33}$$

for all  $t \geq t_1$  for which  $z(t)$  is defined.

Suppose that condition (18) holds, and choose arbitrary  $\varepsilon > 0$ . Put

$$\begin{aligned}
 \Delta & = \max_{s \in [T_1, t_1]} (|\tilde{\gamma}(s)| + |\tilde{c}(s)|), \quad L = \sup_{T \leq t < \infty} \int_T^t \tilde{\Lambda}(s) ds, \\
 \delta & = \mu \varepsilon \Delta^{-1} (1 + m\tilde{\beta}(t_1) (t_1 - T_1))^{-1} \exp \left\{ \int_T^{t_1} \tilde{\Lambda}(s) ds - L \right\}, \tag{34}
 \end{aligned}$$

where  $\mu > 0$ ,  $T_1 \geq t_0$ , and  $T \geq T_1$  are the numbers from condition (i).

If the initial function  $z_0(t)$  of the solution  $z(t)$  satisfies  $\max_{s \in [T_1, t_1]} |z_0(s)| < \delta$ , then the multiplication of (32) by  $\exp\{-\int_{t_1}^t \tilde{\Lambda}(s) ds\}$  and the integration over  $[t_1, t]$  yield

$$V(t) \exp\left\{-\int_{t_1}^t \tilde{\Lambda}(s) ds\right\} - V(t_1) \leq 0, \tag{35}$$

for all  $t \geq t_1$  for which  $z(t)$  is defined. From (33) and (35) we get

$$\begin{aligned} \mu |z(t)| &\leq V(t) \leq V(t_1) \exp\left\{\int_{t_1}^t \tilde{\Lambda}(s) ds\right\} \\ &\leq \left[ (|\tilde{\gamma}(t_1)| + |\tilde{c}(t_1)|) |z(t_1)| \right. \\ &\quad \left. + \tilde{\beta}(t_1) \max_{s \in [T_1, t_1]} |z(s)| \right. \\ &\quad \left. \times \left( \sum_{k=1}^m \int_{\theta_k(t_1)}^{t_1} (|\tilde{\gamma}(s)| + |\tilde{c}(s)|) ds \right) \right] \\ &\quad \times \exp\left\{\int_{t_1}^t \tilde{\Lambda}(s) ds\right\} \\ &\leq \left[ \Delta \max_{s \in [T_1, t_1]} |z_0(s)| \right. \\ &\quad \left. + \tilde{\beta}(t_1) \max_{s \in [T_1, t_1]} |z_0(s)| \Delta \right. \\ &\quad \left. \times \sum_{k=1}^m (t_1 - \theta_k(t_1)) \right] \\ &\quad \times \exp\left\{\int_{t_1}^t \tilde{\Lambda}(s) ds\right\}; \end{aligned} \tag{36}$$

that is,

$$\begin{aligned} \mu |z(t)| &\leq \Delta \max_{s \in [T_1, t_1]} |z_0(s)| (1 + m\tilde{\beta}(t_1)(t_1 - T_1)) \\ &\quad \times \exp\left\{L - \int_T^{t_1} \tilde{\Lambda}(s) ds\right\} < \mu\epsilon. \end{aligned} \tag{37}$$

Thus, we have  $|z(t)| < \epsilon$  for all  $t \geq t_1$ , and we conclude that the trivial solution of (2) is stable.

Now suppose that condition (19) is valid. Then, in view of the first part of Theorem 2, for  $K > 0$ , there is a  $\rho > 0$  such that  $\max_{s \in [T_1, t_1]} |z_0(s)| < \rho$  implies that the solution  $z(t)$  of (2) exists for all  $t \geq t_1$  and satisfies  $|z(t)| < K$ , where  $K$  is arbitrary real constant. Hence, from this and (33), we have

$$|z(t)| \leq \mu^{-1} V(t) \leq \mu^{-1} V(t_1) \exp\left\{\int_{t_1}^t \tilde{\Lambda}(s) ds\right\}, \tag{38}$$

for all  $t \geq t_1$ . This inequality, with condition (19), gives

$$\lim_{t \rightarrow \infty} z(t) = 0, \tag{39}$$

which completes the proof.  $\square$

*Remark 3.* Theorem 2 represents a generalization of previous results.

If we take  $A_1(t) = A(t)$ ,  $A_k \equiv 0$ , for  $k = 2, \dots, m$ ,  $B_1(t) = B(t)$ ,  $B_k \equiv 0$ , for  $k = 2, \dots, m$ , and  $\theta_1(t) = t - r$ , where  $r > 0$ , we get Theorem 4 from [16].

If we take  $\theta_k(t) = t - r_k$ , where  $r_k > 0$ ,  $k = 1, \dots, m$ , we obtain Theorem 1 from [18].

If we take  $A_1(t) = A(t)$ ,  $A_k \equiv 0$ , for  $k = 2, \dots, m$ ,  $B_1(t) = B(t)$ ,  $B_k \equiv 0$ , for  $k = 2, \dots, m$ , and  $\theta_1(t) = \theta(t)$ , we get Theorem 2.2 from [20].

The next theorem involves the function  $\tilde{q}$  in (ii); thus, it is more general than Theorem 2. A part of the proof of Theorem 2 is utilized in the proof of Theorem 4.

**Theorem 4.** *Let the assumptions (i), (ii), (iii), and (iv) hold and*

$$\begin{aligned} V(t) &= |\tilde{\gamma}(t) z(t) + \tilde{c}(t) \bar{z}(t)| \\ &\quad + \tilde{\beta}(t) \sum_{k=1}^m \int_{\theta_k(t)}^t |\tilde{\gamma}(s) z(s) + \tilde{c}(s) \bar{z}(s)| ds, \end{aligned} \tag{40}$$

where  $z(t)$  is any solution of (2) defined for  $t \rightarrow \infty$ . Then

$$\begin{aligned} \mu |z(t)| &\leq V(s) \exp\left(\int_s^t \tilde{\Lambda}(\xi) d\xi\right) \\ &\quad + \int_s^t \tilde{q}(\xi) \exp\left(\int_\xi^t \tilde{\Lambda}(\sigma) d\sigma\right) d\xi, \end{aligned} \tag{41}$$

for  $t \geq s \geq t_1$ , where  $t_1 \geq T$ .

*Proof.* Following the proof of Theorem 2, we have

$$\begin{aligned} V'(t) &\leq |\tilde{\gamma}(t) z(t) + \tilde{c}(t) \bar{z}(t)| \Theta(t) \\ &\quad + \tilde{\beta}'(t) \sum_{k=1}^m \int_{\theta_k(t)}^t |\tilde{\gamma}(s) z(s) + \tilde{c}(s) \bar{z}(s)| ds \\ &\quad + \tilde{q}(t) \\ &\leq \tilde{\Lambda}(t) V(t) + \tilde{q}(t), \end{aligned} \tag{42}$$

a.e. on  $[t_1, \infty)$ . Using this inequality, we get

$$V'(t) - \tilde{\Lambda}(t) V(t) \leq \tilde{q}(t), \tag{43}$$

a.e. on  $[t_1, \infty)$ . Multiplying (43) by  $\exp(-\int_s^t \tilde{\Lambda}(\xi) d\xi)$  gives

$$\left[ V(t) \exp\left(-\int_s^t \tilde{\Lambda}(\xi) d\xi\right) \right] \leq \tilde{q}(t) \exp\left(-\int_s^t \tilde{\Lambda}(\xi) d\xi\right), \tag{44}$$

a.e. on  $[t_1, \infty)$ . Integration over  $[s, t]$  yields

$$\begin{aligned} V(t) \exp\left(-\int_s^t \tilde{\Lambda}(\xi) d\xi\right) - V(s) \\ \leq \int_s^t \tilde{q}(\xi) \exp\left(-\int_s^\xi \tilde{\Lambda}(\sigma) d\sigma\right) d\xi, \end{aligned} \tag{45}$$

and multiplying (45) by  $\exp(\int_s^t \tilde{\Lambda}(\xi)d\xi)$ , we obtain

$$V(t) \leq V(s) \exp\left(\int_s^t \tilde{\Lambda}(\xi) d\xi\right) + \int_s^t \tilde{q}(\xi) \exp\left(\int_\xi^t \tilde{\Lambda}(\sigma) d\sigma\right) d\xi. \tag{46}$$

The statement now follows from (33). □

*Remark 5.* Theorem 4 generalizes theorems contained in previous papers.

If we take  $A_1(t) = A(t)$ ,  $A_k \equiv 0$ , for  $k = 2, \dots, m$ ,  $B_1(t) = B(t)$ ,  $B_k \equiv 0$ , for  $k = 2, \dots, m$ , and  $\theta_1(t) = t - r$ , where  $r > 0$ , we get Theorem 2 from [16].

If we take  $\theta_k(t) = t - r_k$ , where  $r_k > 0$ ,  $k = 1, \dots, m$ , we obtain Theorem 2 from [18].

If we take  $A_1(t) = A(t)$ ,  $A_k \equiv 0$ , for  $k = 2, \dots, m$ ,  $B_1(t) = B(t)$ ,  $B_k \equiv 0$ , for  $k = 2, \dots, m$ , and  $\theta_1(t) = \theta(t)$ , we get Theorem 2.7 from [20].

The last of the main propositions gives the conditions under which all solutions of (2) tend to zero.

**Theorem 6.** *Let the assumptions (i), (ii), (iii), and (iv) be satisfied. Let  $\tilde{\Lambda}(t) \leq 0$  a.e. on  $[T^*, \infty)$ , where  $T^* \in [T, \infty)$ . If*

$$\lim_{t \rightarrow \infty} \int^t \tilde{\Lambda}(s) ds = -\infty, \quad \tilde{q}(t) = o(\tilde{\Lambda}(t)), \tag{47}$$

then any solution  $z(t)$  of (2) existing for  $t \rightarrow \infty$  satisfies

$$\lim_{t \rightarrow \infty} z(t) = 0. \tag{48}$$

*Proof.* Choose arbitrary  $\varepsilon > 0$ . According to (47), there is  $s \geq T^*$  such that  $\tilde{q}(t) \leq (1/2)\mu\varepsilon|\tilde{\Lambda}(t)|$  for  $t \geq s$  and

$$\begin{aligned} & \int_s^t \tilde{q}(\tau) \exp\left(\int_\tau^t \tilde{\Lambda}(\sigma) d\sigma\right) d\tau \\ & \leq \frac{\mu\varepsilon}{2} \int_s^t [-\tilde{\Lambda}(\tau)] \exp\left(\int_\tau^t \tilde{\Lambda}(\sigma) d\sigma\right) d\tau \\ & = \frac{\mu\varepsilon}{2} \int_s^t \left(\frac{d}{d\tau} \left[\exp\left(\int_\tau^t \tilde{\Lambda}(\sigma) d\sigma\right)\right]\right) d\tau \\ & = \frac{\mu\varepsilon}{2} \left[\exp\left(\int_\tau^t \tilde{\Lambda}(\sigma) d\sigma\right)\right]_s^t \\ & = \frac{\mu\varepsilon}{2} \left[1 - \exp\left(\int_s^t \tilde{\Lambda}(\tau) d\tau\right)\right] < \frac{\mu\varepsilon}{2}, \end{aligned} \tag{49}$$

for  $t \geq s$ . From (47) we have  $\exp(\int_s^t \tilde{\Lambda}(\tau)d\tau) \rightarrow 0$  as  $t \rightarrow \infty$ ; hence, there is  $S \geq s$  such that  $\exp(\int_s^t \tilde{\Lambda}(\tau)d\tau) < \mu\varepsilon(2V(s))^{-1}$  for  $t \geq S$ . Considering this fact and (41), we get

$$\mu|z(t)| < V(s) \frac{\mu\varepsilon}{2V(s)} + \frac{\mu\varepsilon}{2} = \mu\varepsilon, \tag{50}$$

for  $t \geq S$ . This completes the proof. □

*Remark 7.* Theorem 6 is a generalization of results published in the papers [16, 18, 20].

If we take  $A_1(t) = A(t)$ ,  $A_k \equiv 0$ , for  $k = 2, \dots, m$ ,  $B_1(t) = B(t)$ ,  $B_k \equiv 0$ , for  $k = 2, \dots, m$ , and  $\theta_1(t) = t - r$ , where  $r > 0$ , we get Theorem 3 from [16].

If we take  $\theta_k(t) = t - r_k$ , where  $r_k > 0$ ,  $k = 1, \dots, m$ , we obtain Theorem 3 from [18].

If we take  $A_1(t) = A(t)$ ,  $A_k \equiv 0$ , for  $k = 2, \dots, m$ ,  $B_1(t) = B(t)$ ,  $B_k \equiv 0$ , for  $k = 2, \dots, m$ , and  $\theta_1(t) = \theta(t)$ , we get Theorem 2.14 from [20].

### 4. Corollaries and Examples

From Theorem 2 we easily obtain several corollaries. We give an example which shows that it is worth to consider the case (5).

**Corollary 8.** *Let  $a(t) \equiv a \in \mathbb{C}$ ,  $b(t) \equiv b \in \mathbb{C}$ ,  $|\operatorname{Im} a| > |b|$ . Suppose that  $\lim_{t \rightarrow \infty} \theta_k(t) = \infty$ ,  $\theta_k(t) \leq t$ , for  $t \geq T_1$ , where  $T_1 \geq t_0$ . Let  $\rho_0, \rho_1, \dots, \rho_m : [T_1, \infty) \rightarrow \mathbb{R}$  be such that*

$$|g(t, z, w_1, \dots, w_m)| \leq \rho_0(t)|z| + \sum_{k=1}^m \rho_k(t)|w_k|, \tag{51}$$

for  $t \geq T_1$ ,  $|z| < R$ ,  $|w_k| < R$ ,  $k = 1, \dots, m$ ,  $R > 0$  and  $\rho_0 \in L_{\text{loc}}([T_1, \infty), \mathbb{R})$ .

Let  $\tilde{\beta} \in AC_{\text{loc}}([T_1, \infty), \mathbb{R}_+)$  satisfy

$$\begin{aligned} & \theta'_k(t) \tilde{\beta}(t) \\ & \geq \left(\frac{|\operatorname{Im} a| + |b|}{|\operatorname{Im} a| - |b|}\right)^{1/2} (\rho_k(t) + |A_k(t)| + |B_k(t)|), \end{aligned} \tag{52}$$

a.e. on  $[T_1, \infty)$  for  $k = 1, \dots, m$ . If

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \int^t \max\left(\operatorname{Re} a + \left(\frac{|\operatorname{Im} a| + |b|}{|\operatorname{Im} a| - |b|}\right)^{1/2} \rho_0(s) \right. \\ & \quad \left. + m\tilde{\beta}(s), \frac{\tilde{\beta}'(s)}{\tilde{\beta}(s)}\right) ds < \infty, \end{aligned} \tag{53}$$

then the trivial solution of (2) is stable. If

$$\begin{aligned} & \lim_{t \rightarrow \infty} \int^t \max\left(\operatorname{Re} a + \left(\frac{|\operatorname{Im} a| + |b|}{|\operatorname{Im} a| - |b|}\right)^{1/2} \rho_0(s) \right. \\ & \quad \left. + m\tilde{\beta}(s), \frac{\tilde{\beta}'(s)}{\tilde{\beta}(s)}\right) ds = -\infty, \end{aligned} \tag{54}$$

then the trivial solution of (2) is asymptotically stable.

*Proof.* Choose  $T \geq T_1$  such that  $\theta_k(t) \geq T_1$  for  $t \geq T$ ,  $k = 1, \dots, m$ . Denote that  $z = z(t)$  and  $w_k = z(\theta_k(t))$  again. Since

$a, b \in \mathbb{C}$  are constants, then also  $\tilde{\gamma}$  and  $\tilde{c}$  are constants, and we have  $\tilde{\vartheta}(t) \equiv 0$ . Using condition (51) we get

$$\begin{aligned} & |\tilde{\gamma}g(t, z, w_1, \dots, w_m) + \tilde{c}\bar{g}(t, z, w_1, \dots, w_m)| \\ & \leq (|\tilde{\gamma}| + |\tilde{c}|) \left( \rho_0(t) |z| + \sum_{k=1}^m \rho_k(t) |w_k| \right) \\ & = \frac{|\tilde{\gamma}| + |\tilde{c}|}{|\tilde{\gamma}| - |\tilde{c}|} (|\tilde{\gamma}| - |\tilde{c}|) \left( \rho_0(t) |z| + \sum_{k=1}^m \rho_k(t) |w_k| \right) \\ & \leq \frac{|\tilde{\gamma}| + |\tilde{c}|}{|\tilde{\gamma}| - |\tilde{c}|} \left( \rho_0(t) |\tilde{\gamma}z + \tilde{c}\bar{z}| + \sum_{k=1}^m \rho_k(t) |\tilde{\gamma}w_k + \tilde{c}\bar{w}_k| \right), \end{aligned} \tag{55}$$

and it follows that condition (ii) holds with

$$\tilde{\kappa}(t) = \frac{|\tilde{\gamma}| + |\tilde{c}|}{|\tilde{\gamma}| - |\tilde{c}|} \rho_0(t), \quad \tilde{\kappa}_k(t) = \frac{|\tilde{\gamma}| + |\tilde{c}|}{|\tilde{\gamma}| - |\tilde{c}|} \rho_k(t), \tag{56}$$

and  $\tilde{q}(t) \equiv 0$ .

Condition (53) implies that  $\operatorname{Re} a \leq 0$ . Since

$$\begin{aligned} \frac{|\tilde{\gamma}| + |\tilde{c}|}{|\tilde{\gamma}| - |\tilde{c}|} &= \frac{|\operatorname{Im} a| + \sqrt{|\operatorname{Im} a|^2 - |b|^2} + |b|}{|\operatorname{Im} a| + \sqrt{|\operatorname{Im} a|^2 - |b|^2} - |b|} \\ &= \left( \frac{|\operatorname{Im} a| + |b|}{|\operatorname{Im} a| - |b|} \right)^{1/2}, \end{aligned} \tag{57}$$

in view of (14) we obtain

$$\begin{aligned} \tilde{\lambda}_k(t) &= \left( \frac{|\operatorname{Im} a| + |b|}{|\operatorname{Im} a| - |b|} \right)^{1/2} \\ & \quad \times \{ \rho_k(t) + |A_k(t)| + |B_k(t)| \}, \\ \tilde{\Theta}(t) &= \operatorname{Re} a + \frac{|\tilde{\gamma}| + |\tilde{c}|}{|\tilde{\gamma}| - |\tilde{c}|} \rho_0(t) + m\tilde{\beta}(t) \\ &= \operatorname{Re} a + \left( \frac{|\operatorname{Im} a| + |b|}{|\operatorname{Im} a| - |b|} \right)^{1/2} \rho_0(t) + m\tilde{\beta}(t), \end{aligned} \tag{58}$$

and the assertion follows from (16) and Theorem 2.  $\square$

Now we show an example that, under certain circumstances, Corollary 8 is more useful than Corollary 1 from [21].

*Example 9.* Consider (2), where  $a(t) \equiv -\sqrt{5} + 2i$ ,  $b(t) \equiv 1$ ,  $A_k(t) \equiv 0$ ,  $B_k(t) \equiv 0$ , for  $k = 1, \dots, m$ , and

$$g(t, z, w_1, \dots, w_m) = \frac{2}{\sqrt{3}} e^{it} z + \sum_{k=1}^m \frac{k}{2mt} (\sqrt{15} - \sqrt{14}) e^{-t} w_k. \tag{59}$$

Assume that  $t_0 = m$  and  $R = \infty$ ,  $\theta_k(t) = k \ln t$ . Put  $T = e^{t_0} = e^m$ . Then,  $\rho_0(t) \equiv 2/\sqrt{3}$ ,  $\rho_k(t) = (k/2mt)(\sqrt{15} - \sqrt{14})e^{-t}$ . We have

$$\begin{aligned} & \max \left( \frac{|a| - |b|}{|a|} \operatorname{Re} a + \left( \frac{|a| + |b|}{|a| - |b|} \right)^{1/2} \rho_0(t) + m\beta(t), \frac{\beta'(t)}{\beta(t)} \right) \\ &= \max \left( -\frac{2}{3} \sqrt{5} + \sqrt{2} \frac{2}{\sqrt{3}} + m\beta(t), \frac{\beta'(t)}{\beta(t)} \right) \\ &\geq \frac{2}{3} (\sqrt{6} - \sqrt{5}) > 0, \end{aligned} \tag{60}$$

for

$$\begin{aligned} \theta'_k(t) \beta(t) &= \frac{k}{t} \beta(t) \geq \left( \frac{|a| + |b|}{|a| - |b|} \right)^{1/2} \\ & \quad \times \{ \rho_k(t) + |A_k(t)| + |B_k(t)| \} \\ &= \frac{k}{mt\sqrt{2}} (\sqrt{15} - \sqrt{14}) e^{-t}, \end{aligned} \tag{61}$$

where  $k \in \{1, \dots, m\}$ ; hence, we cannot apply Corollary 1 from [21].

On the other hand, if we use

$$\begin{aligned} \theta'_k(t) \tilde{\beta}(t) &= \frac{k}{t} \tilde{\beta}(t) = \frac{k\sqrt{3}}{2mt} (\sqrt{15} - \sqrt{14}) e^{-t} \\ &\geq \left( \frac{|\operatorname{Im} a| + |b|}{|\operatorname{Im} a| - |b|} \right)^{1/2} \{ \rho_k(t) + |A_k(t)| + |B_k(t)| \}, \end{aligned} \tag{62}$$

where  $k \in \{1, \dots, m\}$ , we have

$$\begin{aligned} & \max \left( \operatorname{Re} a + \left( \frac{|\operatorname{Im} a| + |b|}{|\operatorname{Im} a| - |b|} \right)^{1/2} \rho_0(t) + m\tilde{\beta}(t), \frac{\tilde{\beta}'(t)}{\tilde{\beta}(t)} \right) \\ &= \max \left( -\sqrt{5} + 2 + m \frac{\sqrt{3}}{2m} (\sqrt{15} - \sqrt{14}) e^{-t}, -1 \right) \\ &\leq -\sqrt{5} + 2 + \frac{\sqrt{3}}{2} (\sqrt{15} - \sqrt{14}) < -\frac{12}{100} < 0; \end{aligned} \tag{63}$$

thus, Corollary 8 guarantees the stability and also asymptotic stability of the trivial solution of the considered equation.

The following corollary gives sufficient conditions for stability of the trivial solution of (2).

**Corollary 10.** Assume that the conditions (i), (ii), and (iii) are valid with  $\tilde{q}(t) \equiv 0$ . If  $\tilde{\beta}(t)$  is monotone and bounded on  $[T, \infty)$  and if

$$\limsup_{t \rightarrow \infty} \int_t^{\infty} [\tilde{\Theta}(s)]_+ ds < \infty, \tag{64}$$

where  $[\tilde{\Theta}(t)]_+ = \max\{\tilde{\Theta}(t), 0\}$ , then the trivial solution of (2) is stable.



*Proof.* Suppose firstly that  $\tilde{\beta}$  is nonincreasing on  $[T, \infty)$ . Then,  $\tilde{\beta}' \leq 0$  a.e. on  $[T, \infty)$ .

If  $\tilde{\beta}(T_2) = 0$  for some  $T_2 \geq T$ , then  $\tilde{\beta}(t) \equiv 0$  on  $[T_2, \infty)$ . Consequently,  $\tilde{\Lambda}$  has to satisfy only the inequality  $\tilde{\Theta}(t) \leq \tilde{\Lambda}(t)$  a.e. on  $[T_2, \infty)$ , so we may choose  $\tilde{\Lambda}(t) = \tilde{\Theta}(t)$  on  $[T_2, \infty)$ . It follows that  $\tilde{\Lambda}(t) = \tilde{\Theta}(t) \leq \max\{\tilde{\Theta}(t), 0\} = [\tilde{\Theta}(t)]_+$ .

On the other hand, if  $\tilde{\beta}(t) > 0$  on  $[T, \infty)$ , we may put  $\tilde{\Lambda}(t) = \max\{\tilde{\Theta}(t), \tilde{\beta}'(t)/\tilde{\beta}(t)\}$ . Then,

$$\tilde{\Lambda}(t) = \max \left\{ \tilde{\Theta}(t), \frac{\tilde{\beta}'(t)}{\tilde{\beta}(t)} \right\} \leq \max \{ \tilde{\Theta}(t), 0 \} = [\tilde{\Theta}(t)]_+. \tag{65}$$

In both cases,  $\tilde{\Lambda}$  satisfies condition (iv) and the inequality  $\tilde{\Lambda}(t) \leq [\tilde{\Theta}(t)]_+$  on  $[T_2, \infty)$ ; hence,

$$\limsup_{t \rightarrow \infty} \int^t \tilde{\Lambda}(s) ds \leq \limsup_{t \rightarrow \infty} \int^t [\tilde{\Theta}(s)]_+ ds < \infty. \tag{66}$$

Now assume that  $\tilde{\beta}$  is nondecreasing on  $[T, \infty)$ . Then,  $\tilde{\beta}' \geq 0$  a.e. on  $[T, \infty)$ .

If  $\tilde{\beta}(t) \equiv 0$  on  $[T, \infty)$ , we may treat it as previously mentioned.

Otherwise, there is some  $T_3 \geq T$  such that  $\tilde{\beta}(t) > 0$  on  $[T_3, \infty)$ , and we may choose  $\tilde{\Lambda}(t) = \max\{\tilde{\Theta}(t), \tilde{\beta}'(t)/\tilde{\beta}(t)\}$  on  $[T_3, \infty)$ . Clearly  $\tilde{\Lambda}$  satisfies condition (iv) on  $[T_3, \infty)$ . Since  $\tilde{\beta}' \geq 0$  a.e. on  $[T, \infty)$ , it follows that  $\tilde{\beta}'/\tilde{\beta} \geq 0$  a.e. on  $[T_3, \infty)$ . Hence,

$$\begin{aligned} \tilde{\Lambda}(t) &= \max \left\{ \tilde{\Theta}(t), \frac{\tilde{\beta}'(t)}{\tilde{\beta}(t)} \right\} \\ &\leq \max \left\{ [\tilde{\Theta}(t)]_+, \frac{\tilde{\beta}'(t)}{\tilde{\beta}(t)} \right\} \\ &\leq [\tilde{\Theta}(t)]_+ + \frac{\tilde{\beta}'(t)}{\tilde{\beta}(t)}, \end{aligned} \tag{67}$$

and then

$$\begin{aligned} \limsup_{t \rightarrow \infty} \int^t \tilde{\Lambda}(s) ds &\leq \limsup_{t \rightarrow \infty} \int^t [\tilde{\Theta}(s)]_+ ds + \limsup_{t \rightarrow \infty} \int^t \frac{\tilde{\beta}'(t)}{\tilde{\beta}(t)} ds \\ &\leq \limsup_{t \rightarrow \infty} \int^t [\tilde{\Theta}(s)]_+ ds + \limsup_{t \rightarrow \infty} (\ln(\tilde{\beta}(t))) \\ &\quad - \ln(\tilde{\beta}(T_3)) < \infty, \end{aligned} \tag{68}$$

since  $\tilde{\beta}$  is bounded on  $[T, \infty)$ .

The statement follows from Theorem 2. □

We can derive several consequences from Theorem 4.

**Corollary 11.** *Let the conditions (i), (ii), (iii), and (iv) be fulfilled and*

$$\limsup_{t \rightarrow \infty} \int_s^t \tilde{q}(\xi) \exp \left( - \int_s^\xi \tilde{\Lambda}(\sigma) d\sigma \right) d\xi < \infty, \tag{69}$$

for some  $s \geq T$ .

If  $z(t)$  is any solution of (2) existing for  $t \rightarrow \infty$ , then

$$z(t) = O \left[ \exp \left( \int_s^t \tilde{\Lambda}(\xi) d\xi \right) \right]. \tag{70}$$

*Proof.* From the assumptions and (45) we can see that there are  $K > 0$  and  $S \geq s$  such that for  $t \geq S$  we have

$$\begin{aligned} V(t) \exp \left( - \int_s^t \tilde{\Lambda}(\xi) d\xi \right) - V(s) &\leq \int_s^t \tilde{q}(\xi) \exp \left( - \int_s^\xi \tilde{\Lambda}(\sigma) d\sigma \right) d\xi \\ &\leq K < \infty. \end{aligned} \tag{71}$$

Then,

$$\mu |z(t)| \leq V(t) \leq (K + V(s)) \exp \left( \int_s^t \tilde{\Lambda}(\xi) d\xi \right). \tag{72}$$

□

**Corollary 12.** *Let the assumptions (i), (ii), (iii), and (iv) hold, and let*

$$\limsup_{t \rightarrow \infty} \tilde{\Lambda}(t) < \infty, \quad \tilde{q}(t) = O(e^{\eta t}), \tag{73}$$

where  $\eta > \limsup_{t \rightarrow \infty} \tilde{\Lambda}(t)$ . If  $z(t)$  is any solution of (2) existing for  $t \rightarrow \infty$ , then  $z(t) = O(e^{\eta t})$ .

*Proof.* In view of (73), there are  $L > 0$ ,  $\eta^* < \eta$ , and  $s > T$  such that  $\eta^* > \tilde{\Lambda}(t)$  for  $t \geq s$  and  $\tilde{q}(t)e^{-\eta t} < L$  for  $t \geq s$ . From (41) we get

$$\begin{aligned} \mu |z(t)| &\leq V(s) e^{\eta^*(t-s)} + L \int_s^t e^{\eta\tau} e^{\eta^*(t-\tau)} d\tau \\ &\leq V(s) e^{\eta^*(t-s)} + L e^{\eta^* t} \frac{e^{(\eta-\eta^*)t} - e^{(\eta-\eta^*)s}}{\eta - \eta^*} \\ &\leq V(s) e^{\eta^*(t-s)} + \frac{L}{\eta - \eta^*} e^{\eta t} = O(e^{\eta t}). \end{aligned} \tag{74}$$

□

*Remark 13.* If  $\tilde{\rho}(t) \equiv 0$ , we can take  $L = 0$  in the proof of Corollary 12, and taking inequalities (74) into account we obtain the following statement: there is an  $\eta^* < \eta_0 < \eta$  such that  $z(t) = o(e^{\eta_0 t})$  holds for the solution  $z(t)$  of (2).

### 5. Conclusion

We studied asymptotic behavior of real two-dimensional differential system with a finite number of nonconstant delays. We considered the case corresponding to the situation

when the equilibrium point 0 of autonomous system (6) is a stable focus or a stable centre. We utilized the method of complexification and the method of Lyapunov-Krasovskii functional. Criteria for stability and asymptotic stability of the solutions as well as conditions ensuring that all solutions of (2) tend to zero are derived. At the end we supplied several corollaries and an example which shows that in some cases the criteria obtained in this paper are more applicable than the criteria presented in [21].

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