Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 850986, 7 pages
http://dx.doi.org/10.1155/2013/850986

Research Article

Modified Preconditioned GAOR Methods for Systems of Linear Equations

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Received 30 January 2013; Accepted 13 May 2013

Academic Editor: Giuseppe Marino

Three kinds of preconditioners are proposed to accelerate the generalized AOR (GAOR) method for the linear system from the generalized least squares problem. The convergence and comparison results are obtained. The comparison results show that the convergence rate of the preconditioned generalized AOR (PGAOR) methods is better than that of the original GAOR methods. Finally, some numerical results are reported to confirm the validity of the proposed methods.

1. Introduction

Consider the generalized least squares problem

$$\min_{x\in\mathbb{R}^n}(Ax-b)^T W^{-1}(Ax-b), \tag{1}$$

where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$, and the variance-covariance matrix $W \in \mathbb{R}^{n\times n}$ is a known symmetric and positive-definite matrix. This problem has many scientific applications and one of the applications is a parameter estimation in mathematical model [1, 2].

In order to solve the problem simply, one has to solve a linear system of the equivalent form as follows:

$$Fy = f, \tag{2}$$

where

$$F = \begin{pmatrix} I-B & H \\ K & I-C \end{pmatrix}, \tag{3}$$

with $B \in \mathbb{R}^{p\times p}$, $C \in \mathbb{R}^{q\times q}$, and $p+q=n$. Without loss of generality, we assume that $F = \mathcal{I} - \mathcal{L} - \mathcal{U}$, where $\mathcal{I}$ is the identity matrix, and $\mathcal{L}$ and $\mathcal{U}$ are strictly lower and upper triangular matrices obtained from F, respectively. So we can pretty easily get that

$$\mathcal{I} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \qquad \mathcal{L} = \begin{pmatrix} 0 & 0 \\ -K & 0 \end{pmatrix}, \qquad \mathcal{U} = \begin{pmatrix} B & -H \\ 0 & C \end{pmatrix}. \tag{4}$$

In order to get the approximate solutions of the linear system (2), a lot of iterative methods such as Jacobi, Gauss-Seidel (GS), successive over relaxation (SOR), and accelerated over relaxation (AOR) have been studied by many authors [3–8]. These iterative methods have very good results, but have a serious drawback because of computing the inverses of $I-B$ and $I-C$ in (3). To avoid this drawback, Darvishi and Hessari [9] proposed the generalized convergence of the generalized AOR (GAOR) method when the coefficient matrix F is a diagonally dominant matrix. The GAOR method [10, 11] can be defined as follows:

$$y_{k+1} = \mathcal{T}_{\gamma\omega} y_k + \omega g, \quad k = 0, 1, 2, \ldots, \tag{5}$$

where

$$\mathcal{T}_{\gamma\omega} = \begin{pmatrix} I & 0 \\ \gamma K & I \end{pmatrix}^{-1} \times \left((1-\omega) I + (\omega-\gamma) \begin{pmatrix} 0 & 0 \\ -K & 0 \end{pmatrix} + \omega \begin{pmatrix} B & -H \\ 0 & C \end{pmatrix} \right),$$

$$g = \begin{pmatrix} I & 0 \\ -\gamma K & I \end{pmatrix} f. \tag{6}$$

Here, ω and γ are real parameters with $\omega \neq 0$. The iteration matrix is rewritten briefly as

$$\mathcal{T}_{\gamma\omega} = \begin{pmatrix} (1-\omega) I + \omega B & -\omega H \\ \omega(\gamma - 1) K - \gamma\omega K B & (1-\omega) I + \omega C + w\gamma K H \end{pmatrix}. \tag{7}$$

To improve the convergence rate of the GAOR iterative method, a preconditioner should be applied. Now we can transform the original linear system (2) into the preconditioned linear system

$$PFy = Pf, \tag{8}$$

where P is the preconditioner. PF can be expressed as

$$PF = \begin{pmatrix} I - B^* & H^* \\ K^* & I - C^* \end{pmatrix}. \tag{9}$$

Meanwhile, the PGAOR method for solving the preconditioned linear system (8) is defined by

$$y_{k+1} = \mathcal{T}^*_{\gamma\omega} y_k + wg^*, \quad k = 0, 1, 2, \ldots, \tag{10}$$

where

$$\mathcal{T}^*_{\gamma\omega} = \begin{pmatrix} (1-\omega) I + \omega B^* & -\omega H^* \\ \omega(\gamma - 1) K^* - \gamma\omega K^* B^* & (1-\omega) I + \omega C^* + \gamma\omega K^* H^* \end{pmatrix},$$
$$g = \begin{pmatrix} I & 0 \\ -\gamma K^* & I \end{pmatrix} Pf. \tag{11}$$

In this paper, we propose three new types of preconditioners and study the convergence rate of the preconditioned GAOR methods for solving the linear system (2). This paper is organized as follows. In Section 2, some notations, definitions, and preliminary results are presented. In Section 3, three new types of preconditioners are proposed and compared with that of the original GAOR methods. Lastly, a numerical example is provided to confirm the theoretical results studied in Section 4.

2. Preliminaries

For vector $x \in \mathbb{R}^n$, $x \geq 0$ ($x > 0$) denotes that all components of x are nonnegative (positive). For two vectors $x, y \in \mathbb{R}^n$, $x \geq y$ ($x > y$) means that $x - y \geq 0$ ($x - y > 0$). These definitions are carried immediately over to matrices. A matrix A is said to be irreducible if the directed graph of A is strongly connected. $\rho(A)$ denotes the spectral radius of A. Some useful results are provided as follows.

Lemma 1 (see [7]). *Let $A \geq 0$ be an irreducible matrix. Then,*

(a) *A has a positive eigenvalue equal to its spectral radius.*

(b) *A has an eigenvector $x > 0$ corresponding to $\rho(A)$.*

(c) *$\rho(A)$ is a simple eigenvalue of A.*

Lemma 2 (see [12]). *Let A be a nonnegative matrix. Then,*

(i) *if $\alpha x \leq Ax$ for some nonnegative vector x, $x \neq 0$, then $\alpha \leq \rho(A)$;*

(ii) *if $Ax \leq \beta x$ for some positive vector x, then $\rho(A) \leq \beta$. Moreover, if A is irreducible and if $0 \neq \alpha x \leq Ax \leq \beta x$ for some nonnegative vector x, then $\alpha \leq \rho(A) \leq \beta$ and x is a positive vector.*

3. Preconditioned GAOR Methods

To solve the linear system (2) with the coefficient matrix F in (3), we consider the preconditioners as follows:

$$P_i = I + \overline{S} = \begin{pmatrix} I & 0 \\ S_i & I \end{pmatrix}, \quad i = 1, 2, 3, \tag{12}$$

where

$$S_1 = \begin{pmatrix} 0 & 0 & \cdots & -k_{1p} \\ 0 & 0 & \cdots & -k_{2p} \\ 0 & \ddots & \vdots & \\ 0 & 0 & \cdots & -k_{qp} \end{pmatrix},$$

$$S_2 = \begin{pmatrix} 0 & -k_{12} & \cdots & 0 & 0 \\ -k_{21} & 0 & -k_{23} & \cdots & 0 \\ 0 & -k_{32} & \ddots & \vdots & \vdots \\ \vdots & \vdots & \cdots & 0 & -k_{q-1,p} \\ 0 & 0 & \cdots & -k_{q,p-1} & 0 \end{pmatrix}, \tag{13}$$

$$S_3 = \begin{pmatrix} 0 & -k_{12} & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & \vdots & 0 & -k_{q-1,p} \\ -\dfrac{k_{q1}}{\alpha} & \cdots & 0 & 0 \end{pmatrix}, \quad (\alpha > 0).$$

The preconditioned coefficient matrix P_iF can be expressed as

$$P_iF = \begin{pmatrix} I - B & H \\ K + S_i(I - B) & S_iH + I - C \end{pmatrix}, \quad i = 1, 2, 3, \tag{14}$$

where

$$K+S_1(I-B)=\begin{pmatrix} k_{11} & k_{12} & \cdots & k_{1p}b_{pp} \\ k_{21} & k_{22} & \cdots & k_{2p}b_{pp} \\ k_{q1} & k_{q2} & \cdots & k_{qp}b_{pp} \end{pmatrix},$$

$$K+S_2(I-B)=\begin{pmatrix} k_{11} & k_{12}b_{22} & \cdots & k_{1,p-1} & k_{1p} \\ k_{21}b_{11}+k_{23}b_{31} & k_{22} & k_{21}b_{13}+k_{23}b_{33} & \cdots & k_{2p} \\ k_{31} & k_{32}b_{22}+k_{34}b_{42} & \ddots & \vdots & \vdots \\ \vdots & \vdots & \cdots & 0 & k_{q-1,p-2}b_{q-2,p}+k_{q-1,p}b_{qp} \\ k_{q1} & k_{q2} & \cdots & k_{q,p-1}b_{q-1,p-1} & k_{qp} \end{pmatrix}, \quad (15)$$

$$K+S_3(I-B)=\begin{pmatrix} k_{11} & k_{12}b_{22} & \cdots & k_{1p} \\ k_{21} & k_{22} & \ddots & k_{2p} \\ \vdots & \vdots & \cdots & k_{q-1,p}b_{pp} \\ k_{q1}\left(1-\dfrac{1-b_{11}}{\alpha}\right) & k_{q2} & \cdots & k_{qp} \end{pmatrix}.$$

Based on the discussed above, P_iF can be spitted as

$$P_iF=\mathscr{I}-\mathscr{L}_i-\mathscr{U}_i, \quad i=1,2,3. \quad (16)$$

Similarly,

$$\mathscr{I}=\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \qquad \mathscr{L}_i=\begin{pmatrix} 0 & 0 \\ -K-S_i(I-B) & 0 \end{pmatrix},$$
$$\mathscr{U}_i=\begin{pmatrix} B & -H \\ 0 & C-S_iH \end{pmatrix}, \quad i=1,2,3. \quad (17)$$

The preconditioned GAOR methods for solving $P_iFy=P_if$ are defined by

$$y_{k+1}=\mathscr{T}^*_{\gamma\omega i}y_k+\omega g_i^*, \quad k=0,1,2,\ldots, \; i=1,2,3, \quad (18)$$

where

$$\omega g_i^*=(\mathscr{I}-\Gamma\mathscr{L}_i)^{-1}\Omega P_i f, \quad i=1,2,3, \quad (19)$$

with

$$\mathscr{T}^*_{\gamma\omega i}=(I-\Gamma\mathscr{L}_i)^{-1}(I-\Omega+(\Omega-\Gamma)\mathscr{L}_i+\Omega\mathscr{U}_i), \quad i=1,2,3, \quad (20)$$

where

$$\Omega=\begin{pmatrix} \omega_1 I & 0 \\ 0 & \omega_2 I, \end{pmatrix}, \qquad \Gamma=\begin{pmatrix} \gamma_1 I & 0 \\ 0 & \gamma_2 I \end{pmatrix}. \quad (21)$$

For $i=1,2,3$, we have

$$\mathscr{T}^*_{\gamma\omega i}=\begin{pmatrix} (1-\omega_1)I+\omega_1 B & -\omega_1 H \\ (\omega_1\gamma_2-\omega_2)[K+S_i(I-B)] & (1-\gamma_2)I+\gamma_2 C-\gamma_2 S_i H \\ -\omega_1\gamma_2[K+S_i(I-B)]B & +\omega_1\gamma_2[K+S_i(I-B)]H \end{pmatrix}. \quad (22)$$

Next, we will study the convergence condition of the PGAOR methods. For simplicity, without loss of generality, we can assume that

$$H\le 0, \quad K\le 0, \quad B\ge 0, \quad C\ge 0,$$
$$0<\omega_1\le 1, \quad 0<\omega_2\le 1, \quad 0<\gamma_2\le\frac{\omega_2}{\omega_1}. \quad (23)$$

Then, we have the following theorem.

Theorem 3. *Let $\mathscr{T}_{\gamma\omega}$ and $\mathscr{T}^*_{\gamma\omega 1}$ be the iteration matrices of the GAOR method and the PGAOR method corresponding to problem (2), which are defined by (7) and (22), respectively. If matrix F in (3) is an irreducible matrix then it holds that $\rho(\mathscr{T}_{\gamma\omega})\ne 1$,*

$$\rho(\mathscr{T}^*_{\gamma\omega 1})>\rho(\mathscr{T}_{\gamma\omega})\ne 1 \quad \text{if } \rho(\mathscr{T}_{\gamma\omega})>1,$$
$$\rho(\mathscr{T}^*_{\gamma\omega 1})<\rho(\mathscr{T}_{\gamma\omega})\ne 1 \quad \text{if } \rho(\mathscr{T}_{\gamma\omega})<1. \quad (24)$$

Proof. By some simple calculations on (7), one can get

$$\mathscr{T}_{\gamma\omega}=\begin{pmatrix} (1-\omega_1 I)+\omega_1 B & -\omega_1 H \\ (\omega_1\gamma_2-\omega_2)K & (1-\omega_2 I+\omega_2)C \end{pmatrix} + \omega_1\gamma_2\begin{pmatrix} 0 & 0 \\ -KB & KH \end{pmatrix}. \quad (25)$$

Since here F is irreducible, one can pretty easily obtain that $\mathscr{T}_{\gamma\omega}$ is nonnegative and irreducible by the above assumptions. And so on, one can also easily prove that $\mathscr{T}^*_{\gamma\omega 1}$ is nonnegative and irreducible. By Lemma 1, there exists a positive vector $x>0$ such that

$$\mathscr{T}_{\gamma\omega}x=\lambda x, \quad (26)$$

where $\lambda=\rho(\mathscr{T}_{\gamma\omega})$.

One can easily have

$$[\mathcal{I} - \Omega + (\Omega - \Gamma)\mathcal{L} + \Omega\mathcal{U}]\,x = \lambda(\mathcal{I} - \Gamma\mathcal{L})\,x. \tag{27}$$

That is,

$$(\mathcal{I} - \Omega)\,x = \lambda(\mathcal{I} - \Gamma\mathcal{L}x - (\Omega - \Gamma)\mathcal{L}x - \Omega\mathcal{U}x). \tag{28}$$

With the same vector $x > 0$, it holds

$$\mathcal{T}^*_{\gamma\omega 1}x - \lambda x = (\mathcal{I} - \Gamma\mathcal{L}_1)^{-1} \times [\mathcal{I} - \Omega + (\Omega - \Gamma)\mathcal{L}_1 - \lambda(\mathcal{I} - \Gamma\mathcal{L}_1)]\,x. \tag{29}$$

Using (22), (26), and (28), we can obtain

$$\begin{aligned}
\mathcal{T}^*_{\gamma\omega 1}x - \lambda x &= (\mathcal{I} - \Gamma\mathcal{L}_1)^{-1} \times [(\Omega - \Gamma)(\mathcal{L}_1 - \mathcal{L}) + \Omega(\mathcal{U}_1 - \mathcal{U}) + \lambda\Gamma(\mathcal{L}_1 - \mathcal{L})]\,x \\
&= (\mathcal{I} - \Gamma\mathcal{L}_1)^{-1}[\Omega(\mathcal{U}_1 - \mathcal{U} + \mathcal{L}_1 - \mathcal{L}) + (\lambda - 1)\Gamma(\mathcal{L}_1 - \mathcal{L})]\,x \\
&= (\mathcal{I} - \Gamma\mathcal{L}_1)^{-1}\Omega(\mathcal{U}_1 - \mathcal{U} + \mathcal{L}_1 - \mathcal{L})\,x + (\lambda - 1)\left((\mathcal{I} - \Gamma\mathcal{L}_1)^{-1}\right)\Gamma(\mathcal{L}_1 - \mathcal{L})\,x \\
&= (\mathcal{I} - \Gamma\mathcal{L}_1)^{-1}\begin{pmatrix} 0 & 0 \\ -\omega_2 S_1(I - B) & -\omega_2 S_1 H \end{pmatrix}x + (\lambda - 1)\left((\mathcal{I} - \Gamma\mathcal{L}_1)^{-1}\right) \times \begin{pmatrix} 0 & 0 \\ -\gamma_2 S_1(I - B) & 0 \end{pmatrix}x.
\end{aligned} \tag{30}$$

Meanwhile, we have

$$\begin{aligned}
&(\mathcal{I} - \Gamma\mathcal{L}_1)^{-1}\begin{pmatrix} 0 & 0 \\ -\omega_2 S_1(I - B) & -\omega_2 S_1 H \end{pmatrix}x \\
&= (\mathcal{I} - \Gamma\mathcal{L}_1)^{-1}\begin{pmatrix} 0 & 0 \\ \frac{\omega_2}{\omega_1}S_1 & 0 \end{pmatrix}\begin{pmatrix} -\omega_1 I + \omega_1 B & -\omega_1 H \\ 0 & 0 \end{pmatrix}x \\
&= (\mathcal{I} - \Gamma\mathcal{L}_1)^{-1}\begin{pmatrix} 0 & 0 \\ \frac{\omega_2}{\omega_1}S_1 & 0 \end{pmatrix} \times \begin{pmatrix} -\omega_1 I + \omega_1 B & -\omega_1 H \\ (\omega_1\gamma_2 K - \omega_1\gamma_2 KB) & -\omega_2 I + \omega_2 C + \omega_1\gamma_2 KH \end{pmatrix}x \\
&= (\mathcal{I} - \Gamma\mathcal{L}_1)^{-1}\begin{pmatrix} 0 & 0 \\ \frac{\omega_2}{\omega_1}S_1 & 0 \end{pmatrix}(\mathcal{T}_{wr} - \mathcal{I})\,x \\
&= (\lambda - 1)(\mathcal{I} - \Gamma\mathcal{L}_1)^{-1}\begin{pmatrix} 0 & 0 \\ \frac{\omega_2}{\omega_1}S_1 & 0 \end{pmatrix}x.
\end{aligned} \tag{31}$$

By far, we can easily get

$$\begin{aligned}
\mathcal{T}^*_{\gamma\omega 1}x - \lambda x &= (\lambda - 1)(\mathcal{I} - \Gamma\mathcal{L}_1)^{-1} \times \left[\begin{pmatrix} 0 & 0 \\ \frac{\omega_2}{\omega_1}S_1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -\gamma_2 S_1(I - B) & 0 \end{pmatrix}\right]x \\
&= (\lambda - 1)\begin{pmatrix} I & 0 \\ -\gamma_2[K + S_1(I - B)] & I \end{pmatrix} \times \begin{pmatrix} 0 & 0 \\ \left(\frac{\omega_2}{\omega_1} - \gamma_2\right)S_1 + \gamma_2 S_1 B & 0 \end{pmatrix}x \\
&= (\lambda - 1)\begin{pmatrix} 0 & 0 \\ \left(\frac{\omega_2}{\omega_1} - \gamma_2\right)S_1 + \gamma_2 S_1 B & 0 \end{pmatrix}x.
\end{aligned} \tag{32}$$

In view of the abovementioned assumptions, we have that

$$\begin{pmatrix} 0 & 0 \\ \left(\frac{\omega_2}{\omega_1} - \gamma_2\right)S_1 + \gamma_2 S_1 B & 0 \end{pmatrix}x > 0. \tag{33}$$

Then, if $\lambda = \rho(\mathcal{T}_{wr}) > 1$, then

$$\mathcal{T}^*_{\gamma\omega 1} - \lambda x > 0, \qquad \mathcal{T}^*_{\gamma\omega 1} - \lambda x \neq 0. \tag{34}$$

From Lemma 2, we can easily get

$$\rho(\mathcal{T}^*_{\gamma\omega 1}) > \rho(\mathcal{T}_{wr}) > 1. \tag{35}$$

Similarly, if $\lambda = \rho(\mathcal{T}_{wr}) < 1$, then

$$\mathcal{T}^*_{\gamma\omega 1} - \lambda x < 0, \qquad \mathcal{T}^*_{\gamma\omega 1} - \lambda x \neq 0. \tag{36}$$

So we have

$$\rho(\mathcal{T}^*_{\gamma\omega 1}) < \rho(\mathcal{T}_{\gamma\omega}) < 1. \tag{37}$$

If $\lambda = \rho(\mathcal{T}_{\gamma\omega}) = 1$, then we may get that $Fy = f$ but $y \neq 0$, which is contradictory to the fact of nonsingular matrix F by assumptions; this completes the conclusion of the theorem. □

Theorem 4. *Let $\mathcal{T}_{\gamma\omega}$ and $\mathcal{T}^*_{\gamma\omega 2}$ be the iteration matrices of the GAOR method and the PGAOR method corresponding to problem* (2), *which are defined by* (7) *and* (22), *respectively. If the matrix F in* (3) *is an irreducible matrix satisfying*

$$b_{11} > 0, \qquad k_{ij} \neq 0 \quad \text{for } |i - j| = 1, \tag{38}$$

then it holds that $\rho(T_{wr}) \neq 0$,

$$\begin{aligned}
&\rho(\mathcal{T}^*_{\gamma\omega 2}) > \rho(\mathcal{T}_{\gamma\omega}) \neq 1 \quad \text{if } \rho(\mathcal{T}_{\gamma\omega}) > 1, \\
&\rho(\mathcal{T}^*_{\gamma\omega 2}) < \rho(\mathcal{T}_{\gamma\omega}) \neq 1 \quad \text{if } \rho(\mathcal{T}_{\gamma\omega}) < 1.
\end{aligned} \tag{39}$$

Proof. One can easily prove this theorem by using similar arguments of Theorem 3. □

Similarly, we have the following theorem.

Theorem 5. *Let $\mathcal{T}_{\gamma\omega}$ and $\mathcal{T}^*_{\gamma\omega 3}$ be the iteration matrices of the GAOR method and the PGAOR method corresponding to problem* (2)*, which are defined by* (7) *and* (22)*, respectively. If the matrix F in* (3) *is an irreducible matrix satisfying*

$$\alpha > 1, \qquad k_{q1} < 0, \qquad b_{11} > 0,$$
$$k_{i,i+1} < 0 \quad \text{for } i = 1, 2, \ldots, q-1, \tag{40}$$

then it holds that $\rho(\mathcal{T}_{\gamma\omega}) \neq 1$,

$$\rho\left(\mathcal{T}^*_{\gamma\omega 3}\right) > \rho\left(\mathcal{T}_{\gamma\omega}\right) \neq 1 \quad \text{if } \rho\left(\mathcal{T}_{\gamma\omega}\right) > 1,$$
$$\rho\left(\mathcal{T}^*_{\gamma\omega 3}\right) < \rho\left(\mathcal{T}_{\gamma\omega}\right) \neq 1 \quad \text{if } \rho\left(\mathcal{T}_{\gamma\omega}\right) < 1. \tag{41}$$

4. Numerical Examples

In this section, we give numerical examples to demonstrate the conclusions drawn above. The numerical experiments were done by using MATLAB 7.0.

Example 1. Consider the following Laplace equation:

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0. \tag{42}$$

Under a uniform square domain, applying the five-point finite difference method with the uniform mesh size, we can get the following linear system:

$$\mathcal{F} x = f, \tag{43}$$

where

$$\mathcal{F} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{8} & 0 & -\frac{1}{8} & -\frac{1}{8} \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{8} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{8} & -\frac{1}{8} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{8} & -\frac{1}{8} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{8} & -\frac{1}{8} \\ 0 & 0 & -\frac{1}{8} & -\frac{1}{8} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{8} & 0 & -\frac{1}{8} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ -\frac{1}{8} & 0 & -\frac{1}{8} & 0 & -\frac{1}{8} & -\frac{1}{8} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ -\frac{1}{8} & 0 & 0 & 0 & 0 & -\frac{1}{8} & 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}. \tag{44}$$

The coefficient matrix $\mathcal{F}$ is spitted as

$$\mathcal{F} = \begin{pmatrix} I - B & H \\ K & I - C \end{pmatrix}, \tag{45}$$

where

$$B = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix},$$

$$C = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix},$$

$$H = \begin{pmatrix} 0 & -\frac{1}{8} & 0 & -\frac{1}{8} & -\frac{1}{8} \\ 0 & -\frac{1}{8} & 0 & 0 & 0 \\ -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & 0 \\ -\frac{1}{8} & -\frac{1}{8} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{8} & -\frac{1}{8} & 0 \\ 0 & 0 & 0 & -\frac{1}{8} & -\frac{1}{8} \end{pmatrix},$$

$$K = \begin{pmatrix} 0 & 0 & -\frac{1}{8} & -\frac{1}{8} & 0 & 0 \\ -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & 0 & 0 \\ 0 & 0 & -\frac{1}{8} & 0 & -\frac{1}{8} & 0 \\ -\frac{1}{8} & 0 & -\frac{1}{8} & 0 & -\frac{1}{8} & -\frac{1}{8} \\ -\frac{1}{8} & 0 & 0 & 0 & 0 & -\frac{1}{8} \end{pmatrix}. \tag{46}$$

Table 1 reveals the spectral radii of the GAOR methods and the PGAOR methods. It tells that the spectral radii of the preconditioned PGAOR methods are smaller than those of the GAOR methods, so we can get that the proposed three

Table 1: Spectral radii of GAOR method and PGAOR method.

Preconditioner	α	ω_1	ω_2	γ_2	ρ_{GAOR}	ρ_{PGAOR}
S_1		0.8912	0.9654	0.8865	0.8478	0.8457
S_2		0.8912	0.9654	0.8865	0.8478	0.8376
S_3	1.5	0.8912	0.9654	0.8865	0.8478	0.8418
	2	0.8912	0.9654	0.8865	0.8478	0.8420

Table 2: Spectral radii of GAOR method and PGAOR method.

Preconditioner	α	ω_1	$\omega_2 = \omega$	$\gamma_2 = \gamma$	ρ_{GAOR}	ρ_{PGAOR}
S_1		0.8	0.6	0.7	0.7505	0.6618
S_2		0.8	0.6	0.7	0.7505	0.7340
S_3	0.1	0.8	0.6	0.7	0.7505	0.7298
	0.5	0.8	0.6	0.7	0.7505	0.7328

preconditioners can accelerate the speed rate of the GAOR method for the linear systems (2). The results in Table 1 are in accordance with Theorems 3–5.

Example 2. The coefficient matrix F in (3) is given by

$$F = \begin{pmatrix} I - B & H \\ K & I - C \end{pmatrix}, \quad (47)$$

where $B = (b_{ij})_{p\times p}$, $C = (c_{ij})_{q\times q}$, and $p + q = n$, $H = (h_{ij})_{p\times q}$, $K = (k_{ij})_{q\times p}$ with

$$\begin{gathered}
b_{ii} = \frac{1}{i+1}, \quad i = 1, \dots p, \\
b_{ij} = \frac{1}{30} - \frac{1}{(30j) + i}, \quad j > i,\ i = 1, \dots p-1,\ j = 2, \dots p, \\
b_{ij} = \frac{1}{30} - \frac{1}{30\left(\left(10\left(i-j+1\right)\right) + i\right)}, \\
i > j,\ i = 2, \dots p,\ j = 1, \dots p-1, \\
c_{ii} = \frac{1}{n+i+1}, \quad i = 1, \dots q, \\
c_{ij} = \frac{1}{30} - \frac{1}{30\left(n+j\right) + n + i}, \\
j > i,\ i = 1, \dots, q-1,\ j = 2, \dots, q, \\
c_{ij} = \frac{1}{30} - \frac{1}{30\left(i-j+1\right) + n + i}, \\
i > j,\ i = 2, \dots, q,\ j = 1, \dots q-1, \\
k_{ij} = \frac{1}{30\left(n+i-j+1\right) + n + i} - \frac{1}{30}, \\
i = 1, \dots, q,\ j = 1, \dots, p, \\
h_{ij} = \frac{1}{30\left(n+j\right) + i} - \frac{1}{30}, \quad i = 1, \dots, p,\ j = 1, \dots, q.
\end{gathered} \quad (48)$$

Obviously, F is irreducible. Table 2 shows the spectral radii of the corresponding iteration matrices with $n = 8$ and $m = 6$.

Similarly, in Table 2, we get that the results are in concord with Theorems 3–5.

Acknowledgments

The authors would like to thank the anonymous referees for their helpful suggestions, which greatly improved the paper. This research of this author was supported by NSFC Tianyuan Mathematics Youth Fund (no. 11026040), by Science and Technology Development Plan of Henan Province (no. 122300410316), and by Natural Science Foundations of Henan Province (no. 13A110022).

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