Research Article

On Inequalities of Lyapunov for Two-Dimensional Nonlinear Dynamic Systems on Time Scales

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We establish some new Lyapunov-type inequalities for two-dimensional nonlinear dynamic systems on time scales. As for application, boundedness of the Emden-Fowler-type equation is proved.

1. Introduction

In this paper, we establish some Lyapunov-type inequalities for the following two-dimensional nonlinear dynamic system:

$$
\begin{aligned}
x^{\Delta_1 \Delta_2}(s,t) &= \alpha_1(s,t)\, x(\sigma(s),\sigma(t)) \\
&\quad + \beta_1(s,t)\, |u(s,t)|^{\gamma-2} u(s,t), \\
u^{\Delta_1 \Delta_2}(s,t) &= -\beta_2(s,t)\, |x(\sigma(s),\sigma(t))|^{\beta-2} x(\sigma(s),\sigma(t)) \\
&\quad - \alpha_1(s,t)\, u(s,t),
\end{aligned}
\tag{1}
$$

where $s,t \in \mathbb{T} =$ a time scale, Δ_1 denotes the delta derivative with respect to s, and Δ_2 denotes the delta derivative with respect to t.

Lyapunov-type inequalities have proven to be very useful in the study of qualitative behavior of solutions such as oscillation, disconjugacy, and eigenvalue problems for differential equations and difference equations. Since the appearance of Lyapunov's fundamental paper [1], considerable attention has been given to various extensions and improvements of the Lyapunov-type inequality from different viewpoints [2–8]. Although Lyapunov-type inequalities are well developed for the continuous cases, their time scale versions are still in early stages and are worth due attention.

Recently, He et al. in [2] considered the linear Hamiltonian system

$$
\begin{aligned}
x^{\Delta}(t) &= \alpha(t)\, x(\sigma(t)) + \beta(t)\, y(t), \\
y^{\Delta}(t) &= -\gamma(t)\, x(\sigma(t)) - \alpha(t)\, y(t)
\end{aligned}
\tag{2}
$$

and obtained several useful Lyapunov-type inequalities.

Chen et al. in [3] considered the nonlinear system

$$
\begin{aligned}
\frac{\partial^2 x(s,t)}{\partial s\, \partial t} &= \alpha_1(s,t)\, x(s,t) + \beta_1(s,t)\, |u(s,t)|^{\gamma-2} u(s,t), \\
\frac{\partial^2 u(s,t)}{\partial s\, \partial t} &= -\beta_2(s,t)\, |x(s,t)|^{\beta-2} x(s,t) - \alpha_1(s,t)\, u(s,t)
\end{aligned}
\tag{3}
$$

and obtained some interesting Lyapunov-type inequalities for partial differential equations.

In this paper, under the assumption of existence of a nontrivial solution $(x(s,t), u(s,t))$ to the 2-dimensional nonlinear dynamic system (1), some new and interesting Lyapunov-type inequalities are established.

2. Main Results

Throughout this paper, the following mild and natural conditions are assumed:

(i) $\gamma > 1$, $\beta > 1$ are real constants,

(ii) $\alpha_1(s,t)$, $\beta_1(s,t)$, $\beta_2(s,t)$: $[s_0,\infty)_\mathbb{T} \times [t_0,\infty)_\mathbb{T} \to R$ are rd-continuous functions such that $\beta_1(s,t) > 0$ for $(s,t) \in [s_0,\infty)_\mathbb{T} \times [t_0,\infty)_\mathbb{T}$, where $[s_0,\infty)_\mathbb{T} := [s_0,\infty) \cap \mathbb{T}$, and σ is the forward jump operator; that is, $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$.

Theorem 1. *If the nonlinear dynamic system (1) has a real solution $(x(s,t), u(s,t))$ which is not identically zero on $[a,b]_\mathbb{T} \times [c,d]_\mathbb{T}$ satisfying $x(a,t) = x(b,t) = x(s,c) = x(s,d) = 0$ and $x^{\Delta_1}(s,\sigma(t))u^{\Delta_2}(s,t) + x^{\Delta_2}(\sigma(s),t)u^{\Delta_1}(s,t) = 0$ for all $(s,t) \in [a,b]_\mathbb{T} \times [c,d]_\mathbb{T}$, where $a,b,c,d \in \mathbb{T}$ with $a < b$, $c < d$, then*

$$2 \leq \int_a^b \int_c^d |\alpha_1(s,t)| \, \Delta t \, \Delta s$$

$$+ M^{\beta/\alpha - 1} \left(\int_a^b \int_c^d \beta_1(s,t) \, \Delta t \, \Delta s \right)^{1/\gamma} \quad (4)$$

$$\cdot \left(\int_a^b \int_c^d \beta_2^+(s,t) \, \Delta t \, \Delta s \right)^{1/\alpha},$$

where $1/\alpha + 1/\gamma = 1$, $M := \max\{|x(s,t)| : \sigma(a) \leq s < b, \sigma(c) \leq t < d\}$, and $\beta_2^+(s,t) := \max\{\beta_2(s,t), 0\}$.

Proof. From the conditions $x(a,t) = x(b,t) = x(s,c) = x(s,d) = 0$ and $x(s,t)$ is not identically zero on $[a,b]_\mathbb{T} \times [c,d]_\mathbb{T}$, there exists (τ_1, τ_2) such that $|x(\tau_1,\tau_2)| = M > 0$.

Multiplying the first equation of (1) by $u(s,t)$ and the second one by $x(\sigma(s),\sigma(t))$ and adding up, we get

$$x^{\Delta_1\Delta_2}(s,t)u(s,t) + u^{\Delta_1\Delta_2}(s,t)x(\sigma(s),\sigma(t))$$

$$= \beta_1(s,t)|u(s,t)|^\gamma - \beta_2(s,t)|x(\sigma(s),\sigma(t))|^\beta, \quad (5)$$

and, hence,

$$(x(s,t)u(s,t))^{\Delta_1\Delta_2} = x^{\Delta_1\Delta_2}(s,t)u(s,t)$$

$$+ x^{\Delta_2}(\sigma(s),t)u^{\Delta_1}(s,t)$$

$$+ x^{\Delta_1}(s,\sigma(t))u^{\Delta_2}(s,t)$$

$$+ x(\sigma(s),\sigma(t))u^{\Delta_1\Delta_2}(s,t)$$

$$= x^{\Delta_1\Delta_2}(s,t)u(s,t) \quad (6)$$

$$+ u^{\Delta_1\Delta_2}(s,t)x(\sigma(s),\sigma(t))$$

$$= \beta_1(s,t)|u(s,t)|^\gamma$$

$$- \beta_2(s,t)|x(\sigma(s),\sigma(t))|^\beta.$$

Integrating the left hand side of (6) over t from c to d and then over s from a to b, we get

$$\int_a^b \int_c^d (x(s,t)u(s,t))^{\Delta_1\Delta_2} \Delta t \, \Delta s$$

$$= \int_a^b (x(s,d)u(s,d))^{\Delta_1} \Delta s - \int_a^b (x(s,c)u(s,c))^{\Delta_1} \Delta s \quad (7)$$

$$= x(b,d)u(b,d) - x(a,d)u(a,d)$$

$$- x(b,c)u(b,c) + x(a,c)u(a,c).$$

Noting that $x(a,t) = x(b,t) = 0$, we have

$$\int_a^b \int_c^d \beta_1(s,t)|u(s,t)|^\gamma \Delta t \, \Delta s$$

$$= \int_a^b \int_c^d \beta_2(s,t)|x(\sigma(s),\sigma(t))|^\beta \Delta t \, \Delta s. \quad (8)$$

On the other hand, integrating the first equation of (1) over t from c to τ_2 and then over s from a to τ_1, we get

$$\int_a^{\tau_1} \int_c^{\tau_2} x^{\Delta_1\Delta_2}(s,t) \, \Delta t \, \Delta s$$

$$= \int_a^{\tau_1} \int_c^{\tau_2} \left(\alpha_1(s,t)x(\sigma(s),\sigma(t)) \right. \quad (9)$$

$$\left. + \beta_1(s,t)|u(s,t)|^{\gamma-2}u(s,t) \right) \Delta t \, \Delta s.$$

By the boundary conditions on x, it is elementary to verify that

$$\int_a^{\tau_1} \int_c^{\tau_2} (x(s,t))^{\Delta_1\Delta_2} \Delta t \, \Delta s = x(\tau_1,\tau_2), \quad (10)$$

and so

$$x(\tau_1,\tau_2) = \int_a^{\tau_1} \int_c^{\tau_2} \left(\alpha_1(s,t)x(\sigma(s),\sigma(t)) \right.$$

$$\left. + \beta_1(s,t)|u(s,t)|^{\gamma-2}u(s,t) \right) \Delta t \, \Delta s. \quad (11)$$

Hence,

$$|x(\tau_1,\tau_2)| \leq \int_a^{\tau_1} \int_c^{\tau_2} |\alpha_1(s,t)||x(\sigma(s),\sigma(t))| \Delta t \, \Delta s$$

$$+ \int_a^{\tau_1} \int_c^{\tau_2} \beta_1(s,t)|u(s,t)|^{\gamma-1} \Delta t \, \Delta s. \quad (12)$$

By similar arguments, we easily get

$$|x(\tau_1,\tau_2)| \leq \int_{\tau_1}^b \int_{\tau_2}^d |\alpha_1(s,t)||x(\sigma(s),\sigma(t))| \Delta t \, \Delta s$$

$$+ \int_{\tau_1}^b \int_{\tau_2}^d \beta_1(s,t)|u(s,t)|^{\gamma-1} \Delta t \, \Delta s. \quad (13)$$

Summing (12) and (13) and by Hölder's inequality with indices γ and α, we obtain

$$
2\left|x\left(\tau_1, \tau_2\right)\right| \le \int_a^b \int_c^d \left|\alpha_1\left(s, t\right)\right|
$$

$$
\times \left|x\left(\sigma\left(s\right), \sigma\left(t\right)\right)\right| \Delta t\, \Delta s
$$

$$
+ \int_a^b \int_c^d \beta_1\left(s, t\right) \left|u\left(s, t\right)\right|^{\gamma-1} \Delta t\, \Delta s
$$

$$
\le \int_a^b \int_c^d \left|\alpha_1\left(s, t\right)\right| \tag{14}
$$

$$
\times \left|x\left(\sigma\left(s\right), \sigma\left(t\right)\right)\right| \Delta t\, \Delta s
$$

$$
+ \left(\int_a^b \int_c^d \beta_1\left(s, t\right) \Delta t\, \Delta s\right)^{1/\gamma}
$$

$$
\times \left(\int_a^b \int_c^d \beta_1\left(s, t\right) \left|u\left(s, t\right)\right|^{\gamma} \Delta t\, \Delta s\right)^{1/\alpha}.
$$

In view of (8), we have

$$
2\left|x\left(\tau_1, \tau_2\right)\right| \le \int_a^b \int_c^d \left|\alpha_1\left(s, t\right)\right| \left|x\left(\sigma\left(s\right), \sigma\left(t\right)\right)\right| \Delta t\, \Delta s
$$

$$
+ \left(\int_a^b \int_c^d \beta_1\left(s, t\right) \Delta t\, \Delta s\right)^{1/\gamma}
$$

$$
\times \left(\int_a^b \int_c^d \beta_2\left(s, t\right) \left|x\left(\sigma\left(s\right), \sigma\left(t\right)\right)\right|^{\beta} \Delta t\, \Delta s\right)^{1/\alpha}, \tag{15}
$$

and so

$$
2 \le \int_a^b \int_c^d \left|\alpha_1\left(s, t\right)\right| \Delta t\, \Delta s
$$

$$
+ M^{\beta/\alpha-1}\left(\int_a^b \int_c^d \beta_1\left(s, t\right) \Delta t\, \Delta s\right)^{1/\gamma} \tag{16}
$$

$$
\cdot \left(\int_a^b \int_c^d \beta_2^+\left(s, t\right) \Delta t\, \Delta s\right)^{1/\alpha}.
$$

The proof is complete. □

Remark 2. It is interesting to note that when $\mathbb{T} = \mathbb{R}$ Theorem 1 reduces to Theorem 2.1 of [3].

Theorem 3. *If the nonlinear dynamic system (1) has a real solution $(x(s, t), u(s, t))$ which is not identically zero on $[a, b]_{\mathbb{T}} \times [c, d]_{\mathbb{T}}$ satisfying $x(s, c) = x(s, d) = 0$, $x(a, t)x(\sigma(a), t) < 0$, $x(b, t)x(\sigma(b), t) < 0$, and*

$x^{\Delta_1}(s, \sigma(t))u^{\Delta_2}(s, t) + x^{\Delta_2}(\sigma(s), t)u^{\Delta_1}(s, t) = 0$ *for all $(s, t) \in [a, b]_{\mathbb{T}} \times [c, d]_{\mathbb{T}}$, where $a, b, c, d \in \mathbb{T}$ with $a < b$, $c < d$, then*

$$
2 \le \int_a^{\sigma(b)} \int_c^d \left|\alpha_1\left(s, t\right)\right| \Delta t\, \Delta s
$$

$$
+ M^{\beta/\alpha-1}\left(\int_a^{\sigma(b)} \int_c^d \beta_1\left(s, t\right) \Delta t\, \Delta s\right)^{1/\gamma} \tag{17}
$$

$$
\cdot \left(\int_a^{\sigma(b)} \int_c^d \beta_2^+\left(s, t\right) \Delta t\, \Delta s\right)^{1/\alpha},
$$

where $1/\alpha + 1/\gamma = 1$ and M, $\beta_2^+(s, t)$ are as defined in Theorem 1.

Proof. Choose (τ_1, τ_2) such that $|x(\tau_1, \tau_2)| = M$. Note that $M > 0$. From (6) and

$$
\int_a^{\sigma(b)} \int_c^d \left(x\left(s, t\right) u\left(s, t\right)\right)^{\Delta_1 \Delta_2} \Delta t\, \Delta s
$$

$$
= x\left(\sigma\left(b\right), d\right) u\left(\sigma\left(b\right), d\right) - x\left(a, d\right) u\left(a, d\right) \tag{18}
$$

$$
- x\left(\sigma\left(b\right), c\right) u\left(\sigma\left(b\right), c\right) + x\left(a, c\right) u\left(a, c\right) = 0,
$$

we have

$$
\int_a^{\sigma(b)} \int_c^d \beta_1\left(s, t\right) \left|u\left(s, t\right)\right|^{\gamma} \Delta t\, \Delta s
$$

$$
= \int_a^{\sigma(b)} \int_c^d \beta_2\left(s, t\right) \left|x\left(\sigma\left(s\right), \sigma\left(t\right)\right)\right|^{\beta} \Delta t\, \Delta s. \tag{19}
$$

By the boundary conditions on x, it is elementary to check that

$$
\int_a^{\tau_1} \int_c^{\tau_2} \left(x\left(s, t\right)\right)^{\Delta_1 \Delta_2} \Delta t\, \Delta s = x\left(\tau_1, \tau_2\right) - x\left(a, \tau_2\right). \tag{20}
$$

So

$$
x\left(\tau_1, \tau_2\right) = x\left(a, \tau_2\right) + \int_a^{\tau_1} \int_c^{\tau_2} \alpha_1\left(s, t\right) x\left(\sigma\left(s\right), \sigma\left(t\right)\right) \Delta t\, \Delta s
$$

$$
+ \int_a^{\tau_1} \int_c^{\tau_2} \beta_1\left(s, t\right) \left|u\left(s, t\right)\right|^{\gamma-2} u\left(s, t\right) \Delta t\, \Delta s. \tag{21}
$$

For the fixed τ_2, by $x(a, \tau_2)x(\sigma(a), \tau_2) < 0$, there exists $\xi \in (0, 1)$ such that

$$
(1 - \xi) x\left(a, \tau_2\right) + \xi x\left(\sigma\left(a\right), \tau_2\right) = 0. \tag{22}
$$

Integrating the first equation of (1) over t from c to τ_2, we obtain

$$
x^{\Delta_1}\left(s, \tau_2\right) - x^{\Delta_1}\left(s, c\right)
$$

$$
= \int_c^{\tau_2} \alpha_1\left(s, t\right) x\left(\sigma\left(s\right), \sigma\left(t\right)\right) \Delta t \tag{23}
$$

$$
+ \int_c^{\tau_2} \beta_1\left(s, t\right) \left|u\left(s, t\right)\right|^{\gamma-2} u\left(s, t\right) \Delta t.
$$

Multiplying (23) by $\mu(s)$ and noting that $x(s,c)=0$, we get

$$
\begin{aligned}
&x\left(\sigma\left(s\right),\tau_2\right)-x\left(s,\tau_2\right)\\
&\quad=\mu\left(s\right)\int_c^{\tau_2}\alpha_1\left(s,t\right)x\left(\sigma\left(s\right),\sigma\left(t\right)\right)\Delta t\\
&\qquad+\mu\left(s\right)\int_c^{\tau_2}\beta_1\left(s,t\right)\left|u\left(s,t\right)\right|^{\gamma-2}u\left(s,t\right)\Delta t.
\end{aligned}\tag{24}
$$

Letting $s=a$, we get

$$
\begin{aligned}
&x\left(\sigma\left(a\right),\tau_2\right)-x\left(a,\tau_2\right)\\
&\quad=\mu\left(a\right)\int_c^{\tau_2}\alpha_1\left(a,t\right)x\left(\sigma\left(a\right),\sigma\left(t\right)\right)\Delta t\\
&\qquad+\mu\left(a\right)\int_c^{\tau_2}\beta_1\left(a,t\right)\left|u\left(a,t\right)\right|^{\gamma-2}u\left(a,t\right)\Delta t.
\end{aligned}\tag{25}
$$

By (22) and (25), we obtain

$$
\begin{aligned}
x\left(a,\tau_2\right)&=-\xi\mu\left(a\right)\int_c^{\tau_2}\alpha_1\left(a,t\right)x\left(\sigma\left(a\right),\sigma\left(t\right)\right)\Delta t\\
&\quad-\xi\mu\left(a\right)\int_c^{\tau_2}\beta_1\left(a,t\right)\left|u\left(a,t\right)\right|^{\gamma-2}u\left(a,t\right)\Delta t.
\end{aligned}\tag{26}
$$

Substituting (26) into (21), we get

$$
\begin{aligned}
&x\left(\tau_1,\tau_2\right)\\
&=-\xi\mu\left(a\right)\int_c^{\tau_2}\alpha_1\left(a,t\right)x\left(\sigma\left(a\right),\sigma\left(t\right)\right)\Delta t\\
&\quad-\xi\mu\left(a\right)\int_c^{\tau_2}\beta_1\left(a,t\right)\left|u\left(a,t\right)\right|^{\gamma-2}u\left(a,t\right)\Delta t\\
&\quad+\int_a^{\tau_1}\int_c^{\tau_2}\alpha_1\left(s,t\right)x\left(\sigma\left(s\right),\sigma\left(t\right)\right)\Delta t\,\Delta s\\
&\quad+\int_a^{\tau_1}\int_c^{\tau_2}\beta_1\left(s,t\right)\left|u\left(s,t\right)\right|^{\gamma-2}u\left(s,t\right)\Delta t\,\Delta s\\
&=-\xi\mu\left(a\right)\int_c^{\tau_2}\alpha_1\left(a,t\right)x\left(\sigma\left(a\right),\sigma\left(t\right)\right)\Delta t\\
&\quad+\mu\left(a\right)\int_c^{\tau_2}\alpha_1\left(a,t\right)x\left(\sigma\left(a\right),\sigma\left(t\right)\right)\Delta t\\
&\quad-\xi\mu\left(a\right)\int_c^{\tau_2}\beta_1\left(a,t\right)\left|u\left(a,t\right)\right|^{\gamma-2}u\left(a,t\right)\Delta t\\
&\quad+\mu\left(a\right)\int_c^{\tau_2}\beta_1\left(a,t\right)\left|u\left(a,t\right)\right|^{\gamma-2}u\left(a,t\right)\Delta t\\
&\quad+\int_{\sigma(a)}^{\tau_1}\int_c^{\tau_2}\alpha_1\left(s,t\right)x\left(\sigma\left(s\right),\sigma\left(t\right)\right)\Delta t\,\Delta s
\end{aligned}
$$

$$
\begin{aligned}
&\quad+\int_{\sigma(a)}^{\tau_1}\int_c^{\tau_2}\beta_1\left(s,t\right)\left|u\left(s,t\right)\right|^{\gamma-2}u\left(s,t\right)\Delta t\,\Delta s\\
&=\left(1-\xi\right)\mu\left(a\right)\int_c^{\tau_2}\alpha_1\left(a,t\right)x\left(\sigma\left(a\right),\sigma\left(t\right)\right)\Delta t\\
&\quad+\left(1-\xi\right)\mu\left(a\right)\int_c^{\tau_2}\beta_1\left(a,t\right)\left|u\left(a,t\right)\right|^{\gamma-2}u\left(a,t\right)\Delta t\\
&\quad+\int_{\sigma(a)}^{\tau_1}\int_c^{\tau_2}\alpha_1\left(s,t\right)x\left(\sigma\left(s\right),\sigma\left(t\right)\right)\Delta t\,\Delta s\\
&\quad+\int_{\sigma(a)}^{\tau_1}\int_c^{\tau_2}\beta_1\left(s,t\right)\left|u\left(s,t\right)\right|^{\gamma-2}u\left(s,t\right)\Delta t\,\Delta s.
\end{aligned}\tag{27}
$$

Immitating the arguments from (21) to (27) step by step, we have

$$
\begin{aligned}
x\left(\tau_1,\tau_2\right)&=x\left(b,\tau_2\right)+\int_{\tau_1}^b\int_{\tau_2}^d\alpha_1\left(s,t\right)x\left(\sigma\left(s\right),\sigma\left(t\right)\right)\Delta t\,\Delta s\\
&\quad+\int_{\tau_1}^b\int_{\tau_2}^d\beta_1\left(s,t\right)\left|u\left(s,t\right)\right|^{\gamma-2}u\left(s,t\right)\Delta t\,\Delta s.
\end{aligned}\tag{28}
$$

By $x(b,\tau_2)x(\sigma(b),\tau_2)<0$, there exists $\eta\in(0,1)$ such that

$$
\left(1-\eta\right)x\left(b,\tau_2\right)+\eta x\left(\sigma\left(b\right),\tau_2\right)=0.\tag{29}
$$

Integrating the first equation of (1) over t from τ_2 to d and using $x(s,d)=0$, we get

$$
\begin{aligned}
x\left(b,\tau_2\right)&=\eta\mu\left(b\right)\int_{\tau_2}^d\alpha_1\left(b,t\right)x\left(\sigma\left(b\right),\sigma\left(t\right)\right)\Delta t\\
&\quad+\eta\mu\left(b\right)\int_{\tau_2}^d\beta_1\left(b,t\right)\left|u\left(b,t\right)\right|^{\gamma-2}u\left(b,t\right)\Delta t,
\end{aligned}\tag{30}
$$

and, hence,

$$
\begin{aligned}
&x\left(\tau_1,\tau_2\right)\\
&=\eta\mu\left(b\right)\int_{\tau_2}^d\alpha_1\left(b,t\right)x\left(\sigma\left(b\right),\sigma\left(t\right)\right)\Delta t\\
&\quad+\eta\mu\left(b\right)\int_{\tau_2}^d\beta_1\left(b,t\right)\left|u\left(b,t\right)\right|^{\gamma-2}u\left(b,t\right)\Delta t\\
&\quad+\int_{\tau_1}^b\int_{\tau_2}^d\alpha_1\left(s,t\right)x\left(\sigma\left(s\right),\sigma\left(t\right)\right)\Delta t\,\Delta s\\
&\quad+\int_{\tau_1}^b\int_{\tau_2}^d\beta_1\left(s,t\right)\left|u\left(s,t\right)\right|^{\gamma-2}u\left(s,t\right)\Delta t\,\Delta s.
\end{aligned}\tag{31}
$$

Let

$$
\tilde\alpha_1\left(s,t\right)=\begin{cases}\left(1-\xi\right)\alpha_1\left(a,t\right),&s=a,\\ \alpha_1\left(s,t\right),&\sigma\left(a\right)\le s<b,\\ \eta\alpha_1\left(b,t\right),&s=b,\end{cases}
$$
$$
\tilde\beta_1\left(s,t\right)=\begin{cases}\left(1-\xi\right)\beta_1\left(a,t\right),&s=a,\\ \beta_1\left(s,t\right),&\sigma\left(a\right)\le s<b,\\ \eta\beta_1\left(b,t\right),&s=b.\end{cases}\tag{32}
$$

Then, (27) can be written as

$$x(\tau_1, \tau_2) = \int_a^{\tau_1} \int_c^{\tau_2} \widetilde{\alpha}_1(s,t) \, x(\sigma(s), \sigma(t)) \, \Delta t \, \Delta s$$
$$+ \int_a^{\tau_1} \int_c^{\tau_2} \widetilde{\beta}_1(s,t) \, |u(s,t)|^{\gamma-2} u(s,t) \, \Delta t \, \Delta s, \tag{33}$$

and (31) can be written as

$$x(\tau_1, \tau_2) = \int_{\tau_1}^{\sigma(b)} \int_{\tau_2}^{d} \widetilde{\alpha}_1(s,t) \, x(\sigma(s), \sigma(t)) \, \Delta t \, \Delta s$$
$$+ \int_{\tau_1}^{\sigma(b)} \int_{\tau_2}^{d} \widetilde{\beta}_1(s,t) \, |u(s,t)|^{\gamma-2} u(s,t) \, \Delta t \, \Delta s. \tag{34}$$

It follows from (33) and (34) that

$$|x(\tau_1, \tau_2)| \le \int_a^{\tau_1} \int_c^{\tau_2} |\widetilde{\alpha}_1(s,t)| \, |x(\sigma(s), \sigma(t))| \, \Delta t \, \Delta s$$
$$+ \int_a^{\tau_1} \int_c^{\tau_2} \widetilde{\beta}_1(s,t) \, |u(s,t)|^{\gamma-1} \Delta t \, \Delta s,$$
$$|x(\tau_1, \tau_2)| \le \int_{\tau_1}^{\sigma(b)} \int_{\tau_2}^{d} |\widetilde{\alpha}_1(s,t)| \, |x(\sigma(s), \sigma(t))| \, \Delta t \, \Delta s \tag{35}$$
$$+ \int_{\tau_1}^{\sigma(b)} \int_{\tau_2}^{d} \widetilde{\beta}_1(s,t) \, |u(s,t)|^{\gamma-1} \Delta t \, \Delta s.$$

Using (19), $|\widetilde{\alpha}_1(s,t)| \le |\alpha_1(s,t)|$, $\widetilde{\beta}_1(s,t) \le \beta_1(s,t)$, and Hölder's inequality, we have

$$2 \, |x(\tau_1, \tau_2)|$$
$$\le \int_a^{\sigma(b)} \int_c^{d} |\widetilde{\alpha}_1(s,t)| \, |x(\sigma(s), \sigma(t))| \, \Delta t \, \Delta s$$
$$+ \int_a^{\sigma(b)} \int_c^{d} \widetilde{\beta}_1(s,t) \, |u(s,t)|^{\gamma-1} \Delta t \, \Delta s$$
$$\le \int_a^{\sigma(b)} \int_c^{d} |\alpha_1(s,t)| \, |x(\sigma(s), \sigma(t))| \, \Delta t \, \Delta s$$
$$+ \int_a^{\sigma(b)} \int_c^{d} \beta_1(s,t) \, |u(s,t)|^{\gamma-1} \Delta t \, \Delta s$$
$$\le \int_a^{\sigma(b)} \int_c^{d} |\alpha_1(s,t)| \, |x(\sigma(s), \sigma(t))| \, \Delta t \, \Delta s$$
$$+ \left(\int_a^{\sigma(b)} \int_c^{d} \beta_1(s,t) \, \Delta t \, \Delta s \right)^{1/\gamma}$$
$$\times \left(\int_a^{\sigma(b)} \int_c^{d} \beta_1(s,t) \, |u(s,t)|^{\gamma} \Delta t \, \Delta s \right)^{1/\alpha}$$

$$\le \int_a^{\sigma(b)} \int_c^{d} |\alpha_1(s,t)| \, |x(\sigma(s), \sigma(t))| \, \Delta t \, \Delta s$$
$$+ \left(\int_a^{\sigma(b)} \int_c^{d} \beta_1(s,t) \, \Delta t \, \Delta s \right)^{1/\gamma}$$
$$\times \left(\int_a^{\sigma(b)} \int_c^{d} \beta_2(s,t) \, |x(\sigma(s), \sigma(t))|^{\beta} \Delta t \, \Delta s \right)^{1/\alpha}$$
$$\le M \int_a^{\sigma(b)} \int_c^{d} |\alpha_1(s,t)| \, \Delta t \, \Delta s$$
$$+ M^{\beta/\alpha} \left(\int_a^{\sigma(b)} \int_c^{d} \beta_1(s,t) \, \Delta t \, \Delta s \right)^{1/\gamma}$$
$$\times \left(\int_a^{\sigma(b)} \int_c^{d} \beta_2^+(s,t) \, \Delta t \, \Delta s \right)^{1/\alpha}. \tag{36}$$

Therefore,

$$2 \le \int_a^{\sigma(b)} \int_c^{d} |\alpha_1(s,t)| \, \Delta t \, \Delta s$$
$$+ M^{\beta/\alpha-1} \left(\int_a^{\sigma(b)} \int_c^{d} \beta_1(s,t) \, \Delta t \, \Delta s \right)^{1/\gamma}$$
$$\cdot \left(\int_a^{\sigma(b)} \int_c^{d} \beta_2^+(s,t) \, \Delta t \, \Delta s \right)^{1/\alpha}. \tag{37}$$

The proof is complete. □

Theorem 4. *Suppose that β and γ are conjugate exponents; that is, $1/\beta + 1/\gamma = 1$. If the nonlinear dynamic system (1) has a real solution $(x(s,t), u(s,t))$ which is not identically zero on $[a,b]_{\mathbb{T}} \times [c,d]_{\mathbb{T}}$ such that $x(a,t) = x(s,c) = x(s,d) = 0$ and $x^{\Delta_1}(s, \sigma(t))u^{\Delta_2}(s,t) + x^{\Delta_2}(\sigma(s),t)u^{\Delta_1}(s,t) = 0$ for all $(s,t) \in [a,b]_{\mathbb{T}} \times [c,d]_{\mathbb{T}}$, where $a,b,c,d \in \mathbb{T}$ with $a < b$, $c < d$, then*

$$1 \le \int_a^{\rho(b)} \int_c^{d} |\alpha_1(s,t)| \, \Delta t \, \Delta s + \left(\int_a^{\rho(b)} \int_c^{d} \beta_1(s,t) \, \Delta t \, \Delta s \right)^{1/\gamma}$$
$$\cdot \left(\int_a^{\rho(b)} \int_c^{d} \beta_2^+(s,t) \, \Delta t \, \Delta s \right)^{1/\beta}. \tag{38}$$

Proof. By (6) and the conditions $x(s,c) = x(s,d) = 0$ for all $s \in [a,b]_{\mathbb{T}}$, we have

$$\int_a^{\rho(b)} \int_c^{d} (x(s,t) \, u(s,t))^{\Delta_1 \Delta_2} \Delta t \, \Delta s$$
$$= x(\rho(b), d) \, u(\rho(b), d) - x(a,d) \, u(a,d) \tag{39}$$
$$- x(\rho(b), c) \, u(\rho(b), c) + x(a,c) \, u(a,c) = 0.$$

So

$$\int_a^{\rho(b)} \int_c^d \beta_1(s,t) |u(s,t)|^\gamma \Delta t \, \Delta s$$
$$= \int_a^{\rho(b)} \int_c^d \beta_2(s,t) |x(\sigma(s),\sigma(t))|^\beta \Delta t \, \Delta s. \tag{40}$$

Fix (τ_1,τ_2) such that $|x(\tau_1,\tau_2)| = \max\{|x(s,t)| : \sigma(a) \le s < b, \sigma(c) \le t < d\} > 0$. Integrating the first equation of (1) over t from c to τ_2 and then over s from a to τ_1, we get

$$x(\tau_1,\tau_2) = \int_a^{\tau_1} \int_c^{\tau_2} \alpha_1(s,t) x(\sigma(s),\sigma(t)) \Delta t \, \Delta s$$
$$+ \int_a^{\tau_1} \int_c^{\tau_2} \beta_1(s,t) |u(s,t)|^{\gamma-2} u(s,t) \Delta t \, \Delta s, \tag{41}$$

and so by Hölder's inequality with indices γ and β we have

$$|x(\tau_1,\tau_2)| \le \int_a^{\rho(b)} \int_c^d |\alpha_1(s,t)| \, |x(\sigma(s),\sigma(t))| \Delta t \, \Delta s$$
$$+ \int_a^{\rho(b)} \int_c^d \beta_1(s,t) |u(s,t)|^{\gamma-1} \Delta t \, \Delta s$$
$$\le \int_a^{\rho(b)} \int_c^d |\alpha_1(s,t)| \, |x(\sigma(s),\sigma(t))| \Delta t \, \Delta s$$
$$+ \left(\int_a^{\rho(b)} \int_c^d \beta_1(s,t) \Delta t \, \Delta s \right)^{1/\gamma}$$
$$\cdot \left(\int_a^{\rho(b)} \int_c^d \beta_1(s,t) |u(s,t)|^\gamma \Delta t \, \Delta s \right)^{1/\beta}$$
$$\le |x(\tau_1,\tau_2)| \cdot \int_a^{\rho(b)} \int_c^d |\alpha_1(s,t)| \Delta t \, \Delta s$$
$$+ \left(\int_a^{\rho(b)} \int_c^d \beta_1(s,t) \Delta t \, \Delta s \right)^{1/\gamma}$$
$$\cdot \left(\int_a^{\rho(b)} \int_c^d \beta_2(s,t) |x(\sigma(s),\sigma(t))|^\beta \Delta t \, \Delta s \right)^{1/\beta}. \tag{42}$$

Hence,

$$1 \le \int_a^{\rho(b)} \int_c^d |\alpha_1(s,t)| \Delta t \, \Delta s$$
$$+ \left(\int_a^{\rho(b)} \int_c^d \beta_1(s,t) \Delta t \, \Delta s \right)^{1/\gamma} \tag{43}$$
$$\cdot \left(\int_a^{\rho(b)} \int_c^d \beta_2^+(s,t) \Delta t \, \Delta s \right)^{1/\beta}.$$

The proof is complete. $\qquad\square$

Theorem 5. *Suppose that β and γ are conjugate exponents. If the nonlinear dynamic system (1) has a real solution*

$(x(s,t),u(s,t))$ *which is not identically zero on* $[a,b]_\mathbb{T} \times [c,d]_\mathbb{T}$ *such that* $x(b,t) = x(s,c) = x(s,d) = 0$ *and* $x^{\Delta_1}(s,\sigma(t))u^{\Delta_2}(s,t) + x^{\Delta_2}(\sigma(s),t)u^{\Delta_1}(s,t) = 0$ *for all* $(s,t) \in [a,b]_\mathbb{T} \times [c,d]_\mathbb{T}$, *where* $a,b,c,d \in \mathbb{T}$ *with* $a < b$, $c < d$, *then*

$$1 \le \int_{\sigma(a)}^b \int_c^d |\alpha_1(s,t)| \Delta t \, \Delta s + \left(\int_{\sigma(a)}^b \int_c^d \beta_1(s,t) \Delta t \, \Delta s \right)^{1/\gamma}$$
$$\cdot \left(\int_{\sigma(a)}^b \int_c^d \beta_2^+(s,t) \Delta t \, \Delta s \right)^{1/\beta}. \tag{44}$$

Proof. By (6) and the assumption that $x(s,c) = x(s,d) = 0$ for all $s \in [a,b]_\mathbb{T}$, we get

$$\int_{\sigma(a)}^b \int_c^d (x(s,t) u(s,t))^{\Delta_1 \Delta_2} \Delta t \, \Delta s$$
$$= x(b,d) u(b,d) - x(\sigma(a),d) u(\sigma(a),d)$$
$$- x(b,c) u(b,c) + x(\sigma(a),c) u(\sigma(a),c) = 0. \tag{45}$$

So

$$\int_{\sigma(a)}^b \int_c^d \beta_1(s,t) |u(s,t)|^\gamma \Delta t \, \Delta s$$
$$= \int_{\sigma(a)}^b \int_c^d \beta_2(s,t) |x(\sigma(s),\sigma(t))|^\beta \Delta t \, \Delta s. \tag{46}$$

Fix (τ_1,τ_2) such that $|x(\tau_1,\tau_2)| = \max\{|x(s,t)| : \sigma(a) \le s \le \rho(b), \sigma(t) \le t \le \rho(b)\} > 0$. Integrating the first equation of (1) over t from τ_2 to d and then over s from τ_1 to b, we get

$$x(\tau_1,\tau_2) = \int_{\tau_1}^b \int_{\tau_2}^d \alpha_1(s,t) x(\sigma(s),\sigma(t)) \Delta t \, \Delta s$$
$$+ \int_{\tau_1}^b \int_{\tau_2}^d \beta_1(s,t) |u(s,t)|^{\gamma-2} u(s,t) \Delta t \, \Delta s, \tag{47}$$

and so by Hölder's inequality with indices γ and β we have

$$|x(\tau_1,\tau_2)| \le \int_{\sigma(a)}^b \int_c^d |\alpha_1(s,t)| \, |x(\sigma(s),\sigma(t))| \Delta t \, \Delta s$$
$$+ \int_{\sigma(a)}^b \int_c^d \beta_1(s,t) |u(s,t)|^{\gamma-1} \Delta t \, \Delta s$$
$$\le \int_{\sigma(a)}^b \int_c^d |\alpha_1(s,t)| \, |x(\sigma(s),\sigma(t))| \Delta t \, \Delta s$$
$$+ \left(\int_{\sigma(a)}^b \int_c^d \beta_1(s,t) \Delta t \, \Delta s \right)^{1/\gamma}$$

$$\cdot \left(\int_{\sigma(a)}^{b} \int_{c}^{d} \beta_1(s,t) \left| u(s,t) \right|^{\gamma} \Delta t \, \Delta s \right)^{1/\beta}$$

$$\leq \left| x(\tau_1, \tau_2) \right| \int_{\sigma(a)}^{b} \int_{c}^{d} \left| \alpha_1(s,t) \right| \Delta t \, \Delta s$$

$$+ \left(\int_{\sigma(a)}^{b} \int_{c}^{d} \beta_1(s,t) \Delta t \, \Delta s \right)^{1/\gamma}$$

$$\cdot \left(\int_{\sigma(a)}^{b} \int_{c}^{d} \beta_2(s,t) \left| x(\sigma(s), \sigma(t)) \right|^{\beta} \Delta t \, \Delta s \right)^{1/\beta}.$$

$$(48)$$

Hence,

$$1 \leq \int_{\sigma(a)}^{b} \int_{c}^{d} \left| \alpha_1(s,t) \right| \Delta t \, \Delta s$$

$$+ \left(\int_{\sigma(a)}^{b} \int_{c}^{d} \beta_1(s,t) \Delta t \, \Delta s \right)^{1/\gamma} \qquad (49)$$

$$\cdot \left(\int_{\sigma(a)}^{b} \int_{c}^{d} \beta_2^+(s,t) \Delta t \, \Delta s \right)^{1/\beta}.$$

The proof is complete. □

Remark 6. Analogously, we can also consider the cases (i) $x(a,t) = x(b,t) = x(s,d) = 0$ and (ii) $x(a,t) = x(b,t) = x(s,c) = 0$. Similar results to those in Theorem 4 and Theorem 5 can easily be arrived at. The detailed proofs are omitted here.

Next, we exhibit an application of our results. Consider the following special case of (1):

$$x^{\Delta_1 \Delta_2}(s,t) = \beta_1(s,t) \left| u(s,t) \right|^{\gamma-2} u(s,t),$$

$$u^{\Delta_1 \Delta_2}(s,t) = -\beta_2(s,t) \left| x(\sigma(s), \sigma(t)) \right|^{\beta-2} x(\sigma(s), \sigma(t)).$$

$$(50)$$

Definition 7. A nontrivial solution $(x(s,t), u(s,t))$ of the dynamic equation (50) defined on $[s_0, \infty)_{\mathbb{T}} \times [t_0, \infty)_{\mathbb{T}}$ is said to be *proper* if

$$\sup \left\{ \left| x(s,t) \right| + \left| u(s,t) \right| : s \in [a, \infty)_{\mathbb{T}}, t \in [c, \infty)_{\mathbb{T}} \right\} > 0 \quad (51)$$

for all $a \geq s_0$, $c \geq t_0$. A proper solution $(x(s,t), u(s,t))$ of the dynamic system (50) is called *weakly oscillatory* if at least one argument has a sequence of zeros diverging to ∞.

Theorem 8. *Assume that $u(\tau,t)$ is bounded on $[t_0, \infty)_{\mathbb{T}}$ for a fixed τ, $u(s,\eta)$ is bounded on $[s_0, \infty)_{\mathbb{T}}$ for a fixed η,*

$$\int_{s_0}^{\infty} \int_{t_0}^{\infty} \beta_1(s,t) \Delta t \, \Delta s < \infty,$$

$$\int_{s_0}^{\infty} \int_{t_0}^{\infty} \left| \beta_2(s,t) \right| \Delta t \, \Delta s < \infty,$$

$$(52)$$

then every weakly oscillatory proper solution of (50) is bounded on $[s_0, \infty)_{\mathbb{T}} \times [t_0, \infty)_{\mathbb{T}}$.

Proof. Let $(x(s,t), u(s,t))$ be a nontrivial weakly oscillatory proper solution of (50) on $[s_0, \infty)_{\mathbb{T}} \times [t_0, \infty)_{\mathbb{T}}$, and $x(s,t)$ have a sequence of zeros diverging to ∞. Suppose that $\limsup |x(s,t)| = \infty$; then, for any positive constant M_0, there exists $\{S_n, T_n\}$ such that $|x(S_n, T_n)| > M_0$, $n = 1, 2, \ldots$. Since $x(s,t)$ is an oscillatory solution, there exists $I := (a,b)_{\mathbb{T}} \times (c,d)_{\mathbb{T}}$ such that $x(a,t) = x(b,t) = x(s,c) = x(s,d) = 0$, $|x(s,t)| > 0$ on I, and $|x(\tau_1, \tau_2)| = \max\{|x(s,t)| : \sigma(a) \leq s < d, \sigma(c) \leq t < d\} := M > M_0$. From (52), we can choose a, c sufficiently large such that

$$\int_{a}^{\infty} \int_{c}^{\infty} \beta_1(s,t) \Delta t \, \Delta s < M^{(\alpha-\beta)/(\alpha-1)},$$

$$\int_{a}^{\infty} \int_{c}^{\infty} \left| \beta_2(s,t) \right| \Delta t \, \Delta s < 1.$$

$$(53)$$

By Theorem 1, we have

$$2 \leq M^{\beta/\alpha-1} \left(\int_{a}^{b} \int_{c}^{d} \beta_1(s,t) \Delta t \, \Delta s \right)^{1/\gamma}$$

$$\cdot \left(\int_{a}^{b} \int_{c}^{d} \beta_2^+(s,t) \Delta t \, \Delta s \right)^{1/\alpha},$$

$$(54)$$

and so

$$2^{\alpha} \leq M^{\beta-\alpha} \left(\int_{a}^{b} \int_{c}^{d} \beta_1(s,t) \Delta t \, \Delta s \right)^{\alpha-1}$$

$$\cdot \left(\int_{a}^{b} \int_{c}^{d} \beta_2^+(s,t) \Delta t \, \Delta s \right)$$

$$\leq M^{\beta-\alpha} \cdot M^{\alpha-\beta} = 1,$$

$$(55)$$

which contradicts $\alpha > 1$. Hence, there exists a positive constant K such that $|x(s,t)| \leq K$.

Integrating the second equation of the dynamic equation (50) over t from τ_2 to t and then over s from τ_1 to s, respectively, we get

$$\int_{\tau_1}^{s} \int_{\tau_2}^{t} u^{\Delta_1 \Delta_2}(s,t) \Delta t \, \Delta s$$

$$= u(s,t) - u(\tau_1, t) - u(s, \tau_2) + u(\tau_1, \tau_2)$$

$$= - \int_{\tau_1}^{s} \int_{\tau_2}^{t} \beta_2(s,t) \left| x(\sigma(s), \sigma(t)) \right|^{\beta-2} x(\sigma(s), \sigma(t)) \Delta t \, \Delta s.$$

$$(56)$$

Since $u(\tau,t)$ is bounded on $[t_0, \infty)_{\mathbb{T}}$ for a fixed τ and $u(s,\eta)$ is bounded on $[s_0, \infty)_{\mathbb{T}}$ for a fixed η, we get

$$\left| u(s,t) \right| \leq \left| u(\tau_1, t) + u(s, \tau_2) - u(\tau_1, \tau_2) \right|$$

$$+ \int_{\tau_1}^{s} \int_{\tau_2}^{t} \left| \beta_2(s,t) \right| \left| x(\sigma(s), \sigma(t)) \right|^{\beta-1} \Delta t \, \Delta s$$

$$\leq \left| u(\tau_1, t) + u(s, \tau_2) - u(\tau_1, \tau_2) \right|$$

$$+ K^{\beta-1} \int_{\tau_1}^{\infty} \int_{\tau_2}^{\infty} \left| \beta_2(s,t) \right| \Delta t \, \Delta s.$$

$$(57)$$

Since $\int_{s_0}^{\infty} \int_{t_0}^{\infty} |\beta_2(s,t)| \Delta t \Delta s < \infty$, $|u(s,t)|$ is bounded. So $\limsup\{|x(s,t)| + u(s,t)|\}$ is bounded on $[s_0, \infty)_{\mathbb{T}} \times [t_0, \infty)_{\mathbb{T}}$. The proof is complete. $\qquad\square$

Acknowledgments

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