Research Article

Stationary Patterns of a Cross-Diffusion Epidemic Model

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We investigate the complex dynamics of cross-diffusion *SI* epidemic model. We first give the conditions of the local and global stability of the nonnegative constant steady states, which indicates that the basic reproduction number determines whether there is an endemic outbreak or not. Furthermore, we prove the existence and nonexistence of the positive nonconstant steady states, which guarantees the existence of the stationary patterns.

1. Introduction

In epidemiology, epidemic compartmental models, since the pioneer work of Kermack and McKendrick [1], are widely used for increasing the understanding of infectious disease dynamics and for determining preventive measures to control infection spread qualitatively and quantitatively [2]. More recently, many studies have shown that a spatial epidemic model is an appropriate tool for investigating the fundamental mechanism of complex spatiotemporal epidemic dynamics. In these studies, reaction-diffusion equations have been intensively used to describe spatiotemporal dynamics [3–21]. Spatial epidemiology with diffusion has become a principal scientific discipline aiming at understanding the causes and consequences of spatial heterogeneity in disease transmission [20].

In addition, from a biological perspective, the diffusion of individuals may be connected with other things, such as searching for food, escaping high infection risks. In the first case, individuals tend to diffuse in the direction of lower density of a population, where there are richer resources. In the second, individuals may move along the gradient of infectious individuals to avoid higher infection [12, 22]. Keeping these in view, cross-diffusion arises, which was proposed first by Kerner [23] and first applied in competitive population system by Shigesada et al. [24]. In particular, Sun et al. [16],

by using the standard linear analysis, studied the pattern formation in a cross-diffusion *SI* epidemic model. And in [19], the authors presented Turing pattern selection in a cross-diffusion *SI* epidemic model with zero-flux boundary conditions, gave the conditions of Hopf and Turing bifurcations, and derived the amplitude equations for the excited modes.

In the past decades, it has been shown that the reaction-diffusion system is capable to generate complex spatiotemporal patterns, and the existence of stationary patterns induced by diffusion has attracted the extensive attention of a great number of biologists and mathematicians, and lots of fascinating and important phenomena have been observed [25–33]. In particular, in the field of epidemiology, there are many contributions to the existence of steady states in the diffusive epidemic models [34–45]. But in the studies on the steady states of diffusive epidemic models, little attention has been paid to study on the effect of cross-diffusion.

The main focus of this paper is to investigate how cross-diffusion affects disease's dynamics through studying the existence of the constant and nonconstant steady states of a cross-diffusion *SI* epidemic model.

The rest of this paper is organized as follows. In Section 2, we derive a cross-diffusion *SI* epidemic model. In Section 3, we give the global existence and positivity of the solution. In Section 4, we study the local and global stability of the nonnegative steady states of the model. In Section 5, we first

give a priori estimates for the positive solutions of the model and then give some results on the existence and nonexistence of positive nonconstant steady states of the model. The paper ends with a brief discussion in Section 6.

2. Model Derivation

Assume that the habitat $\Omega \subset \mathbb{R}^m$ ($m \geq 1$) is a bounded domain with smooth boundary $\partial\Omega$ (when $m > 1$), and $\mathbf{n}$ is the outward unit normal vector on $\partial\Omega$. We consider the following cross-diffusion SI epidemic model:

$$
\begin{aligned}
\frac{\partial S}{\partial t} - d_S \Delta S &= rS\left(1 - \frac{S}{K}\right) - \frac{\beta SI}{S+I}, \quad x \in \Omega, \ t > 0, \\
\frac{\partial I}{\partial t} - D\Delta S - d_I \Delta I &= \frac{\beta SI}{S+I} - \mu I, \quad x \in \Omega, \ t > 0, \\
\frac{\partial S}{\partial \mathbf{n}} &= \frac{\partial I}{\partial \mathbf{n}} = 0, \quad x \in \partial\Omega, \\
S(x,0) = S_0(x) > 0, \quad I&(x,0) = I_0(x) \geq 0, \quad x \in \Omega,
\end{aligned}
\tag{1}
$$

where $S(x,t)$ and $I(x,t)$ denote the density of susceptible and infected individuals at location $x \in \Omega$ and time t, respectively, d_S and d_I are the self-diffusion coefficients for the susceptible and infected individuals, and D is the cross-diffusion coefficient. r stands for the susceptible population intrinsic growth rate, β the rate of transmission, μ the death rate of the infected population I, and K the carrying capacity. The symbol Δ is the Laplacian operator. The homogeneous Neumann boundary condition implies that the above model is self-contained and there is no infection across the boundary.

It is worthy to note that the diffusion coefficients d_S, d_I, and D are such that $d_S, d_I > 0$, $D \geq 0$ and $d_S > d_I$, $D^2 < 4d_S d_I$ which is the parabolic condition.

For model (1), the basic reproduction number is defined as

$$
R_0 = \frac{\beta}{\mu}. \tag{2}
$$

The steady states of model (1) satisfy

$$
\begin{aligned}
-d_S \Delta S &= rS\left(1 - \frac{S}{K}\right) - \frac{\beta SI}{S+I}, \quad x \in \Omega, \\
-D\Delta S - d_I \Delta I &= \frac{\beta SI}{S+I} - \mu I, \quad x \in \Omega, \\
\frac{\partial S}{\partial \mathbf{n}} &= \frac{\partial I}{\partial \mathbf{n}} = 0, \quad x \in \partial\Omega, \ t > 0.
\end{aligned}
\tag{3}
$$

Throughout this paper, the positive solution (S, I) satisfying (1) refers to a classical one with $S > 0$, $I > 0$ on $\overline{\Omega}$. Clearly, model (1) has a semitrivial constant solution (disease-free equilibrium) $E_0 = (K, 0)$ and a positive constant solution (endemic equilibrium)

$$
\begin{aligned}
E^* &= (S^*, I^*) \\
&= \left(\frac{K}{r}(r - (R_0 - 1)\mu), \frac{K}{r}(r - (R_0 - 1)\mu)(R_0 - 1)\right)
\end{aligned}
\tag{4}
$$

if $1 < R_0 < 1 + (r/\mu)$.

3. Global Existence and Positivity of the Solution

In this section, we show the existence of unique positive global solution of model (1).

First, we convert model (1) into an abstract first-order system $C(\overline{\Omega}) \times C(\overline{\Omega})$ of the form

$$
\begin{aligned}
U'(t) &= AU(t) + F(U(t)), \quad t > 0, \\
U(0) &= U_0 \in C(\overline{\Omega}) \times C(\overline{\Omega}),
\end{aligned}
\tag{5}
$$

where

$$
\begin{aligned}
AU(t) &= (d_S \Delta S, D\Delta S + d_I \Delta I), \\
F(U(t)) &= \left(rS\left(1 - \frac{S}{K}\right) - \frac{\beta SI}{S+I}, \frac{\beta SI}{S+I} - \mu I\right).
\end{aligned}
\tag{6}
$$

Since F is locally Lipschitz in $C(\overline{\Omega}) \times C(\overline{\Omega})$, for every initial date $U_0 \in C(\overline{\Omega}) \times C(\overline{\Omega})$, system (5) admits a unique local solution on $[0, T_{\max})$, where $T_{\max}$ is the maximal existence time for solution of system (5) [46].

Set $Z = I - (D/(d_S - d_I))S$. Then model (1) leads to

$$
\begin{aligned}
\frac{\partial S}{\partial t} - d_S \Delta S &= rS\left(1 - \frac{S}{K}\right) \\
&\quad - \frac{\beta SI}{S+I} := \Lambda(S, Z), \quad x \in \Omega, \ t > 0, \\
\frac{\partial Z}{\partial t} - d_I \Delta Z &= \frac{D}{d_S - d_I}\left(rS\left(1 - \frac{S}{K}\right) - \frac{\beta SI}{S+I}\right) \\
&\quad + \frac{\beta SI}{S+I} - \mu I := \Upsilon(S, Z), \quad x \in \Omega, \ t > 0, \\
\frac{\partial S}{\partial \mathbf{n}} &= \frac{\partial Z}{\partial \mathbf{n}} = 0, \quad x \in \partial\Omega, \ t > 0, \\
S(x,0) &= S_0(x) > 0, \\
Z(x,0) = Z_0(x) &= I_0(x) - \frac{D}{d_S - d_I}S_0(x), \quad x \in \Omega.
\end{aligned}
\tag{7}
$$

A simple application of a comparison theorem to model (7) implies (see [47]) that for positive initial date $S_0(x) > 0$ and $Z_0(x) \geq 0$ we have that

$$
S(x,t) > 0, \quad I(x) \geq \frac{D}{d_S - d_I}S(x,t), \quad \forall x \in \Omega, \ t > 0.
\tag{8}
$$

Applying the comparison principle we get that $S(x,t) \leq \max\{\|S_0\|_\infty, K\}$. To establish the uniform boundedness of $I(x,t)$, it is sufficient to show the uniform boundedness of $Z(x,t)$. This task is carried out using a result found in Henry [48], from which it is sufficient to derive a uniform estimate for $\|\Upsilon(S, Z)\|_p$. Hence, we apply the same method of [49, 50] to study the existence of global solution of model (1). And we have the following result.

Theorem 1. *The solution of model* (1) *is global and uniformly bounded in* $[0, \infty[$.

For the sake of simplicity, we omit the proof, and the interested readers may refer to [49, 50] for details.

4. Stability of Nonnegative Constant Steady States

In this section, we consider the stability behavior of nonnegative constant steady states to model (1).

4.1. Local Stability of Nonnegative Steady States. In this subsection, we will discuss the local stability of the constant steady states $E_0 = (K, 0)$ and $E^* = (S^*, I^*)$. For this purpose, we need to introduce some notations.

Let $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$ be the eigenvalues of the operator-Δ on Ω with the homogeneous Neumann boundary conditions. Let $\mathbf{X} = \{(S, I) \in [C^2(\Omega)]^2 \mid \partial S/\partial \mathbf{n} = \partial I/\partial \mathbf{n} = 0, x \in \partial\Omega\}$, $\{\phi_{ij} \mid j = 1, \ldots, \dim E(\lambda_i)\}$ be an orthonormal basis of $E(\lambda_i)$, and $\mathbf{X}_{ij} = \{\mathbf{c}\phi_{ij} \mid \mathbf{c} \in \mathbb{R}^2\}$; then $\mathbf{X} = \bigoplus_{i=1}^{\infty} \mathbf{X}_i$ where $\mathbf{X}_i = \bigoplus_{j=1}^{\dim E(\lambda_i)} \mathbf{X}_{ij}$.

Theorem 2. *For model* (1),

(a) *if* $1 < R_0 < 1 + (r/\mu)$ *and* $d_I(\mu R_0^2 - rR_0 - \mu) + D\mu(R_0 - 1)^2 < d_S\mu(R_0-1)$, *the positive constant steady state* $E^* = (S^*, I^*)$ *is locally asymptotically stable;*

(b) *if* $R_0 \leq 1$, *the semitrivial constant steady state* $E_0 = (K, 0)$ *is locally asymptotically stable.*

Proof. (a) The linearization of model (1) at the positive constant steady state $E^* = (S^*, I^*)$ can be expressed by

$$\frac{\partial \xi}{\partial t} = d_S\Delta\xi + \frac{\mu R_0^2 - rR_0 - \mu}{R_0}\xi - \frac{\mu}{R_0}\zeta, \quad x \in \Omega, \ t > 0,$$

$$\frac{\partial \zeta}{\partial t} = D\Delta\xi + d_I\Delta\zeta + \frac{\mu(R_0 - 1)^2}{R_0}\xi$$

$$- \frac{\mu(R_0 - 1)}{R_0}\zeta, \quad x \in \Omega, \ t > 0,$$

$$\frac{\partial \xi}{\partial \mathbf{n}} = \frac{\partial \zeta}{\partial \mathbf{n}} = 0, \quad x \in \partial\Omega.$$

(9)

Let

$$£ = \begin{pmatrix} d_S\Delta + \dfrac{\mu R_0^2 - rR_0 - \mu}{R_0} & -\dfrac{\mu}{R_0} \\ D\Delta + \dfrac{\mu(R_0 - 1)^2}{R_0} & d_I\Delta - \dfrac{\mu(R_0 - 1)}{R_0} \end{pmatrix}. \quad (10)$$

For each $i \geq 0$, $\mathbf{X}_i$ is invariant under the operator $£$, and η is an eigenvalue of $£$ if and only if η is an eigenvalue of the matrix

$$A_i = \begin{pmatrix} -d_S\lambda_i + \dfrac{\mu R_0^2 - rR_0 - \mu}{R_0} & -\dfrac{\mu}{R_0} \\ -D\lambda_i + \dfrac{\mu(R_0 - 1)^2}{R_0} & -d_I\lambda_i - \dfrac{\mu(R_0 - 1)}{R_0} \end{pmatrix}, \quad (11)$$

for some $i \geq 0$. Thus the stability of the positive constant steady state is reduced to consider the characteristic equation:

$$\det(\eta I - A_i) = \eta^2 - \mathrm{trace}(A_i)\eta + \det(A_i) := \varphi_i(\eta), \quad (12)$$

where

$$\mathrm{trace}(A_i) = -\lambda_i(d_S + d_I) - (r - \mu(R_0 - 1)) < 0,$$

$$\det(A_i) = d_S d_I \lambda_i^2 + \frac{1}{R_0}\left(d_S\mu(R_0 - 1) - d_I(\mu R_0^2 - rR_0 - \mu)\right.$$

$$\left. - D\mu(R_0 - 1)^2\right)\lambda_i$$

$$+ \frac{\mu(R_0 - 1)(r - \mu(R_0 - 1))}{R_0}.$$

(13)

It follows from $d_I(\mu R_0^2 - rR_0 - \mu) + D\mu(R_0 - 1)^2 < d_S\mu(R_0 - 1)$ that $\det(A_i) > 0$. Therefore, the eigenvalues of the matrix A_i have negative real parts. It thus follows from the Routh-Hurwitz criterion that, for each $i \geq 0$, the two roots η_{i1} and η_{i2} of $\varphi_i(\eta) = 0$ all have negative real parts.

In the following, we prove that there exists $\kappa > 0$ such that

$$\mathfrak{R}\{\eta_{i1}\}, \mathfrak{R}\{\eta_{i2}\} \leq -\kappa. \quad (14)$$

Let $\eta = \lambda_i\xi$; then $\varphi_i(\eta) = \lambda_i^2\xi^2 - \mathrm{tr}(A_i)\lambda_i\xi + \det(A_i) := \widetilde{\varphi}_i(\eta)$. Since $\lambda_i \to \infty$ as $i \to \infty$, it follows that

$$\lim_{i \to \infty} \frac{\widetilde{\varphi}_i(\eta)}{\lambda_i^2} = \xi^2 + (d_S + d_I)\xi + d_S d_I := \widetilde{\varphi}(\xi). \quad (15)$$

Clearly, $\widetilde{\varphi}(\xi) = 0$ has two negative roots: $-d_S, -d_I$. Let $d = \min\{d_S, d_I\}$. By continuity, we see that there exists i_0 such that the two roots ξ_{i1}, ξ_{i2} of $\widetilde{\varphi}_i(\eta) = 0$ satisfy $\mathfrak{R}\{\xi_{i1}\}, \mathfrak{R}\{\xi_{i2}\} \leq -(d/2), \forall i \geq i_0$. In turn, $\mathfrak{R}\{\eta_{i1}\}, \mathfrak{R}\{\eta_{i2}\} \leq -(\lambda_i d/2) \leq -(d/2)$, $\forall i \geq i_0$. Let $-\widetilde{\kappa} = \max_{1 \leq i \leq i_0}\{\mathfrak{R}\{\eta_{i1}\}, \mathfrak{R}\{\eta_{i2}\}\}$. Then $\widetilde{\kappa} > 0$ and (14) holds for $\kappa = \min\{\widetilde{\kappa}, d/2\}$.

Consequently, the spectrum of $£$ which consists of eigenvalues lies in $\{\mathfrak{R}\{\eta\} \leq -\kappa\}$. In the sense of [48], we obtain that the positive constant steady state solution $E^* = (S^*, I^*)$ of model (1) is uniformly asymptotically stable.

(b) The stability of the semitrivial constant steady state $E_0 = (K, 0)$ is reduced to consider the characteristic equation:

$$\det(\eta I - A_i) = \eta^2 - \mathrm{trace}(A_i)\eta + \det(A_i), \quad (16)$$

where

$$\text{trace}\,(A_i)$$
$$= -\lambda_i\,(d_S + d_I) - (r + \mu\,(R_0 - 1)) < 0,$$

$$\det\,(A_i)$$
$$= d_S d_I \lambda_i^2 + (d_S \mu\,(1 - R_0) + d_I r)\,\lambda_i + r\mu\,(1 - R_0) > 0. \tag{17}$$

The remaining arguments are rather similar as above. The proof is complete. □

4.2. Global Stability of the Nonnegative Steady States.

This subsection is devoted to the global stability of $E_0 = (K, 0)$ and $E^* = (S^*, I^*)$ for model (1).

First, we have the following lemma regarding the persistence property of the susceptible individuals which will play a critical role in the proof of the global stability of $E^* = (S^*, I^*)$.

Lemma 3. If $R_0 < r/\mu$, $S(x, t)$ satisfies

$$\liminf_{t \to \infty} \min_{\overline{\Omega}} S\,(x, t) \geq K\left(1 - \frac{R_0 \mu}{r}\right). \tag{18}$$

Proof. For all $t \geq 0$, $S(x, t)$ is an upper solution of the following problem:

$$\frac{\partial z}{\partial t} - d_S \Delta z = rz\left(1 - \frac{\beta}{r} - \frac{z}{K}\right), \quad x \in \Omega,\ t > 0,$$

$$\frac{\partial z}{\partial \mathbf{n}} = 0, \quad x \in \partial\Omega,\ t > 0, \tag{19}$$

$$z\,(x, 0) = S_0\,(x) > 0, \quad x \in \overline{\Omega}.$$

Let $S(t)$ be the unique positive solution of the problem

$$\frac{\partial w}{\partial t} = rw\left(1 - \frac{\beta}{r} - \frac{w}{K}\right), \quad t > 0,$$

$$w\,(0) = \max_{\overline{\Omega}} S_0\,(x) > 0. \tag{20}$$

Then $S(t)$ is a lower solution of (20). Since $\beta < r$ we have

$$\lim_{t \to \infty} S\,(t) = K\left(1 - \frac{\beta}{r}\right) = K\left(1 - \frac{R_0 \mu}{r}\right). \tag{21}$$

It follows by a comparison argument that

$$\liminf_{t \to \infty} \min_{\overline{\Omega}} S\,(x, t) \geq K\left(1 - \frac{R_0 \mu}{r}\right). \tag{22}$$

The proof is complete. □

Now, we give the result of the global stability of E_0 and E^*.

Theorem 4. For model (1),

(a) *if $\mu < r$ and $1 < R_0 < (r + \mu)/2\mu$, then positive constant steady state $E^* = (S^*, I^*)$ of model (1) globally asymptotically stable;*

(b) *if $R_0 \leq 1$, the semitrivial constant steady state $E_0 = (K, 0)$ of model (1) is globally asymptotically stable.*

Proof. (a) We adopt the Lyapunov function:

$$V\,(t) = \int_\Omega \left[V_1\,(S\,(x, t)) + \nu V_2\,(I\,(x, t))\right] dx, \tag{23}$$

where $V_1(S) = \int_{S^*}^{S}((\xi - S^*)/\xi)d\xi$, $V_2(I) = \nu \int_{I^*}^{I}((\eta - I^*)/\eta)d\eta$, and $\nu = S^*/I^* = 1/(R_0 - 1)$. Then $V(t) \geq 0$ and $V(t) = 0$ if and only if $(S, I) = (S^*, I^*)$. Then,

$$\frac{dV}{dt} = \int_\Omega \left(\frac{S - S^*}{S}\frac{\partial S}{\partial t} + \frac{\nu\,(I - I^*)}{I}\frac{\partial I}{\partial t}\right)dx$$

$$= \int_\Omega (S - S^*)\left(r - \frac{rS}{K} - \frac{\beta I}{S + I} + d_S\frac{\Delta S}{S}\right)dx$$

$$+ \int_\Omega \nu\,(I - I^*)\left(\frac{\beta S}{S + I} - \mu + d_I\frac{\Delta I}{I} + D\frac{\Delta S}{I}\right)dx$$

$$= \int_\Omega P\,(S, I)\,dx$$

$$- \int_\Omega \left(\frac{d_S S^*}{S^2}|\nabla S|^2 + \frac{d_I \nu I^*}{I^2}|\nabla I|^2 + \frac{D\nu I^*}{I^2}|\nabla S|\,|\nabla I|\right)dx, \tag{24}$$

where

$$P\,(S, I) = (S - S^*)\left(r - \frac{rS}{K} - \frac{\beta I}{S + I}\right)$$
$$+ \nu\,(I - I^*)\left(\frac{\beta S}{S + I} - \mu\right). \tag{25}$$

It follows from Lemma 3 that, for any $\varepsilon > 0$, there exists $t_0 > 0$, such that $S + I \geq S \geq K(1 - (\mu R_0/r)) - \varepsilon$ for all $x \in \overline{\Omega}$ and $t \geq t_0$. And by some computational analysis, we have

$$P\,(S, I) = (S - S^*)^2\left(-\frac{r}{K} + \frac{\beta I^*}{(S^* + I^*)\,(S + I)}\right)$$

$$- \frac{\nu S^*}{(S^* + I^*)\,(S + I)}(I - I^*)^2$$

$$= -(S - S^*)^2\left(\frac{r}{K} - \frac{\mu\,(R_0 - 1)}{S + I}\right)$$

$$- \frac{1}{R_0\,(R_0 - 1)\,(S + I)}(I - I^*)^2 \tag{26}$$

$$\leq -(S - S^*)^2\left(\frac{r}{K} - \frac{\mu\,(R_0 - 1)}{K\,(1 - \mu R_0/r) - \varepsilon}\right)$$

$$- \frac{1}{R_0\,(R_0 - 1)\,(S + I)}(I - I^*)^2.$$

Hence, in view of the conditions of the theorem and the arbitrariness of ε, we have $P(S, I) \leq 0$; that is, $dV/dt \leq 0$ for all $x \in \overline{\Omega}$ and $t \geq t_0$. So $V(t)$ decreases monotonically

along a solution orbit and E^* is globally asymptotically stable under the assumptions of the theorem.

(b) We adopt the Lyapunov function:

$$V(t) = \int_\Omega I\,dx. \tag{27}$$

Then $V(t) \geq 0$ and $V(t) = 0$ if and only if $I = 0$. Then, if $R_0 \leq 1$, we obtain

$$\begin{aligned}
\frac{dV}{dt} &= \int_\Omega \frac{\partial I}{\partial t}\,dx \\
&= \int_\Omega \left(\frac{\beta SI}{S+I} - \mu I + d_I \Delta I + D\Delta S \right) dx \\
&\leq \int_\Omega \mu I\,(R_0 - 1)\,dx \leq 0.
\end{aligned} \tag{28}$$

It follows from $I = 0$ and the second equation of model (1) that S is a constant. As a consequence, from the first equation of model (1) and $S \neq 0$, we have $S = K$. Hence, $E_0 = (K, 0)$ is globally asymptotically stable. $\qquad\square$

5. Existence and Nonexistence of Positive Nonconstant Steady States

In this section, we provide some sufficient conditions for the existence and nonexistence of nonconstant positive solution of model (3) by using the Leray-Schauder degree theory [51]. For the purpose, it is necessary to establish a priori positive upper and lower bounds for the positive solution of model (3).

5.1. A Priori Estimates. In order to obtain the desired bounds, we recall the following maximum principle [22] and Harnack inequality [52].

Lemma 5 (maximum principle, see [22]). *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^m$ and $g \in C(\overline{\Omega} \times \mathbb{R})$.*

(a) *Assume that $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and satisfies*

$$\Delta w(x) + g(x, w(x)) \geq 0 \quad \text{in } \Omega, \qquad \frac{\partial w}{\partial \mathbf{n}} \leq 0 \quad \text{on } \partial\Omega. \tag{29}$$

If $w(x_0) = \max_{\overline{\Omega}} w(x)$, then $g(x_0, w(x_0)) \geq 0$.

(b) *Assume that $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and satisfies*

$$\Delta w(x) + g(x, w(x)) \leq 0 \quad \text{in } \Omega, \qquad \frac{\partial w}{\partial \mathbf{n}} \geq 0 \quad \text{on } \partial\Omega. \tag{30}$$

If $w(x_0) = \min_{\overline{\Omega}} w(x)$, then $g(x_0, w(x_0)) \leq 0$.

Lemma 6 (Harnack inequality, see [52]). *Let $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a positive solution to $\Delta w(x) + c(x)w(x) = 0$, where $c \in C(\overline{\Omega})$, satisfying the homogeneous Neumann boundary*

conditions. Then there exists a positive constant $C^* = C^*(\|c\|_\infty, \Omega)$, such that

$$\max_{\overline{\Omega}} w \leq C^* \min_{\overline{\Omega}} w. \tag{31}$$

For convenience, let us denote the constants $r, K, \beta,$ and μ collectively by Λ. The positive constants $C, \underline{C}, \overline{C},$ and so forth will depend only on the domains Ω and Λ.

Theorem 7. *Assume that $1 < R_0 < r/\mu$. Let D_1 be an arbitrary fixed positive number. Then there exist positive constants $\underline{C} = \underline{C}(\Lambda, \Omega, D_1)$ and $\overline{C} = \overline{C}(\Lambda, \Omega, D_1)$, such that if $d_S, d_I \geq D_1$, any positive solution $(S(x), I(x))$ of model (3) satisfies*

$$\underline{C} \leq S(x), \qquad I(x) \leq \overline{C}. \tag{32}$$

Proof. By applying Lemma 5, we have $K(1 - (\mu R_0/r)) < S(x) < K$ for all $x \in \overline{\Omega}$. Let $\phi = DS + d_I I > 0$ and set $\phi(x_0) = \max_{\overline{\Omega}} \phi$. Then, by applying (i) of Lemma 5 again, we have that

$$I(x_0) \leq (R_0 - 1)S(x_0) < (R_0 - 1)K. \tag{33}$$

Thus,

$$\max_{\overline{\Omega}} I \leq d_I^{-1} \max_{\overline{\Omega}} \phi \leq K\left(R_0 - 1 + Dd_I^{-1}\right). \tag{34}$$

Let $c_1(x) = (1/d_S)(rS(1 - (S/K)) - (\beta SI/(S + I)))$. Then, there exists a positive constant $C = C(\Lambda)$ such that $\|c_1(x)\|_\infty \leq C$ provided that $d_S > D_1$. It follows from Lemma 6 that there exists a positive constant $C^* = C^*(\Lambda, D_1)$ such that $\max_{\overline{\Omega}} S \leq C^* \min_{\overline{\Omega}} S$. On the other hand, ϕ satisfies

$$\begin{aligned}
-\Delta\phi &= \frac{(\beta - \mu)SI - \mu I^2}{(S + I)(DS + d_I I)}\phi, \quad x \in \Omega, \\
\frac{\partial\phi}{\partial \mathbf{n}} &= 0, \quad x \in \partial\Omega.
\end{aligned} \tag{35}$$

Set $c(x) = ((\beta - \mu)SI - \mu I^2)/(S + I)(DS + d_I I)$. Then

$$\|c(x)\|_\infty \leq \frac{\beta - \mu}{d_I} + \frac{\mu}{d_I} = \frac{\beta}{d_I} \leq \frac{\beta}{D_1}. \tag{36}$$

It follows from Lemma 6 that there exists a positive constant $C^* = C^*(\Lambda, D_1)$ such that $\max_{\overline{\Omega}} \phi \leq C^* \min_{\overline{\Omega}} \phi$. As a consequence,

$$\frac{\max_{\overline{\Omega}} I}{\min_{\overline{\Omega}} I} = \frac{\max_{\overline{\Omega}} \phi - D\min_{\overline{\Omega}} S}{\min_{\overline{\Omega}} \phi - D\max_{\overline{\Omega}} S} \leq \frac{C^*\min_{\overline{\Omega}} \phi}{\min_{\overline{\Omega}} \varphi - D\max_{\overline{\Omega}} S} \leq C. \tag{37}$$

Now, it suffices to verify the lower bounds of $I(x)$. We will verify the conclusion by a contradiction argument.

On the contrary, suppose that the conclusion is not true; then there exist sequences $\{d_{S,i}\}_{i=1}^\infty$, $\{d_{I,i}\}_{i=1}^\infty$, and $\{D_i\}_{i=1}^\infty$ with $d_{S,i} \geq d, d_{I,i} > D_1, D_i \geq 0$, and the positive solution (S_i, I_i) of model (3) corresponding to $(d_S, d_I, D) = (d_{S,i}, d_{I,i}, D_i)$, such that

$$\min_{\overline{\Omega}} I_i(x) \longrightarrow 0 \quad \text{as } i \longrightarrow \infty. \tag{38}$$

It follows from Lemma 6 that

$$I_i(x) \longrightarrow 0 \quad \text{uniformly on } \overline{\Omega} \text{ as } i \longrightarrow \infty. \tag{39}$$

(S_i, I_i) satisfies

$$-d_{S,i}\Delta S_i = rS_i\left(1 - \frac{S_i}{K}\right) - \frac{\beta S_i I_i}{S_i + I_i}, \quad x \in \Omega,$$

$$-D_i\Delta S_i - d_{I,i}\Delta I_i = \frac{\beta S_i I_i}{S_i + I_i} - \mu I_i, \quad x \in \Omega, \tag{40}$$

$$\frac{\partial S_i}{\partial \mathbf{n}} = \frac{\partial I_i}{\partial \mathbf{n}} = 0, \quad x \in \partial\Omega.$$

Integrating by parts, we obtain that, for $i = 1, 2, \ldots,$

$$\int_\Omega S_i\left(r\left(1 - \frac{S_i}{K}\right) - \frac{\beta I_i}{S_i + I_i}\right) dx = 0,$$

$$\int_\Omega I_i\left(\frac{\beta S_i}{S_i + I_i} - \mu\right) dx = 0. \tag{41}$$

By the regularity theory for elliptic equations [53], we see that there exist a subsequence of $\{(S_i, I_i)\}_i^\infty$, which we will still denote by $\{(S_i, I_i)\}_i^\infty$ and two nonnegative functions $\widetilde{S}, \widetilde{I} \in C^2(\Omega)$, such that $(S_i, I_i) \to (\widetilde{S}, \widetilde{I})$ in $[C^2(\Omega)]^2$ as $i \to \infty$. By (39), we have that $\widetilde{I} \equiv 0$. Letting $i \to \infty$ in (41) we obtain that

$$\int_\Omega \widetilde{S}\left(r - \frac{r}{K}\widetilde{S} - \frac{\beta\widetilde{I}}{\widetilde{S} + \widetilde{I}}\right) dx = 0,$$

$$\int_\Omega \widetilde{I}\left(\frac{\beta\widetilde{S}}{\widetilde{S} + \widetilde{I}} - \mu\right) dx = 0. \tag{42}$$

Since $\widetilde{I} = 0$, the first equation of (42) becomes $\int_\Omega \widetilde{S}(r - (r/K)\widetilde{S})dx = 0$. As $K(1 - \beta) < S(x) < K$, we derive a contradiction. This completes the proof. □

5.2. Nonexistence of Positive Nonconstant Steady States. This subsection is devoted to the consideration of the nonexistence for the nonconstant positive solutions of model (3), and, in the below results, the diffusion coefficients do play a significant role.

Theorem 8. *Assume that $R_0 > 1$. Let $\varepsilon > 0$ be an arbitrary constant with $4d_S\varepsilon > D^2$ and $d_I > \varepsilon$. Then model (3) has no positive nonconstant solution provided that $(4d_S\varepsilon - D^2)\lambda_1 > 4(r\varepsilon + \beta^2)$ and $(d_I - \varepsilon)\lambda_1 > \mu(R_0 - 1) + \varepsilon$.*

Proof. Let $(S(x), I(x))$ be any positive solution of model (3) and denote $\overline{g} = |\Omega|^{-1}\int_\Omega g \, dx$. Then, multiplying the first

equation of model (3) by $(S - \overline{S})$, integrating over Ω, we have that

$$d_S\int_\Omega |\nabla S|^2 dx$$

$$= \int_\Omega (S - \overline{S}) S\left(r - \frac{rS}{K} - \frac{\beta I}{S + I}\right) dx$$

$$= \int_\Omega (S - \overline{S})^2 \left(r - \frac{r}{K}(S + \overline{S}) - \frac{\beta I\overline{I}}{(\overline{S} + \overline{I})(S + I)}\right) dx$$

$$- \int_\Omega \frac{\beta S\overline{S}}{(\overline{S} + \overline{I})(S + I)}(S - \overline{S})(I - \overline{I}) dx$$

$$\leq r\int_\Omega (S - \overline{S})^2 + \beta\int_\Omega |S - \overline{S}||I - \overline{I}| dx. \tag{43}$$

In a similar manner, we multiply the second equation in model (3) by $(I - \overline{I})$ to have

$$\int_\Omega \left(d_I|\nabla I|^2 + D\nabla S \cdot \nabla I\right) dx$$

$$= \int_\Omega (I - \overline{I})\left(\frac{\beta SI}{S + I} - \mu I\right) P \, dx$$

$$= \int_\Omega (I - \overline{I})^2\left(-\mu + \frac{\beta\overline{S}S}{(\overline{S} + \overline{I})(S + I)}\right) dx$$

$$+ \int_\Omega \frac{\beta\overline{I}I}{(\overline{S} + \overline{I})(S + I)}(S - \overline{S})(I - \overline{I}) dx \tag{44}$$

$$\leq \mu(R_0 - 1)\int_\Omega (I - \overline{I})^2 dx$$

$$+ \int_\Omega \beta|S - \overline{S}||I - \overline{I}| dx.$$

It follows from (43), (44), and the ε-Young inequality that

$$\int_\Omega \left(d_S|\nabla S|^2 + d_I|\nabla I|^2\right) dx$$

$$\leq \int_\Omega \left(\left(r + \frac{\beta^2}{\varepsilon}\right)(S - \overline{S})^2 + (\mu(R_0 - 1) + \varepsilon)(I - \overline{I})^2\right) dx$$

$$+ \int_\Omega \left(\frac{D^2}{4\varepsilon}|\nabla S|^2 + \varepsilon|\nabla I|^2\right) dx. \tag{45}$$

Thanks to the well-known Poincaré inequality

$$\lambda_1\int_\Omega (g - \overline{g})^2 dx \leq \int_\Omega |\nabla(g - \overline{g})|^2 dx, \tag{46}$$

it follows that

$$\int_\Omega \left(d_S |\nabla S|^2 + d_I |\nabla I|^2 \right) dx$$

$$\leq \int_\Omega \frac{4\left(r\varepsilon + \beta^2\right) + D^2\lambda_1}{4\varepsilon\lambda_1} |\nabla S|^2 dx \tag{47}$$

$$+ \frac{\mu\left(R_0 - 1\right) + \varepsilon\left(1 + \lambda_1\right)}{\lambda_1} \int_\Omega |\nabla I|^2 dx.$$

Since $4 d_S \varepsilon \lambda_1 > 4(r\varepsilon + \beta^2) + D^2 \lambda_1$ and $d_I \lambda_1 > \mu(R_0 - 1) + \varepsilon(1 + \lambda_1)$ from the assumption, one can conclude that $S = \overline{S}$ and $I = \overline{I}$, which asserts our results. □

5.3. Existence of Positive Nonconstant Steady States. In this section, we discuss the global existence of nonconstant positive classical solutions to model (3), which guarantees the existence of the stationary patterns [25, 27, 29, 30].

Unless otherwise specified, in this section, we always require that $1 < R_0 < 1 + (r/\mu)$, which guarantees that model (3) has one positive constant solution E^*. From now on, let us denote

$$W = DS + d_I I, \qquad W^* = DS^* + d_I I^*. \tag{48}$$

We also define

$$\mathbf{u} = (S, W), \tag{49}$$

$$\mathbf{u}^* = (S^*, W^*)$$

$$= \left(\frac{K}{r} \left(r - \mu\left(R_0 - 1\right)\right), \tag{50} \right.$$

$$\left. \frac{K}{r} \left(r - \mu\left(R_0 - 1\right)\right)\left(d_I\left(R_0 - 1\right) + D\right) \right).$$

Let $\mathbf{X} = \{ \mathbf{u} \in [C^2(\Omega)]^2 \mid \partial\mathbf{u}/\partial\mathbf{n} = 0, x \in \partial\Omega \}$ and $\mathbf{X}^+ = \{ \mathbf{u} \in \mathbf{X} \mid S, W > 0, x \in \overline{\Omega} \}$. Then we write model (3) in the form

$$-\Delta\mathbf{u} = \mathbf{G}(\mathbf{u}), \quad x \in \Omega,$$

$$\frac{\partial\mathbf{u}}{\partial\mathbf{n}} = 0, \quad x \in \partial\Omega, \tag{51}$$

where

$$\mathbf{G}(\mathbf{u}) = \begin{pmatrix} \dfrac{S}{d_S}\left(r - \dfrac{r}{K}S - \dfrac{\beta(W - DS)}{(d_I - D)S + W}\right) \\ (W - DS)\left(\dfrac{\beta S}{(d_I - D)S + W} - \dfrac{\mu}{d_I}\right) \end{pmatrix}. \tag{52}$$

Define a compact operator $\mathscr{F} : \mathbf{X}^+ \to \mathbf{X}^+$ by

$$\mathscr{F}(\mathbf{u}) := (\mathbf{I} - \Delta)^{-1} \{ \mathbf{G}(\mathbf{u}) + \mathbf{u} \}, \tag{53}$$

where $(\mathbf{I} - \Delta)^{-1}$ is the inverse operator of $\mathbf{I} - \Delta$ subject to the zero-flux boundary condition. Then $\mathbf{u}$ is a positive solution of model (51) if and only if $\mathbf{u}$ satisfies

$$(\mathbf{I} - \mathscr{F})\mathbf{u} = 0, \quad x \in \Omega. \tag{54}$$

To apply the index theory, we investigate the eigenvalue of the problem

$$-\left(\mathbf{I} - \mathscr{F}_{\mathbf{w}}\left(\mathbf{w}^*\right)\right)\Psi = \xi\Psi, \quad \Psi \neq \mathbf{0}, \tag{55}$$

where $\Psi = (\Psi_1, \Psi_2)^T$ and $\mathscr{F}_{\mathbf{w}}(\mathbf{w}^*) = (\mathbf{I} - \Delta)^{-1}(\mathbf{I} + \mathbf{A})$ with

$$\mathbf{A} = \begin{pmatrix} \dfrac{\mu d_I R_0^2 - r d_I R_0 - \mu d_I + \mu D}{d_S d_I R_0} & -\dfrac{\mu}{d_S d_I R_0} \\ \dfrac{\mu\left(d_I R_0 - d_I + D\right)\left(R_0 - 1\right)}{d_I R_0} & -\dfrac{\mu\left(R_0 - 1\right)}{d_I R_0} \end{pmatrix}. \tag{56}$$

If 0 is not an eigenvalue of (55), by Theorem 2.8.1 in [51], the index of $\mathbf{I} - \mathscr{F}$ at $\mathbf{u}^*$ is given by

$$\text{index}\left(\mathbf{I} - \mathscr{F}, \mathbf{u}^*\right) = (-1)^\gamma, \tag{57}$$

where $\gamma = \sum_{\lambda > 0} n_\lambda$ and n_λ is the algebraic multiplicity of the positive eigenvalue ξ of (55).

In fact, after calculation, (55) can be rewritten as

$$-(\xi + 1)\Delta\Psi + (\xi\mathbf{I} - \mathbf{A}) = 0, \quad x \in \Omega,$$

$$\frac{\partial\Psi}{\partial\mathbf{n}} = 0, \quad x \in \partial\Omega. \tag{58}$$

Observe that (58) has a nontrivial solution if and only if $\det(\xi\mathbf{I} + (\lambda_i + 1)^{-1}(\lambda_i\mathbf{I} - \mathbf{A})) = 0$ for some $\xi \geq 0$ and $i \geq 0$. That is to say, ξ is an eigenvalue of (55), and so (58), if and only if ξ is an eigenvalue of the matrix $(\lambda_i + 1)^{-1}(\lambda_i\mathbf{I} - \mathbf{A})$ for any $i \geq 0$. Therefore, $\mathbf{I} - \mathscr{F}_{\mathbf{u}}(\mathbf{u}^*)$ is invertible if and only if, for any $i \geq 0$ the matrix $\lambda_i\mathbf{I} - \mathbf{A}$ is invertible.

Let $m(\lambda_i)$ be the multiplicity of λ_i. For the sake of convenience, we denote

$$H(\lambda_i) = \det\left(\lambda_i\mathbf{I} - \mathbf{A}\right). \tag{59}$$

Then if $\lambda_i\mathbf{I} - \mathbf{A}$ is invertible for any $i \geq 0$, with the same arguments as in [25], we have

$$\text{index}\left(\mathscr{F}, \mathbf{u}^*\right) = (-1)^\gamma, \quad \text{where } \gamma = \sum_{i \geq 0, H(\lambda_i) < 0} m\left(\lambda_i\right). \tag{60}$$

To compute $\text{index}(\mathscr{F}, \mathbf{u}^*)$, we have to consider the sign of $H(\lambda_i)$. A straightforward computation yields

$$H(\lambda_i) = \lambda_i^2 - \text{trace}(\mathbf{A})\lambda_i + \det(\mathbf{A}), \tag{61}$$

where

$$\text{trace}(\mathbf{A})$$

$$= \frac{\mu d_I R_0^2 - \left(\mu d_S + r d_I\right)R_0 + \mu d_S - \mu d_I + \mu D}{d_S d_I R_0}, \tag{62}$$

$$\det(\mathbf{A}) = \frac{\mu\left(R_0 - 1\right)\left(r - \mu\left(R_0 - 1\right)\right)}{d_I R_0}.$$

If $\text{trace}(\mathbf{A})^2 - 4\det(\mathbf{A}) \geq 0$, then $H(\lambda) = 0$ has two positive solutions $\lambda^\pm$ given by

$$\lambda^\pm = \frac{1}{2}\left(\text{trace}(\mathbf{A}) \pm \sqrt{\text{trace}(\mathbf{A})^2 - 4\det(\mathbf{A})}\right). \tag{63}$$

Theorem 9. *Assume that* $trace(\mathbf{A})^2 - 4\det(\mathbf{A}) \geq 0$. *Then if* $\lambda^- \in (\lambda_i, \lambda_{i+1})$ *and* $\lambda^+ \in (\lambda_j, \lambda_{j+1})$ *for some* $0 \leq i < j$ *and* $\sum_{k=i+1}^{j} m(\lambda_k)$ *is odd, then model (3) has at least one nonconstant solution.*

Proof. Suppose that model (3) has no nonconstant positive solution. By Theorem 8, we can fix $\overline{D} > d_S$ such that,

    (i) model (3) with diffusion coefficients $\overline{D}, d_I$, and D has no nonconstant solutions;

    (ii) $H(\lambda_i) > 0$ for all $\lambda_i \geq 0$.

By virtue of Theorem 7, there exists a positive constant $C = C(\Lambda, \Omega)$ such that, for $\hat{d}_S \geq d_S$, any solution $(S(x), I(x))$ of model (3) with diffusion coefficients $\hat{d}_S, d_I$, and D satisfies $C^{-1} < S, I < C$.

Set

$$\mathcal{M} = \left\{ (S, W) \in C\left(\overline{\Omega}\right) \times C\left(\overline{\Omega}\right) : C^{-1} < S, W < C, x \in \overline{\Omega} \right\}, \tag{64}$$

and define

$$\Phi : \mathcal{M} \times [0,1] \longrightarrow C\left(\overline{\Omega}\right) \times C\left(\overline{\Omega}\right) \tag{65}$$

by

$$\Phi(\mathbf{u}, \theta) = (\mathbf{I} - \Delta)^{-1} \left\{ \mathbf{G}(\mathbf{u}, \theta) + \mathbf{u} \right\}, \tag{66}$$

where

$$\mathbf{G}(\mathbf{u}, \theta)$$

$$= \begin{pmatrix} \left(\theta d_S + (1-\theta)\overline{D}\right)^{-1} \left(S\left(r - \dfrac{r}{K}S - \dfrac{\beta(W - DS)}{(d_I - D)S + W}\right) \right) \\ (W - DS)\left(\dfrac{\beta S}{(d_I - D)S + W} - \dfrac{\mu}{d_I} \right) \end{pmatrix}. \tag{67}$$

It is clear that finding the positive solution of model (51) becomes equivalent to finding the fixed point of $\Phi(\mathbf{u}, 1)$ in $\mathcal{M}$. $\Phi(\mathbf{u}, \theta)$ has no fixed points in $\partial\mathcal{M}$ for all $0 \leq \theta \leq 1$.

Since $\Phi(\mathbf{u}, t)$ is compact, the Leray-Schauder topological degree $\deg(\mathbf{I} - \Phi(\mathbf{u}, \theta), \mathcal{M}, 0)$ is well defined. From the invariance of Leray-Schauder degree at the homotopy, we deduce

$$\deg(\mathbf{I} - \Phi(\mathbf{u}, 1), \mathcal{M}, 0) = \deg(\mathbf{I} - \Phi(\mathbf{u}, 0), \mathcal{M}, 0). \tag{68}$$

Clearly, $\mathbf{I} - \Phi(\mathbf{u}, 1) = \mathbf{I} - \mathcal{F}$. Thus, if model (3) has no other solutions except the constant one $\mathbf{u}^*$, we have

$$\deg(\mathbf{I} - \Phi(\mathbf{u}, 1), \mathcal{M}, 0)$$
$$= \mathrm{index}(\mathbf{I} - \mathcal{F}, \mathbf{u}^*) = (-1)^{\sum_{k=i+1}^{j} m(\lambda_k)} = -1. \tag{69}$$

On the contrary, by the choice of $\overline{D}$, we have that $\mathbf{u}^*$ is the only solution of $\Phi(\mathbf{u}, 0) = 0$ and therefore

$$\deg(\mathbf{I} - \Phi(\mathbf{u}, 0), \mathcal{M}, 0) = \mathrm{index}(\mathbf{I} - \mathcal{F}, \mathbf{u}^*) = (-1)^0 = 1. \tag{70}$$

From (68) to (70), we get a contradiction. Therefore, there exists a nonconstant solution of model (3). The proof is completed. □

6. Concluding Remarks

In this paper, we investigate the effect of cross-diffusion on the disease's dynamics through studying the existence and nonexistence positive constant steady states of a spatial *SI* epidemic model. The values of this study lie in twofolds. First, we show the local and global stability of the nonnegative steady states, which indicates that the disease reproduction number R_0 determines whether there is an endemic outbreak or not: the disease free dynamics occurs if $R_0 \leq 1$ while the unique endemic steady state is globally stable if $\mu < r$ and $1 < R_0 < (r + \mu)/2\mu$. Second, we show that even though the unique positive constant steady state (endemic state) is uniformly asymptotically stable for (1), nonconstant positive steady states can exist due to the emergence of cross-diffusion, which demonstrates that stationary patterns can be found as a result of cross-diffusion.

On the other hand, there have been studies of pattern formation in the spatial epidemic model, starting with the pioneering work of Turing [54]. Turing's revolutionary idea was that passive diffusion could interact with the chemical reaction in such a way that even if the reaction by itself has no symmetry-breaking capabilities, diffusion can destabilize the symmetry so that the system with diffusion can have them. Spatial epidemiology with self-diffusion has become a principal scientific discipline aiming at understanding the causes and consequences of spatial heterogeneity in disease transmission. And in the present paper, we prove the existence and nonexistence of the positive nonconstant steady states, which guarantees the existence of the stationary Turing patterns. The numerical results about the Turing patterns for model (1) can be found in [19].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

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